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On systems of nonlinear functional differential equations of fractional multi-order

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Abstract

This paper is devoted to a nonlinear system of multi-order fractional differential equations with variable delays. Using the contraction principle, we show the existence and uniqueness of the solution. Furthermore, we prove the stability of the solution.

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1. Introduction

In the last few years the theory of fractional differential equations had an important role in the real world. Indeed, this type of equations have various applications in physics, biology, chemistry, engineering, etc. see for example [1, 2, 3, 4, 6,7].

In the present work, we initiate the study of the nonlinear system of fractional differential equations with variable delays $\tau_j = \tau_j(t)$, and of multi-order $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ of the form

$$ {}^{c}D^{\alpha_i} x_i(t) = \sum_{j=1}^{n} f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right), \quad i = \underline{1, n}, \quad t > 0, \tag{1.1} $$

$$ x(t) = \Phi(t) = \left(\phi_1(t), \ldots, \phi_n(t)\right), \quad t \in [-\tau, 0]. \tag{1.2} $$

Where ${}^{c}D^{\alpha_i}$ denotes the Caputo fractional derivation of order $\alpha_i \in]0, 1[$ for $i = 1, \ldots, n$. Here, we assume that $f_{ij} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and

τ_j are continuous real-valued functions defined on $\mathbb{R}^+$, such that
$\tau = \max \left\{ \sup_{t \in \mathbb{R}^+} \tau_j(t) : j = 1, 2, \dots, n \right\} > 0$, $\Phi(t)$ is a real-vector function defined on$[-\tau, 0]$ with values in $\mathbb{R}^n$.

Firstly in section 2, the definitions of Riemann–Liouville integral, and Caputo fractional derivative are introduced. Secondly in section 3, sufficient conditions for the existence and uniqueness of the solution of the problem (1.1)-(1.2) are given. Finally in section 4, we discuss the stability of the solution with respect to the orders of derivation and the initial condition.

2. Preliminaries

This section contains the definitions and properties of Riemann-Liouville fractional integral and Caputo fractional derivative (see [4, 7]), which will be used throughout this paper.

Definition 2.1. For all $T > 0$, the Riemann–Liouville fractional integral of a function $f \in L^1[0, T]$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0$$

where Γ is the gamma function.

For $\alpha = 0$, we set $I^0 := Id$, the identity operator.

The semigroup property of Riemann–Liouville fractional integral is given by

Theorem 2.2. If $\alpha, \beta > 0$ and $f \in L^p[0, T]$ $(1 \le p \le \infty)$, then

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t) = I^\beta I^\alpha f(t)$$

at almost every point $t \in [0, T]$.

Definition 2.3. The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of the function f with $D^n f \in L^1[0, T]$ is defined by

$${}^c D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds,$$

where $n - 1 < \alpha \le n$, $n \in \mathbb{N}^*$ and $D = \dfrac{d}{dt}$.

Theorem 2.4. If $f \in C[a, b]$ and $\alpha > 0$ $(n - 1 < \alpha \le n)$, then

$${}^c D^\alpha I^\alpha f(t) = f(t).$$

3. Existence and uniqueness

In this section we give sufficient conditions for the existence and uniqueness of the solution of the problem (1.1)-(1.2). Firstly, based on the semigroup property we prove the following lemma.

Lemma 3.1. The vector function $x(t) := (x_1(t), \dots, x_n(t))$ is a solution of the problem (1.1)-(1.2) if and only if

$$x_i(t) = \begin{cases} \phi_i(0) + \sum_{j=1}^{n} I^{\alpha_i} f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right), & t > 0 \\ \phi_i(t), & t \in [-\tau, 0], \ i = 1, \dots, n. \end{cases} \tag{3.3}$$

Proof. For $t > 0$ and $i = \overline{1, n}$, the equation (1.1) can be written as

$$I^{1-\alpha_i} D x_i(t) = \sum_{j=1}^{n} f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right).$$

Applying the operator I^{α_i} on both sides, we obtain

$$\begin{aligned} I^1 D x_i(t) &= \sum_{j=1}^{n} I^{\alpha_i} f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right). \\ x_i(t) - x_i(0) &= \sum_{j=1}^{n} I^{\alpha_i} f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right). \end{aligned}$$

Then

$$x_i(t) = \phi_i(0) + \sum_{j=1}^{n} I^{\alpha_i} f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right).$$

■

Now, to show the existence and uniqueness of the solution of the problem (1.1)−(1.2) based on the contraction principle, let us fix firstly the appropriate functional framework see [5].

Let the Banach space $E = \left\{ v \in C\left([-\tau, +\infty[, \mathbb{R}^n\right) : v(t) = \Phi(t), \ t \in [-\tau, 0] \right\}$ equipped with the distance defined for all x and y in E by

$$d_\lambda(x, y) := \sum_{i=1}^{n} \sup_{t \in \mathbb{R}^+} \left\{ e^{-\lambda t} \left| x_i(t) - y_i(t) \right| \right\},$$

where $\lambda \in \mathbb{R}^+$ will be chosen later.

We define the continuous operator $F : E \to E$ by

$$(Fx)_i(t) = \begin{cases} \phi_i(0) + \sum_{j=1}^{n} I^{\alpha_i} f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right), & t > 0 \\ \phi_i(t), & t \in [-\tau, 0],\ i = \overline{1, n}. \end{cases}$$

Theorem 3.2. Assume that the following hypotheses are satisfied:

(H_1) Let $f_{ij} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, satisfying the Lipschitz condition

$$\left| f_{ij}\left(t, x_i, y_j\right) - f_{ij}\left(t, u_i, v_j\right) \right| \leq k_i \left|x_i - u_i\right| + h_j \left|y_j - v_j\right|,$$

where $k_i, h_j > 0$, $i, j = \overline{1, n}$.
(H_2) For $j = \overline{1, n}$, $\tau_j \in C\left(\mathbb{R}^+, \mathbb{R}\right)$ and

$$\tau_j(t) > -\tau, \quad t > 0.$$

(H_3) For $j = \overline{1, n}$, $\exists t_j > 0$ such that

$$\begin{cases} \tau_j(t) \geq t, & \forall t \in \left[0, t_j\right], \\ \tau_j(t) < t, & \forall t \in \left]t_j, +\infty\right[. \end{cases}$$

(H_4)

$$\sum_{i=1}^{n} \tau^{\alpha_i} \left(n k_i + e \sum_{j=1}^{n} h_j \right) < 1.$$

Then the problem (1.1)-(1.2) has a unique solution.

Proof. Let $x, y \in E$, for $i = \overline{1, n}$ and $t > 0$ we have

$$\begin{aligned} &\left| F x_i(t) - F y_i(t) \right| \\ &= \left| \sum_{j=1}^{n} I^{\alpha_i} \left\{ f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right) - f_{ij}\left(t, y_i(t), y_j\left(t - \tau_j(t)\right)\right) \right\} \right| \\ &= \left| \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \left\{ f_{ij}\left(s, x_i(s), x_j\left(s - \tau_j(s)\right)\right) - f_{ij}\left(s, y_i(s), y_j\left(s - \tau_j(s)\right)\right) \right\} ds \right| \\ &\leq \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \left| f_{ij}\left(s, x_i(s), x_j\left(s - \tau_j(s)\right)\right) - f_{ij}\left(s, y_i(s), y_j\left(s - \tau_j(s)\right)\right) \right| ds. \end{aligned}$$

From (H_1)

$$\begin{aligned}
&\left|Fx_i(t) - Fy_i(t)\right| \\
\leq\ & \sum_{j=1}^{n} k_i \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \left|x_i(s) - y_i(s)\right| ds \\
+\ & \sum_{j=1}^{n} h_j \int_0^{t_j} \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \left|\phi_j\left(r_j(s)\right) - \phi_j\left(r_j(s)\right)\right| ds \\
+\ & \sum_{j=1}^{n} h_j \int_{t_j}^{t} \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \left|x_j\left(r_j(s)\right) - y_j\left(r_j(s)\right)\right| ds \\
\leq\ & nk_i \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \left|x_i(s) - y_i(s)\right| ds \\
+\ & \sum_{j=1}^{n} h_j \int_{t_j}^{t} \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \left|x_j\left(r_j(s)\right) - y_j\left(r_j(s)\right)\right| ds,
\end{aligned}$$

where $r_j(s) = s - \tau_j(s)$.

So

$$\begin{aligned}
&e^{-\lambda t}\left|Fx_i(t) - Fy_i(t)\right| \\
\leq\ & nk_i \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} e^{-\lambda(t-s)} e^{-\lambda s} \left|x_i(s) - y_i(s)\right| ds \\
+\ & \sum_{j=1}^{n} h_j \int_{t_j}^{t} \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} e^{-\lambda\left(t - r_j(s)\right)} e^{-\lambda r_j(s)} \left|x_j\left(r_j(s)\right) - y_j\left(r_j(s)\right)\right| ds
\end{aligned}$$

$$
\begin{aligned}
& e^{-\lambda t}\left|Fx_i(t)-Fy_i(t)\right| \\
\leq\ & nk_i \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_i(\zeta)-y_i(\zeta)\right|\right\}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\lambda(t-s)}ds \\
+\ & \sum_{j=1}^n h_j \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda r_j(\zeta)}\left|x_j\left(r_j(\zeta)\right)-y_j\left(r_j(\zeta)\right)\right|\right\}\int_{t_j}^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\lambda\left(t-r_j(s)\right)}ds.
\end{aligned}
$$

Using a change of variable, we find

$$
\begin{aligned}
& e^{-\lambda t}\left|Fx_i(t)-Fy_i(t)\right| \\
\leq\ & nk_i \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_i(\zeta)-y_i(\zeta)\right|\right\}\frac{1}{\lambda^{\alpha_i}}\int_0^{\lambda t} \frac{u^{\alpha_i-1}e^{-u}}{\Gamma(\alpha_i)}du \\
+\ & \sum_{j=1}^n h_j \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_j(\zeta)-y_j(\zeta)\right|\right\}\frac{1}{\lambda^{\alpha_i}}\int_0^{\lambda(t-t_j)} \frac{u^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-u}e^{-\lambda\tau_j\left(t-\frac{u}{\lambda}\right)}du \\
\leq\ & nk_i \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_i(\zeta)-y_i(\zeta)\right|\right\}\frac{1}{\lambda^{\alpha_i}}\int_0^{\lambda t} \frac{u^{\alpha_i-1}e^{-u}}{\Gamma(\alpha_i)}du \\
+\ & \sum_{j=1}^n h_j \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_j(\zeta)-y_j(\zeta)\right|\right\}\frac{1}{\lambda^{\alpha_i}}\int_0^{\lambda(t-t_j)} \frac{u^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-u}e^{\lambda\tau}du \\
\leq\ & \frac{nk_i}{\lambda^{\alpha_i}}\sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_i(\zeta)-y_i(\zeta)\right|\right\}+\sum_{j=1}^n\frac{h_je^{\lambda\tau}}{\lambda^{\alpha_i}}\sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_j(\zeta)-y_j(\zeta)\right|\right\} \\
\leq\ & \frac{nk_i}{\lambda^{\alpha_i}}d_\lambda(x,y)+\sum_{j=1}^n\frac{h_je^{\lambda\tau}}{\lambda^{\alpha_i}}d_\lambda(x,y).
\end{aligned}
$$

Then

$$\sum_{i=1}^{n} \sup_{t\in\mathbb{R}^+} \left\{e^{-\lambda t} \left|Fx_i(t) - Fy_i(t)\right|\right\}$$

$$\leq \left(\sum_{i=1}^{n} \frac{nk_i}{\lambda^{\alpha_i}} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{h_j e^{\lambda\tau}}{\lambda^{\alpha_i}}\right) d_\lambda(x, y)$$

$$\sum_{i=1}^{n} \sup_{t\in\mathbb{R}^+} \left\{e^{-\lambda t} \left|Fx_i(t) - Fy_i(t)\right|\right\}$$

$$\leq \left(n\sum_{i=1}^{n} k_i\tau^{\alpha_i} + \sum_{i=1}^{n} \tau^{\alpha_i} \sum_{j=1}^{n} h_j e^{\lambda\tau}\right) d_\lambda(x, y), \qquad \text{with } \tau = \frac{1}{\lambda}$$

$$\leq \left(n\sum_{i=1}^{n} k_i\tau^{\alpha_i} + \sum_{i=1}^{n} \tau^{\alpha_i} \sum_{j=1}^{n} h_j e\right) d_\lambda(x, y).$$

Finally

$$d_\lambda(Fx, Fy) \leq \sum_{i=1}^{n} \tau^{\alpha_i} \left(nk_i + e\sum_{j=1}^{n} h_j\right) d_\lambda(x, y).$$

As $\sum_{i=1}^{n} \tau^{\alpha_i} \left(nk_i + e\sum_{j=1}^{n} h_j\right) < 1$, then from the contraction principle the continuous operator $F : E \to E$ has a unique fixed point $x = Fx$, which is the unique solution of the problem $(1.1) - (1.2)$. ■

4. Stability

Now, we study the stability of the solution of the problem $(1.1) - (1.2)$ with respect to the orders of derivation and the initial condition in the sense of the following definition.

Definition 4.1. The solution x of the problem $(1.1) - (1.2)$ is stable with respect to the initial condition and the orders of derivation if: $\forall\epsilon_1, \epsilon_2 > 0$, there exists a real constant $\delta > 0$ not depending on ϵ_1 and ϵ_2, such that

$$d(\boldsymbol{\alpha}, \bar{\boldsymbol{\alpha}}) < \epsilon_1 \text{ and } \mathbf{d}\left(\Phi, \bar{\Phi}\right) < \epsilon_2 \Rightarrow d_\lambda(x, \bar{x}) < \delta \max\{\epsilon_1, \epsilon_2\},$$

where $\bar{x}$ is a solution of the following problem

$$ {}^{c}D_i^{\bar{\alpha}_i}\bar{x}(t) = \sum_{j=1}^{n} f_{ij}\left(t, \bar{x}_i(t), \bar{x}_j\left(t - \tau_j(t)\right)\right), \ \bar{\alpha}_i \in]0, 1[\ , \quad i = \overline{1, n}, \quad t > 0, \quad (4.4) $$

$$ \bar{x}(t) = \bar{\Phi}(t) = \left(\bar{\phi}_1(t), \ldots, \bar{\phi}_n(t)\right), \quad t \in [-\tau, 0], \qquad (4.5) $$

with

$$ d(\boldsymbol{\alpha}, \bar{\boldsymbol{\alpha}}) = \max_{i=\overline{1,n}} |\alpha_i - \bar{\alpha}_i| \qquad \text{and } \mathbf{d}\left(\Phi, \bar{\Phi}\right) = \sum_{i=1}^{n} \max_{t\in[-\tau,0]} \left|\phi_i(t) - \bar{\phi}_i(t)\right|. $$

Theorem 4.2. Assume that hypotheses $(H_1) - (H_4)$ of the precedent theorem are satisfied, and there exists a positive constant M such that

$$ M = \max_{t\in\mathbb{R}^+} \left| f_{ij}(t, u, v)\right|, \ i, j = 1, \ldots, n. $$

Then for all $\eta \in]0, 1[$, the solution of the problem (1.1)-(1.2) is stable with respect to the initial condition and the orders of derivation in $[\eta, 1[$.

Proof. Let $x(t)$ and $\bar{x}(t)$ be the solutions of the problems (1.1)-(1.2) and (4.4)-(4.5), with $\alpha_i, \bar{\alpha}_i \in [\eta, 1[$, $i = \overline{1, n}$, and for $t > 0$ we have

$$ \begin{aligned} x_i(t) - \bar{x}_i(t) &= \phi_i(0) - \bar{\phi}_i(0) \\ &\quad + \sum_{j=1}^{n} I^{\alpha_i} f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right) \\ &\quad - \sum_{j=1}^{n} I^{\bar{\alpha}_i} f_{ij}\left(t, \bar{x}_i(t), \bar{x}_j\left(t - \tau_j(t)\right)\right) \\ x_i(t) - \bar{x}_i(t) &= \phi_i(0) - \bar{\phi}_i(0) \\ &\quad + \sum_{j=1}^{n} I^{\alpha_i} f_{ij}\left(t, x_i(t), x_j\left(t - \tau_j(t)\right)\right) \\ &\quad - \sum_{j=1}^{n} I^{\alpha_i} f_{ij}\left(t, \bar{x}_i(t), \bar{x}_j\left(t - \tau_j(t)\right)\right) \end{aligned} $$

$$
\begin{aligned}
& +\sum_{j=1}^{n} I^{\alpha_i} f_{ij}\left(t, \bar{x}_i(t), \bar{x}_j\left(t-\tau_j(t)\right)\right) \\
& -\sum_{j=1}^{n} I^{\bar{\alpha}_i} f_{ij}\left(t, \bar{x}_i(t), \bar{x}_j\left(t-\tau_j(t)\right)\right) \\
x_i(t)-\bar{x}_i(t) \;=\; & \phi_i(0)-\bar{\phi}_i(0) \\
& +\sum_{j=1}^{n} I^{\alpha_i}\left\{f_{ij}\left(t, x_i(t), x_j\left(t-\tau_j(t)\right)\right)-f_{ij}\left(t, \bar{x}_i(t), \bar{x}_j\left(t-\tau_j(t)\right)\right)\right\} \\
& +\sum_{j=1}^{n}\left\{I^{\alpha_i}-I^{\bar{\alpha}_i}\right\} f_{ij}\left(t, \bar{x}_i(t), \bar{x}_j\left(t-\tau_j(t)\right)\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \left|x_i(t)-\bar{x}_i(t)\right| \\
\leq\; & \left|\phi_i(0)-\bar{\phi}_i(0)\right| \\
+\; & \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\left|f_{ij}\left(s, x_i(s), x_j\left(s-\tau_j(s)\right)\right)-f_{ij}\left(s, \bar{x}_i(s), \bar{x}_j\left(s-\tau_j(s)\right)\right)\right| ds \\
+\; & \sum_{j=1}^{n} \int_0^t\left|\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}-\frac{(t-s)^{\bar{\alpha}_i-1}}{\Gamma(\bar{\alpha}_i)}\right|\left|f_{ij}\left(s, \bar{x}_i(s), \bar{x}_j\left(s-\tau_j(s)\right)\right)\right| ds
\end{aligned}
$$

$$
\begin{aligned}
\left|x_i(t)-\bar{x}_i(t)\right| \;\leq\; & \left|\phi_i(0)-\bar{\phi}_i(0)\right| \\
& +\sum_{j=1}^{n} k_i \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\left|x_i(s)-\bar{x}_i(s)\right| ds \\
& +\sum_{j=1}^{n} h_j \int_0^{t_j} \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\left|\phi_j\left(s-\tau_j(s)\right)-\bar{\phi}_j\left(s-\tau_j(s)\right)\right| ds \\
& +\sum_{j=1}^{n} h_j \int_{t_j}^{t} \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\left|x_j\left(s-\tau_j(s)\right)-\bar{x}_j\left(s-\tau_j(s)\right)\right| ds \\
& +\sum_{j=1}^{n} \int_0^t\left|\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}-\frac{(t-s)^{\bar{\alpha}_i-1}}{\Gamma(\bar{\alpha}_i)}\right|\left|f_{ij}\left(s, \bar{x}_i(s), \bar{x}_j\left(s-\tau_j(s)\right)\right)\right| ds
\end{aligned}
$$

$$\begin{aligned}
|x_i(t)-\bar{x}_i(t)| &\leq \max_{s\in[-\tau,0]}\left|\phi_i(s)-\bar{\phi}_i(s)\right| \\
&+\sum_{j=1}^{n}\max_{s\in[-\tau,0]}\left|\phi_j(s)-\bar{\phi}_j(s)\right| h_j\int_0^{t_j}\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}ds \\
&+\sum_{j=1}^{n}k_i\int_0^{t}\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}|x_i(s)-\bar{x}_i(s)|\,ds \\
&+\sum_{j=1}^{n}h_j\int_{t_j}^{t}\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\left|x_j\left(s-\tau_j(s)\right)-\bar{x}_j\left(s-\tau_j(s)\right)\right|ds \\
&+\sum_{j=1}^{n}\max_{\zeta\in\mathbb{R}^+}\left|f_{ij}\left(\zeta,\bar{x}_i(\zeta),\bar{x}_j\left(\zeta-\tau_j(\zeta)\right)\right)\right|\int_0^{t}\left|\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}-\frac{(t-s)^{\bar{\alpha}_i-1}}{\Gamma(\bar{\alpha}_i)}\right|ds
\end{aligned}$$

$$\begin{aligned}
|x_i(t)-\bar{x}_i(t)| &\leq \max_{s\in[-\tau,0]}\left|\phi_i(s)-\bar{\phi}_i(s)\right| \\
&+\sum_{j=1}^{n}\max_{s\in[-\tau,0]}\left|\phi_j(s)-\bar{\phi}_j(s)\right| h_j\frac{\left[t^{\alpha_i}-\left(t-t_j\right)^{\alpha_i}\right]}{\Gamma(\alpha_i+1)} \\
&+nk_i\int_0^{t}\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}|x_i(s)-\bar{x}_i(s)|\,ds \\
&+\sum_{j=1}^{n}h_j\int_{t_j}^{t}\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\left|x_j\left(s-\tau_j(s)\right)-\bar{x}_j\left(s-\tau_j(s)\right)\right|ds \\
&+nM\int_0^{t}\left|\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}-\frac{(t-s)^{\bar{\alpha}_i-1}}{\Gamma(\bar{\alpha}_i)}\right|ds.
\end{aligned}$$

Then

$$\begin{aligned}
e^{-\lambda t}|x_i(t)-\bar{x}_i(t)| &\leq e^{-\lambda t}\max_{s\in[-\tau,0]}\left|\phi_i(s)-\bar{\phi}_i(s)\right| \\
&+e^{-\lambda t}\sum_{j=1}^{n}\max_{s\in[-\tau,0]}\left|\phi_j(s)-\bar{\phi}_j(s)\right| h_j\frac{\left[t^{\alpha_i}-\left(t-t_j\right)^{\alpha_i}\right]}{\Gamma(\alpha_i+1)} \\
&+nk_i\int_0^{t}\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\lambda(t-s)}e^{-\lambda s}|x_i(s)-\bar{x}_i(s)|\,ds \\
&+\sum_{j=1}^{n}h_j\int_{t_j}^{t}\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\lambda\left(t-r_j(s)\right)}e^{-\lambda r_j(s)}\left|x_j\left(r_j(s)\right)-\bar{x}_j\left(r_j(s)\right)\right|ds \\
&+e^{-\lambda t}nM\int_0^{t}\left|\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}-\frac{(t-s)^{\bar{\alpha}_i-1}}{\Gamma(\bar{\alpha}_i)}\right|ds,
\end{aligned}$$

with $r_j(s) = s - \tau_j(s)$.

$$\begin{aligned}
e^{-\lambda t}\left|x_i(t) - \bar{x}_i(t)\right| \leq \; & \max_{s\in[-\tau,0]}\left|\phi_i(s) - \bar{\phi}_i(s)\right| \\
& + \sum_{j=1}^{n} \max_{s\in[-\tau,0]}\left|\phi_j(s) - \bar{\phi}_j(s)\right| \frac{h_j t_j^{\alpha_i}}{\Gamma(\alpha_i+1)} \\
& + nk_i \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_i(\zeta) - \bar{x}_i(\zeta)\right|\right\} \frac{1}{\lambda^{\alpha_i}} \int_0^{\lambda t} \frac{u^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-u} du \\
& + e^{-\lambda t} nM \int_0^t \left| \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} - \frac{(t-s)^{\bar{\alpha}_i-1}}{\Gamma(\bar{\alpha}_i)} \right| ds \\
& + \sum_{j=1}^{n} h_j \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda(r_j(\zeta))}\left|x_j\left(r_j(s)\right) - \bar{x}_j\left(r_j(s)\right)\right|\right\} \\
& \times \int_0^{\lambda(t-t_j)} \frac{u^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-u} e^{-\lambda\tau_j\left(t-\frac{u}{\lambda}\right)} du.
\end{aligned}$$

Applying the mean value theorem, there exists an intermediate value $\tilde{\alpha}_i$ between α_i and $\bar{\alpha}_i$, $i = \overline{1, n}$, such that

$$\begin{aligned}
e^{-\lambda t}\left|x_i(t) - \bar{x}_i(t)\right| \leq \; & \max_{s\in[-\tau,0]}\left|\phi_i(s) - \bar{\phi}_i(s)\right| \\
& + \sum_{j=1}^{n} \max_{s\in[-\tau,0]}\left|\phi_j(s) - \bar{\phi}_j(s)\right| \frac{h_j t_j^{\alpha_i}}{\Gamma(\alpha_i+1)} \\
& + \frac{nk_i}{\lambda^{\alpha_i}} \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_i(\zeta) - \bar{x}_i(\zeta)\right|\right\} \\
& + \sum_{j=1}^{n} \frac{h_j e^{\lambda\tau}}{\lambda^{\alpha_i}} \sup_{\zeta\in\mathbb{R}^+}\left\{e^{-\lambda\zeta}\left|x_j(\zeta) - \bar{x}_j(\zeta)\right|\right\} \\
& + nMe^{-\lambda t} \int_0^t \left|g'(s)(\tilde{\alpha}_i)\right| ds \left|\alpha_i - \bar{\alpha}_i\right|,
\end{aligned}$$

with $g(s)(\beta) = \dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}$ for $\beta \in [\eta, 1[$.

$$\begin{aligned} e^{-\lambda t}\left|x_i(t) - \bar{x}_i(t)\right| \leq & \max_{s\in[-\tau,0]}\left|\phi_i(s) - \bar{\phi}_i(s)\right| \\ & + \sum_{j=1}^{n} \mathbf{d}\left(\Phi,\ \bar{\Phi}\right)\frac{h_j t_j^{\alpha_i}}{\Gamma(\alpha_i+1)} \\ & + \frac{nk_i}{\lambda^{\alpha_i}} d_\lambda(x, \bar{x}) + \sum_{j=1}^{n}\frac{h_j e^{\lambda\tau}}{\lambda^{\alpha_i}} d_\lambda(x, \bar{x}) \\ & + nMe^{-\lambda t}\int_0^t \left\{\frac{\left|\ln(t-s)\right| - \psi(\tilde{\alpha}_i)}{\Gamma(\tilde{\alpha}_i)}(t-s)^{\tilde{\alpha}_i - 1}\right\} ds\left|\alpha_i - \bar{\alpha}_i\right|. \end{aligned}$$

Where ψ is the digamma function. So

$$\begin{aligned} & \sum_{i=1}^{n}\sup_{t\in\mathbb{R}^+}\left\{e^{-\lambda t}\left|x_i(t) - \bar{x}_i(t)\right|\right\} \\ \leq & \sum_{i=1}^{n}\max_{s\in[-\tau,0]}\left|\phi_i(s) - \bar{\phi}_i(s)\right| + \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{d}\left(\Phi,\ \bar{\Phi}\right)\frac{h_j t_j^{\alpha_i}}{\Gamma(\alpha_i+1)} \\ & + \sum_{i=1}^{n}\frac{nk_i}{\lambda^{\alpha_i}} d_\lambda(x, \bar{x}) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_j e^{\lambda\tau}}{\lambda^{\alpha_i}} d_\lambda(x, \bar{x}) \\ & + nM\sum_{i=1}^{n}\sup_{t\in\mathbb{R}^+} e^{-\lambda t}\int_0^t \left\{\frac{\left|\ln(t-s)\right| - \psi(\tilde{\alpha}_i)}{\Gamma(\tilde{\alpha}_i)}(t-s)^{\tilde{\alpha}_i - 1}\right\} ds\left|\alpha_i - \bar{\alpha}_i\right|. \end{aligned}$$

Thus, there exists a positive constant K depending on η ($K = K(\eta)$), such that

$$\begin{aligned} d_\lambda(x, \bar{x}) \leq & \mathbf{d}\left(\Phi,\ \bar{\Phi}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_j t_j^{\alpha_i}}{\Gamma(\alpha_i+1)}\mathbf{d}\left(\Phi,\ \bar{\Phi}\right) \\ & + n\sum_{i=1}^{n}\frac{k_i}{\lambda^{\alpha_i}} d_\lambda(x, \bar{x}) + e^{\lambda\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_j}{\lambda^{\alpha_i}} d_\lambda(x, \bar{x}) \\ & + nM\sum_{i=1}^{n} K\left|\alpha_i - \bar{\alpha}_i\right| \end{aligned}$$

$$
\begin{aligned}
d_\lambda(x,\bar{x}) &\leq \left(1+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_j t_j^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\mathbf{d}\left(\Phi,\ \bar{\Phi}\right) \\
&\quad +\left(n\sum_{i=1}^{n}\frac{k_i}{\lambda^{\alpha_i}}+e^{\lambda\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_j}{\lambda^{\alpha_i}}\right)d_\lambda(x,\bar{x}) \\
&\quad +n^2MKd(\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}})
\end{aligned}
$$

$$
\begin{aligned}
&\left[1-\sum_{i=1}^{n}\tau^{\alpha_i}\left(nk_i+e\sum_{j=1}^{n}h_j\right)\right]d_\lambda(x,\bar{x}) \\
&\leq \left(1+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_j t_j^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\mathbf{d}\left(\Phi,\ \bar{\Phi}\right)+n^2MK\,d(\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}),
\end{aligned}
$$

with $\dfrac{1}{\lambda}=\tau$.

Then

$$
\begin{aligned}
&\left[1-\sum_{i=1}^{n}\tau^{\alpha_i}\left(nk_i+e\sum_{j=1}^{n}h_j\right)\right]d_\lambda(x,\bar{x}) \\
&\leq \max\left[\left(1+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_j t_j^{\alpha_i}}{\Gamma(\alpha_i+1)}\right),n^2MK\right]\left[\mathbf{d}\left(\Phi,\ \bar{\Phi}\right)+d(\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}})\right].
\end{aligned}
$$

So, for all $\epsilon_1,\epsilon_2>0$ and $d(\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}})<\epsilon_1$, $\mathbf{d}\left(\Phi,\bar{\Phi}\right)<\epsilon_2$, there exists

$$
\delta=2\left[1-\sum_{i=1}^{n}\tau^{\alpha_i}\left(nk_i+e\sum_{j=1}^{n}h_j\right)\right]^{-1}\max\left[\left(1+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_j t_j^{\alpha_i}}{\Gamma(\alpha_i+1)}\right),n^2MK\right],
$$

such that $d_\lambda(x,\bar{x})\leq\delta\max\{\epsilon_1,\epsilon_2\}$. Thus, according to the stability definition given in the beguining of this section we deduce that the solution of the problem (1.1)-(1.2) is stable with respect to the orders of derivation and the initial condition. ■

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