Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 4 (2016), pp. 3395–3400
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http://www.ripublication.com/gjpam.htm

Conditional energy stability of the zero solution to the boundary-initial value problem

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Abstract

In this paper we examine the stability of the zero solution to the boundary-initial value problem. In order to investigate the effect of a nonlinear term on the stability of a solution to a partial differential equation (PDE).

AMS subject classification:
Keywords: The energy method, stability of the solution.

1. Introduction

We apply the energy method to study the stability of the solution to the boundary-initial value problem,

$$u_t = u_{xx} + au^3 - \frac{1}{2}tu, \quad x \in (0,1), \quad t > 0, \tag{1}$$

where a is a positive constant, suppose_now (1) holds on the spatial region (0,1) with boundary conditions

$$u(0,t) = u(1,t) = 0 \tag{2}$$

and suppose we wish to investigate the behaviour of u subject to initial data

$$u(x,0) = \varphi(x) \tag{3}$$

Even though we know $u \equiv 0$ is a stable solution to (1), (3) we shall obtain this result directly using an energy method, choosing a problem to illustrate the technique.

Let now u be a solution to (1), (3) that satisfies arbitrary initial data $\varphi(x)$. We define an energy $F(t)$ by

$$F(t) = \frac{1}{2}t \parallel u \parallel^2 \tag{4}$$

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where $\| \,.\, \|$ denotes the norm on $L^2(0, 1)$ more precisely,

$$\| u \|^2 = \int_0^1 u^2 dx.$$

Multiply the differential equation (1) by u and integrate over (0,1) to find

$$\int_0^1 u_t u dx = \int_0^1 u u_{xx} dx + a \int_0^1 u^4 dx - \frac{1}{2} t \int_0^1 u^2 dx$$

by use of the Cauchy-Schwarz inequality. From the Sobolev embedding inequality we know that (see the [1])

$$\int_0^1 u^4 dx \leq \frac{1}{4} \left(\int_0^1 u_x^2 dx \right)^2. \tag{5}$$

Hence, with

$$E(t) = \frac{1}{2} \| u \|^2 \left(= \int_0^1 u^2 dx \right).$$

this equation becomes (after integration by parts as before)

$$\frac{d}{dt} E(t) \leq - \| u_x \|^2 + a \| u_x \|^4 - t E(t).$$

this equation becomes

$$\frac{d}{dt} F(t) \leq -a \| u_x \|^2 \left(\frac{1}{a} - \| u_x \|^2 \right). \tag{6}$$

Next, from Poincare's inequality, $\pi^2 \| u \|^2 \leq \| u_x \|^2$. Now, use this in (0.6),

$$\frac{d}{dt} F(t) \leq -a\pi^2 \| u \|^2 \left(\frac{1}{a} - \pi^2 \| u \|^2 \right), \tag{7}$$

if $0 < \frac{1}{a} - \pi^2 \| u \|^2 = \eta$ then

$$\frac{d}{dt} F(t) \leq -a\pi^2 \| u \|^2 \left(\frac{1}{a^{\frac{1}{2}}} - \pi \| u \| \right) \left(\frac{1}{a^{\frac{1}{2}}} + \pi \| u \| \right),$$

$$\frac{d}{dt} F(t) \leq -\eta a \pi^2 \| u \|^2, \tag{8}$$

so (0.8) shows

$$\frac{d}{dt} F(t) \leq 0, \;\; for \; \epsilon < t \tag{9}$$

Hence,

$$\| u(t) \| \leq \| u(\epsilon) \|$$

with same consideration that we have

$$\frac{d}{dt}F(t) \leq -a\pi^2 \parallel u \parallel^2 \eta \tag{10}$$

Further,

$$\frac{dF}{F} \leq -c\frac{dt}{t}, \quad \text{where } c = 2a\pi^2\eta$$

we find

$$dln\,(F(t)) \leq dln(t^{-c}),$$

Using an integrating factor we may obtain from this inequality

$$ln\left(\frac{F(t)}{F(\epsilon)}\right) \leq ln(\frac{t^{-c}}{\epsilon^{-c}}),$$

Next,

$$F(t) \leq F(\epsilon) \times \frac{\epsilon^c}{t^c} \tag{11}$$

which in turn integrates to

$$\parallel u(t) \parallel^2 \leq 2\frac{\epsilon^\gamma}{t^\gamma} \parallel u(\epsilon) \parallel^2, \quad \text{where } \gamma = c + 1. \tag{12}$$

What we have shown is that if $\parallel u(\epsilon) \parallel < \frac{1}{2a}$, then $\parallel u(t) \parallel \longrightarrow 0$ as $t \longrightarrow \infty$. And the zero solution to (0.1), (0.3) is stable.

Again, the equation (0.4) becomes (after integration by parts as before)

$$\frac{d}{dt}E(t) \leq - \parallel u_x \parallel^2 + a \parallel u^2 \parallel^2 - tE(t).$$

this equation becomes

$$\frac{d}{dt}F(t) \leq - \parallel u_x \parallel^2 + a \parallel u^2 \parallel^2 .$$

$$\frac{d}{dt}F(t) \leq -a \parallel u_x \parallel^2 \left(\frac{1}{a} - \max_{S_{adm}} \frac{\parallel u^2 \parallel^2}{\parallel u_x \parallel^2}\right)$$

where S_{adm} is the space of admissible functions over which we seek a maximum. Set

$$S_{adm} = \{u \in C^2 \mid u = 0 \text{ when } x = 0, 1\}.$$

Now define R_F by

$$\frac{1}{R_F} = \max_{S_{adm}} \frac{\parallel u^2 \parallel^2}{\parallel u_x \parallel^2},$$

then the energy inequality (0.5) may be rewritten

$$\frac{d}{dt}F(t) \leq -a \parallel u_x \parallel^2 \left(\frac{1}{a} - \frac{1}{R_F}\right),$$

The problem remains to find R_F. Recall that

$$R_F^{-1} = \max_{S_{adm}} \frac{\parallel u^2 \parallel^2}{\parallel u_x \parallel^2},$$

Let $\Lambda_1 = \parallel u^2 \parallel^2$, $\Lambda_2 = \parallel u_x \parallel^2$. The Euler–Lagrange equations are found from

$$\begin{aligned} \frac{d}{d\epsilon}\frac{\Lambda_1(u+\epsilon\tau)}{\Lambda_1(u_x+\epsilon\tau_x)} \mid_{\epsilon=0} &= \delta\left(\frac{\Lambda_1}{\Lambda_2}\right), \\ &= \frac{\Lambda_2\delta\Lambda_1 - \Lambda_1\delta\Lambda_2}{\Lambda_2^2} \\ &= \frac{1}{\Lambda_2}\left(\delta\Lambda_1 - \frac{\Lambda_1}{\Lambda_2}|_{\max}\delta\Lambda_2\right) \\ &= \frac{1}{\Lambda_2}\left(\delta\Lambda_1 - \frac{1}{R_F}\delta\Lambda_2\right) \end{aligned}$$

(Since δ refers to the *derivative* evaluated at $\epsilon = 0$. $\dfrac{\Lambda_1}{\Lambda_2}$ Is here understood to be at the stationary value.) Therefore,

$$\delta\Lambda_1 - \frac{1}{R_F}\delta\Lambda_2 = 0 \tag{13}$$

Here

$$\delta\Lambda_1 = \frac{d}{d\epsilon}\int_0^1 (u+\epsilon\tau)^4 dx \mid_{\epsilon=0},$$

where τ is an arbitrary $C^2(0, 1)$ function with $\tau(0) = \tau(1) = 0$, and

$$\delta\Lambda_1 = \frac{d}{d\epsilon}\int_0^1 (u_x+\epsilon\tau_x)^2 dx \mid_{\epsilon=0} .$$

So (0.13) leads to

$$\int_0^1 2u^3\tau - R_F^{-1}u_x\tau_x dx = 0.$$

Integration by parts shows that

$$\int_0^1 \left(u_{xx} + 2R_F u^3\right)\tau dx = 0.$$

Since τ is arbitrary apart from the continuity and boundary condition requirements, we must have

$$u_{xx} + 2R_F u^3 = 0, \qquad u(0) = u(1) = 0 \tag{14}$$

We now examine Eq. (0.14) within the framework of a $2 - dim$ phase−space.

The second-order differential equation, given in Eq. (0.14), may be reformulated to two first-order system equations

$$\frac{du}{dx} = v, \quad \frac{dv}{dx} = -2R_F u^3$$

subject to initial conditions

$$u(0) = u(1) = 0$$

then

$$\frac{dv}{du} = -2R_F \frac{u^3}{v}.$$

In general, this is a first−order, nonlinear differential equation, whose solutions are the curves of the solution trajectories in phase-space.

2. The harmonic balance method

The method of harmonic balance provides a general technique for calculating approximations to the periodic solutions of differential equations. It corresponds to a truncated Fourier series and allows for the systematic determination of the coefficients to the various harmonics and the angular frequency.

We begin by calculating the first-order harmonic balance approximation to the periodic solutions of

$$\frac{dv}{du} = -2R_F \frac{u^3}{v}, \tag{15}$$

$$u(0) = 0, \quad v(0) = a. \tag{16}$$

This approximation takes the form

$$u_1 = asin(\omega_1 x),$$

Observe that this expression automatically satisfies the initial conditions. Substituting Eq. (1.1) into Eq. (1.2) gives ($\theta = \omega_1 x - \frac{\pi}{2}$)

$$-a\omega_1^2 cos(\theta) + a^3 \alpha cos^3(\theta) \simeq 0, \quad \alpha = 2R_F.$$

$$-a\omega_1^2 cos(\theta) + a^3 \alpha \left(\frac{3}{4} cos(\theta) + \frac{1}{4} cos(3\theta)\right) \simeq 0.$$

$$\left(\frac{3}{4} a^3 \alpha - a\omega_1^2\right) cos(\theta) + higher\ harmonic \simeq 0.$$

Setting the coefficient of $cos(\theta)$ to zero gives the first−approximation to the angular frequency

$$\omega_1(a) = a\sqrt{\frac{3}{4}\alpha}.$$

and

$$u_1 = a \quad sin(a\sqrt{\frac{3}{4}\alpha x}),$$

where we have taken $a = 1$, since we are primarily interested in R_F not u. The condition $u(1) = 0$ then shows that

$$\sqrt{\frac{3}{2}R_F} = n\pi, \quad n = \pm 1, \pm 2 \cdots .$$

This gives an infinite sequence of values for R_F $\left(\text{corresponding to stationary values of the quotient } \frac{\| u^2 \|^2}{\| u_x \|^2}\right)$,

$$R_F = \frac{2}{3}\pi^2, \frac{8}{3}\pi^2 \cdots .$$

For stability, we need $a < R_F(\min)$, and so $R_F = \frac{2}{3}\pi^2$. In particular, therefore.

$$a < \frac{2}{3}\pi^2.$$

yields stability of the zero solution to (0.1), (0.3).

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