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Blow-up for Discretization of some Semilinear Parabolic Equations with a Convection Term

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Abstract

This paper concerns the study of the numerical approximation for the following parabolic equations with a convection term

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - u(x,t)u_x(x,t) + u^p(x,t), \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \end{cases}$$

where $p > 1$.

We obtain some conditions under which the solution of the discrete form of the above problem blows up in a finite time and estimate its numerical blow-up time. We also prove that the numerical blow-up time converges to the real one, when the mesh size goes to zero. Finally, we give some numerical experiments to illustrate ours analysis.

AMS subject classification: 35K58, 35B44, 35B50,35B51, 65M06, 35K28.
Keywords: Burgers' equation, discretization, parabolic equations, convection term, blow-up, blow-up time, convergence.

1. Introduction

Consider the following boundary value problem

$$u_t(x,t) = u_{xx}(x,t) - u(x,t)u_x(x,t) + u^p(x,t), \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \leq x \leq 1, \tag{3}$$

where $p > 1$, $u_0 \in C^2([0,1])$, u_0 is decreasing on (0,1) and verifies

$$u_0^{'}(0) = 0, \quad u_0^{'}(1) = 0, \tag{4}$$

$$u_0^{''}(x) - u_0(x)u_0^{'}(x) + u_0^p(x) \geq 0, \quad 0 \leq x \leq 1, \tag{5}$$

$$u_0(x) > -p(p-1)u_0^{'}(x), \quad 0 < x < 1. \tag{6}$$

Definition 1.1. We say that the solution u of (1)–(3) blows up in a finite time if there exists a finite time T_b such that $\|u(.,t)\|_\infty < \infty \; for \; t \in [0, T_b)$ but

$$\lim_{t \to T_b} \|u(.,t)\|_\infty = \infty.$$

The time T_b is called the blow-up time of the solution u.

The above problem models viscous Burgers' equation in one dimension with a reaction term. The solution $u(x,t)$ represents the motion field of the fluid in space and time. The term uu_x is called convection term. It's a nonlinear term that ensures the movement, generates instability and also responsible for the turbulent appearance (here we'll refer to it as intermittent since we are in one dimension) when it happens. The term u_{xx} is the diffusion term. In the general case the term u_{xx} is replaced by νu_{xx} with $\nu > 0$. The term νu_{xx} is the viscous term, which has the opposite effect of slicking and making it appear laminar that is ordered. The constant ν, coefficient of the viscous term, is called the kinematic viscosity (normalized by the density) of the fluid. The fluid's flow ability is inversely proportional to the size of the viscosity. The term u^p (the reaction term) is the external force which is generally a white and Gaussian noise within the time scale which forces the fluid to flow faster, slower or make it mill around. It's the quantitative relation between the convection term and viscous, called Reynolds number that will condition the appearance of the flow in the case when there is no external force. Burger's equation occurring in various areas applied mathematics, such as nonlinear acoustics, gas dynamics, traffic flow and fluid mechanics. At the end of the thirties, the Dutch scientist J. M. Burger, introduced a one dimensional model for

pressure less gas dynamics. He was hoping that the use of a simple model having much in common with the Navier-Stokes equation would significantly contribute to the study of fluid turbulence(see [2], [25]).

The theoretical study of blow-up solutions for the parabolic equations with a convection term has been the subject of investigations of many authors (see [3], [6], [7], [8], [9], [21], [22], [23] and the references cited therein). Local in time existence and uniqueness of the solution have been proved(see [4], [5], [26], [28] and the references cited therein). Here, we are interesting in the numerical study using a discrete form of (1)–(3). We give some assumptions under which the solution of a discrete form of (1)–(3) blows up in a finite time and estimate its numerical blow-up time. We also show that the numerical blow-up time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken in [10] and [28].

The paper is organized as follows. In the next section, we present a discrete scheme of (1)–(3) and give some lemmas which will be used throughout the paper. In the third section, under some conditions, we prove that the solution of the discrete form of (1)–(3) blows up in a finite time. In the fourth section, we study the convergence of the numerical blow-up time. Finally, in last section, we give some numerical experiments.

2. Properties of the discrete scheme

In this section, we give some lemmas which will be used later. We start by the construction of the discrete scheme. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$ and approximate the solution u of (1)–(3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \dots, U_I^{(n)})^T$ of the following discrete equations

$$\delta_t U_i^{(n)} = \delta^2 U_i^{(n)} - U_i^{(n)} \delta^0 U_i^{(n)} + (U_i^{(n)})^p, \ 1 \le i \le I-1, \tag{7}$$

$$\delta_t U_0^{(n)} = \delta^2 U_0^{(n)} + (U_0^{(n)})^p, \tag{8}$$

$$\delta_t U_I^{(n)} = \delta^2 U_I^{(n)} + (U_I^{(n)})^p, \tag{9}$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \le i \le I, \tag{10}$$

where

$$n \ge 0, \ \ p \ge 2,$$

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}, \quad 0 \le i \le I,$$

$$\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}, \quad 1 \le i \le I-1,$$

$$\delta^2 U_0^{(n)} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2}, \quad \delta^2 U_I^{(n)} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2},$$

$$\delta^0 U_i^{(n)} = \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h}, \quad 1 \le i \le I-1,$$

$$\delta^0 U_0^{(n)} = 0, \qquad \delta^0 U_I^{(n)} = 0,$$

$$\delta^+ \varphi_i \le 0, \quad 0 \le i \le I-1,$$

$$\varphi_i^{p-1} > -p(p-1)h(\delta^0 \varphi_i)\varphi_{i-1}^{p-2}, \qquad 1 \le i \le I-1.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the blow-up time T_b , we need to adapt the size of the time step. We choose

$$\Delta t_n = \min(\frac{h^2}{2}, \tau \|U_h^{(n)}\|_\infty^{1-p}) \text{ with } \tau \in (0,1).$$

Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution when this one is decreasing. To facilitate our discussion, we need to define the notion of numerical blow-up.

Definition 2.1. We say that the solution $U_h^{(n)}$, $n \ge 0$, of the discrete problem (7)–(10) blows up in a finite time, if

$$\lim_{n\to\infty} \|U_h^{(n)}\|_\infty = \infty,$$

and

$$T_h^{\Delta t} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \triangle t_i < \infty.$$

The number $T_h^{\Delta t}$ is called the numerical blow-up time of the discrete solution.

The following Lemma is a discrete form of the maximum principle.

Lemma 2.2. Let $a_h^{(n)}$, $b_h^{(n)}$ and $V_h^{(n)}$ be three sequences, with $n \ge 0$, $a_h^{(n)} \le 0$ and $b_h^{(n)}\delta^0 V_h^{(n)} \le 0$, such that

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + b_i^{(n)}\delta^0 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \ge 0, \; 0 \le i \le I, \tag{11}$$

$$V_i^{(0)} \ge 0. \tag{12}$$

Then, we have

$$V_i^{(n)} \geq 0, \quad 0 \leq i \leq I, \; n > 0 \; when \;\; \Delta t_n \leq \frac{h^2}{2}.$$

Proof. A straightforward computation shows that for $1 \leq i \leq I-1$,

$$V_i^{(n+1)} \geq \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_i^{(n)} + \frac{\Delta t_n}{h^2}(V_{i+1}^{(n)} + V_{i-1}^{(n)}) - \Delta t_n b_i^{(n)} \delta^0 V_i^{(n)} - \Delta t_n a_i^{(n)} V_i^{(n)},$$

$$V_0^{(n+1)} \geq \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_0^{(n)} + \frac{2\Delta t_n}{h^2} V_1^{(n)} - \Delta t_n a_0^{(n)} V_0^{(n)},$$

$$V_I^{(n+1)} \geq \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_I^{(n)} + \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} - \Delta t_n a_I^{(n)} V_I^{(n)}.$$

If $V_h^{(n)} \geq 0$, then using an argument of recursion, we easily see that $V_h^{(n+1)} \geq 0$, because $1 - \dfrac{2\Delta t_n}{h^2} \geq 0$, $b_h^{(n)} \delta^0 V_h^{(n)} \leq 0$ and $a_h^{(n)} \leq 0$. This ends the proof. ■

A direct consequence of the above result is the following comparison Lemma.

Lemma 2.3. Let $a_h^{(n)}$, $V_h^{(n)}$ and $W_h^{(n)}$ be three sequences, with $n \geq 0$, $a_h^{(n)} \leq 0$, $\delta^0 V_h^{(n)} \leq 0$ and $\delta^0 W_h^{(n)} \leq 0$, such that

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + V_i^{(n)} \delta^0 V_i^{(n)} + a_i^{(n)} V_i^{(n)} < \delta_t W_i^{(n)} -$$

$$\delta^2 W_i^{(n)} + W_i^{(n)} \delta^0 W_i^{(n)} + a_i^{(n)} W_i^{(n)}, \quad 0 \leq i \leq I, \tag{13}$$

$$V_i^{(0)} < W_i^{(0)}, \quad 0 \leq i \leq I. \tag{14}$$

Then, we have

$$V_i^{(n)} < W_i^{(n)}, \; 0 \leq i \leq I, \quad n > 0 \quad \text{when} \quad \Delta t_n \leq \frac{h^2}{2}.$$

Proof. Define the sequence $Z_h^{(n)} = W_h^{(n)} - V_h^{(n)}$. A straightforward calculation gives

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + W_i^{(n)} \delta^0 W_i^{(n)} - V_i^{(n)} \delta^0 V_i^{(n)} + a_i^{(n)} Z_i^{(n)} > 0, \; 0 \leq i \leq I,$$

which is equivalent to

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + W_i^{(n)} \delta^0 Z_i^{(n)} + (a_i^{(n)} + \delta^0 V_i^{(n)}) Z_i^{(n)} > 0, \; 0 \leq i \leq I.$$

Knowing that $Z_h^{(0)} > 0$, from Lemma 2.2, we have $Z_h^{(n)} > 0$, which implies that $V_i^{(n)} < W_i^{(n)}$, $0 \le i \le I$ and the proof is complete. ■

We need the following result about the operator δ_t.

Lemma 2.4. Let $U_h^{(n)}$, $n \ge 0$, be a sequence such that $U_h^{(n)} > 0$. Then

$$\delta_t (U_i^{(n)})^p \ge p (U_i^{(n)})^{p-1} \delta_t U_i^{(n)},\ 0 \le i \le I.$$

Proof. Using Taylor's expansion, we get

$$\delta_t (U_i^{(n)})^p = p (U_i^{(n)})^{p-1} \delta_t U_i^{(n)} + \frac{p(p-1)}{2} \Delta t_n (\delta_t U_i^{(n)})^2 (\theta_i^{(n)})^{p-2},\ 0 \le i \le I,$$

where $\theta_i^{(n)}$ is an intermediate value between $U_i^{(n)}$ and $U_i^{(n+1)}$, $0 \le i \le I$.

Next, we use the fact that $U_h^{(n)} > 0$, $n \ge 0$, to complete the proof. ■

The following lemma shows the decreasing in space of the discrete solution.

Lemma 2.5. Let $U_h^{(n)}$, $n \ge 0$, be the solution of the discrete problem (7)–(10). Then

$$U_{i+1}^{(n)} < U_i^{(n)},\ 0 \le i \le I-1. \tag{15}$$

Proof. Define the vector $Z_h^{(n)}$ such that $Z_i^{(n)} = U_i^{(n)} - U_{i+1}^{(n)}$, $0 \le i \le I-1$. We have

$$Z_i^{(n)} = U_i^{(n)} - U_{i+1}^{(n)},\ 1 \le i \le I-2,$$

$$Z_0^{(n)} = U_0^{(n)} - U_1^{(n)},$$

$$Z_{I-1}^{(n)} = U_{I-1}^{(n)} - U_I^{(n)}.$$

A straightforward computations reveals that

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + U_i^{(n)} \delta^0 Z_i^{(n)} + Z_i^{(n)} \delta^0 U_{i+1}^{(n)} - p(\beta_i^{(n)})^{p-1} Z_i^{(n)} = 0,\ 1 \le i \le I-2,$$

$$\delta_t Z_0^{(n)} - \delta^2 Z_0^{(n)} - U_1^{(n)} \delta^0 U_1^{(n)} - p(\beta_0^{(n)})^{p-1} Z_0^{(n)} = 0,$$

$$\delta_t Z_{I-1}^{(n)} - \delta^2 Z_{I-1}^{(n)} + U_{I-1}^{(n)} \delta^0 U_{I-1}^{(n)} - p(\beta_{I-1}^{(n)})^{p-1} Z_{I-1}^{(n)} = 0,$$

which are equivalent to

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + U_i^{(n)} \delta^0 Z_i^{(n)} + (\delta^0 U_{i+1}^{(n)} - p(\beta_i^{(n)})^{p-1}) Z_i^{(n)} = 0,\ 1 \le i \le I-2,$$

$$\delta_t Z_0^{(n)} - \delta^2 Z_0^{(n)} + U_1^{(n)} \delta^0 Z_0^{(n)} - p(\beta_0^{(n)})^{p-1} Z_0^{(n)} = 0,$$

$$\delta_t Z_{I-1}^{(n)} - \delta^2 Z_{I-1}^{(n)} + U_{I-1}^{(n)} \delta^0 Z_{I-1}^{(n)} - p(\beta_{I-1}^{(n)})^{p-1} Z_{I-1}^{(n)} = 0,$$

where $\beta_i^{(n)}$ is an intermediate value between $U_{i+1}^{(n)}$ and $U_i^{(n)}$, $0 \le i \le I-1$.
Knowing that $Z_h^{(0)} > 0$, from Lemma 2.2, we have $Z_h^{(n)} > 0$, which implies that $U_{i+1}^{(n)} < U_i^{(n)}$, $0 \le i \le I-1$, and we obtain the desired result. ■

The lemma below reveals the positivity of the discrete solution.

Lemma 2.6. Let $U_h^{(n)}$, $n \ge 0$, be the solution of the discrete problem (7)–(10). Then

$$U_i^{(n)} > 0, \quad 0 \le i \le I \quad when \quad \Delta t_n \le \frac{h^2}{2}. \tag{16}$$

Proof. A routine calculation reveals that for $1 \le i \le I-1$,

$$U_i^{(n+1)} = (1 - \frac{\Delta t_n}{h^2}) U_i^{(n)} + \frac{2\Delta t_n}{h^2}(U_{i+1}^{(n)} + U_{i-1}^{(n)}) - \Delta t_n U_i^{(n)} \delta^0 U_i^{(n)} + \Delta t_n (U_i^{(n)})^p,$$

$$U_0^{(n+1)} = \left(1 - \frac{2\Delta t_n}{h^2}\right) U_0^{(n)} + \frac{2\Delta t_n}{h^2} U_1^{(n)} + \Delta t_n (U_0^{(n)})^p,$$

$$U_I^{(n+1)} = \left(1 - \frac{2\Delta t_n}{h^2}\right) U_I^{(n)} + \frac{2\Delta t_n}{h^2} U_{I-1}^{(n)} + \Delta t_n (U_I^{(n)})^p.$$

If $U_h^{(n)} > 0$, then using an argument of recursion, we easily see that $U_h^{(n+1)} > 0$, because $1 - \dfrac{2\Delta t_n}{h^2} \ge 0$ and $\delta^0 U_h^{(n)} < 0$. ■

The following lemma gives the increasing in time of the discrete solution.

Lemma 2.7. Let $U_h^{(n)}$, $n \ge 0$, be the solution of the discrete problem (7)–(10). Then

$$\delta_t U_i^{(n)} > 0, \quad 0 \le i \le I.$$

Proof. Consider the vector $Z_h^{(n)}$ such that $Z_i^{(n)}$=$\delta_t U_i^{(n)}$, $0 \le i \le I$.
A straightforward calculation gives

$$\delta_t Z_i^{(n)} = \delta^2 Z_i^{(n)} - U_i^{(n)} \delta^0 Z_i^{(n)} - Z_i^{(n)} \delta^0 U_i^{(n)} + \delta_t (U_i^{(n)})^p, \ 1 \le i \le I-1,$$

$$\delta_t Z_0^{(n)} = \delta^2 Z_0^{(n)} + \delta_t (U_0^{(n)})^p,$$

$$\delta_t Z_I^{(n)} = \delta^2 Z_I^{(n)} + \delta_t (U_I^{(n)})^p.$$

Using Lemma 2.4, We finally have

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + U_i^{(n)} \delta^0 Z_i^{(n)} + (\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1}) Z_i^{(n)} \geq 0,\ 1 \leq i \leq I-1,$$

$$\delta_t Z_0^{(n)} - \delta^2 Z_0^{(n)} - p(U_0^{(n)})^{p-1} Z_0^{(n)} \geq 0,$$

$$\delta_t Z_I^{(n)} - \delta^2 Z_I^{(n)} - p(U_I^{(n)})^{p-1} Z_I^{(n)} \geq 0.$$

Knowing that $Z_h^{(0)} > 0$, from Lemma 2.2, we have $Z_h^{(n)} > 0$, which implies that $\delta_t U_i^{(n)} > 0$, $0 \leq i \leq I$. We have the wished result. ■

The following lemma is a discrete generalization of the condition (6).

Lemma 2.8. Let $U_h^{(n)}$, $n \geq 0$, be the solution of the discrete problem (7)–(10). Then

$$(U_i^{(n)})^{p-1} > -p(p-1)h(\delta^0 U_i^{(n)})(U_{i-1}^{(n)})^{p-2},\ 1 \leq i \leq I-1,\ p \geq 2.$$

Proof. Consider the vectors $Z_h^{(n)}$, $K_h^{(n)}$ and $V_h^{(n)}$ such that $Z_i^{(n)} = K_i^{(n)} - V_i^{(n)}$ with $K_i^{(n)} = (U_i^{(n)})^{p-1}$ and $V_i^{(n)} = -p(p-1)h(\delta^0 U_i^{(n)})(U_{i-1}^{(n)})^{p-2}$. We have

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + K_i^{(n)} \delta^0 Z_i^{(n)} + Z_i^{(n)} \delta^0 V_i^{(n)} - p(\beta_i^{(n)})^{p-1} Z_i^{(n)} = 0,\ 1 \leq i \leq I-1,$$

which is equivalent to

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + K_i^{(n)} \delta^0 Z_i^{(n)} + (\delta^0 V_i^{(n)} - p(\beta_i^{(n)})^{p-1}) Z_i^{(n)} = 0,\ 1 \leq i \leq I-1,$$

where $\beta_i^{(n)}$ is an intermediate value between $V_i^{(n)}$ and $K_i^{(n)}$, $1 \leq i \leq I-1$. Knowing that $Z_h^{(0)} > 0$, from Lemma 2.2, we have $Z_h^{(n)} > 0$, which implies that $V_i^{(n)} < K_i^{(n)}$, $1 \leq i \leq I-1$, and we obtain the desired result. ■

We need the following result about the operator δ^2.

Lemma 2.9. Let $U_h^{(n)}$, $n \geq 0$, be a sequence such that $U_h^{(n)} > 0$. Then, we have for $0 \leq i \leq I$,

$$\delta^2 (U_i^{(n)})^p \geq p(U_i^{(n)})^{p-1} \delta^2 (U_i^{(n)}).$$

Proof. Applying Taylor's expansion, we obtain

$$\delta^2 (U_i^{(n)})^p = p(U_i^{(n)})^{p-1} \delta^2 U_i^{(n)} + (U_{i-1}^{(n)} - U_i^{(n)})^2 \frac{p(p-1)}{2h^2} (\theta_i^{(n)})^{p-2}$$

$$+ (U_{i+1}^{(n)} - U_i^{(n)})^2 \frac{p(p-1)}{2h^2} (\xi_i^{(n)})^{p-2} \ \ if \ \ 1 \leq i \leq I-1,$$

$$\delta^2(U_0^{(n)})^p = p(U_0^{(n)})^{p-1}\delta^2 U_0^{(n)} + (U_1^{(n)} - U_0^{(n)})^2 \frac{p(p-1)}{2h^2}(\theta_0^{(n)})^{p-2},$$

$$\delta^2(U_I^{(n)})^p = p(U_I^{(n)})^{p-1}\delta^2 U_I^{(n)} + (U_{I-1}^{(n)} - U_I^{(n)})^2 \frac{p(p-1)}{2h^2}(\theta_I^{(n)})^{p-2},$$

where $p \geq 2$, $\theta_0^{(n)}$ is an intermediate value between $U_0^{(n)}$ and $U_1^{(n)}$, $\theta_i^{(n)}$ is an intermediate value between $U_{i-1}^{(n)}$ and $U_i^{(n)}$, $1 \leq i \leq I-1$, $\theta_I^{(n)}$ is an intermediate value between $U_{I-1}^{(n)}$ and $U_I^{(n)}$, $\xi_i^{(n)}$ is an intermediate value between $U_i^{(n)}$ and $U_{i+1}^{(n)}$, $1 \leq i \leq I-1$. The result follows taking into account the fact that $U_h^{(n)} > 0$. ■

Lemma 2.10. Let $U_h^{(n)}$, $n \geq 0$, be the solution of the discrete problem (7)–(10). Then, we have for $1 \leq i \leq I-1$,

$$-U_i^{(n)}\delta^0(U_i^{(n)})^p \geq -p(U_i^{(n)})^p\delta^0 U_i^{(n)} - p(p-1)h(\delta^0 U_i^{(n)})^2 U_i^{(n)}(U_{i-1}^{(n)})^{p-2}.$$

Proof. Using Taylor's expansion, we get for $1 \leq i \leq I-1$ and $p = 2$,

$$\delta^0(U_i^{(n)})^p = p(U_{i-1}^{(n)})^{p-1}\delta^0 U_i^{(n)} + (U_{i+1}^{(n)} - U_{i-1}^{(n)})^2 \frac{p(p-1)}{4h}(U_{i-1}^{(n)})^{p-2},$$

for $1 \leq i \leq I-1$ and $p \geq 3$,

$$\delta^0(U_i^{(n)})^p = p(U_{i-1}^{(n)})^{p-1}\delta^0 U_i^{(n)} + (U_{i+1}^{(n)} - U_{i-1}^{(n)})^2 \frac{p(p-1)}{4h}(U_{i-1}^{(n)})^{p-2} +$$

$$(U_{i+1}^{(n)} - U_{i-1}^{(n)})^3 \frac{p(p-1)(p-2)}{12h}(\zeta_i^{(n)})^{p-3},$$

where $\zeta_i^{(n)} \in (U_{i+1}^{(n)}, U_{i-1}^{(n)})$. From Lemma 2.5 and the fact that $U_h^{(n)} > 0$, we obtain the desired result. ■

3. Discrete Blow-up solutions

In this section under some assumptions, we show that the solution $U_h^{(n)}$ of the discrete problem (7)–(10) blows up in a finite time and estimate its numerical blow-up time.

Theorem 3.1. Let $U_h^{(n)}$ be the solution of the discrete problem (7)–(10). Suppose that there exists a positive integer $\lambda \in (0, 1)$ such that

$$\delta^2\varphi_i - \varphi_i\delta^0\varphi_i + \varphi_i^p \geq \lambda\varphi_i^p, \quad 0 \leq i \leq I. \tag{17}$$

Then, the solution $U_h^{(n)}$ blows up in a finite time $T_h^{\Delta t}$ and we have the following estimate

$$T_h^{\Delta t} \leq \frac{\tau(1+\tau')^{p-1}}{\|\varphi_h\|_\infty^{p-1}((1+\tau')^{p-1}-1)},$$

where

$$\tau' = \lambda \min\left\{\frac{h^2\|\varphi_h\|_\infty^{p-1}}{2}, \tau\right\},\ 0<\tau<1.$$

Proof. Consider the vector $J_h^{(n)}$, $n \geq 0$, such that

$$J_i^{(n)} = \delta_t U_i^{(n)} - \lambda (U_i^{(n)})^p,\ 0 \leq i \leq I. \tag{18}$$

A straightforward calculation gives

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + U_i^{(n)}\delta^0 J_i^{(n)} + (\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1})J_i^{(n)} =$$

$$(1-\lambda)\delta_t(U_i^{(n)})^p + \lambda\delta^2(U_i^{(n)})^p - \lambda U_i^{(n)}\delta^0(U_i^{(n)})^p - \lambda(U_i^{(n)})^p\delta^0 U_i^{(n)}$$

$$-p(U_i^{(n)})^{p-1}\delta_t U_i^{(n)} + p\lambda(U_i^{(n)})^{p-1}(U_i^{(n)})^p,\ 1 \leq i \leq I-1,$$

$$\delta_t J_0^{(n)} - \delta^2 J_0^{(n)} - p(U_0^{(n)})^{p-1}J_0^{(n)} = (1-\lambda)\delta_t(U_0^{(n)})^p$$

$$+\lambda\delta^2(U_0^{(n)})^p - p(U_0^{(n)})^{p-1}\delta_t U_0^{(n)} + \lambda p(U_0^{(n)})^{p-1}(U_0^{(n)})^p,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} - p(U_I^{(n)})^{p-1}J_I^{(n)} = (1-\lambda)\delta_t(U_I^{(n)})^p$$

$$+\lambda\delta^2(U_I^{(n)})^p - p(U_I^{(n)})^{p-1}\delta_t U_I^{(n)} + \lambda p(U_I^{(n)})^{p-1}(U_I^{(n)})^p.$$

From Lemma 2.4, $\delta_t(U_i^{(n)})^p \geq p(U_i^{(n)})^{p-1}\delta_t U_i^{(n)}$, $0 \leq i \leq I$ and the fact that $0<\lambda<1$, we have $(1-\lambda)\delta_t(U_i^{(n)})^p \geq (1-\lambda)p(U_i^{(n)})^{p-1}\delta_t U_i^{(n)},\ 0 \leq i \leq I$. Using also Lemma 2.9, $\delta^2(U_i^{(n)})^p \geq p(U_i^{(n)})^{p-1}\delta^2 U_i^{(n)},\ 0 \leq i \leq I$, we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + U_i^{(n)}\delta^0 J_i^{(n)} + (\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1})J_i^{(n)}$$

$$\geq \lambda p(U_i^{(n)})^{p-1}(-\delta_t U_i^{(n)} + \delta^2 U_i^{(n)} + (U_i^{(n)})^p) - \lambda U_i^{(n)}\delta^0(U_i^{(n)})^p$$

$$-\lambda(U_i^{(n)})^p\delta^0 U_i^{(n)},\ 1 \leq i \leq I-1,$$

$$\delta_t J_0^{(n)} - \delta^2 J_0^{(n)} - p(U_0^{(n)})^{p-1} J_0^{(n)} \geq \lambda p (U_0^{(n)})^{p-1}(-\delta_t U_0^{(n)} + \delta^2 U_0^{(n)} + (U_0^{(n)})^p),$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} - p(U_I^{(n)})^{p-1} J_I^{(n)} \geq \lambda p (U_I^{(n)})^{p-1}(-\delta_t U_I^{(n)} + \delta^2 U_I^{(n)} + (U_I^{(n)})^p).$$

Using (7)–(9), we have these equalities

$$-\delta_t U_i^{(n)} + \delta^2 U_i^{(n)} + (U_i^{(n)})^p = U_i^{(n)} \delta^0 U_i^{(n)},\ 1 \leq i \leq I-1,$$

$$-\delta_t U_0^{(n)} + \delta^2 U_0^{(n)} + (U_0^{(n)})^p = 0,$$

$$-\delta_t U_I^{(n)} + \delta^2 U_I^{(n)} + (U_I^{(n)})^p = 0.$$

We get,

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + U_i^{(n)} \delta^0 J_i^{(n)} + (\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1}) J_i^{(n)}$$

$$\geq \lambda(-U_i^{(n)} \delta^0 (U_i^{(n)})^p + p(U_i^{(n)})^p \delta^0 U_i^{(n)} - (U_i^{(n)})^p \delta^0 U_i^{(n)}),\ 1 \leq i \leq I-1,$$

$$\delta_t J_0^{(n)} - \delta^2 J_0^{(n)} - p(U_0^{(n)})^{p-1} J_0^{(n)} \geq 0,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} - p(U_I^{(n)})^{p-1} J_I^{(n)} \geq 0.$$

Using the Lemma 2.10, for $1 \leq i \leq I-1$,

$$-U_i^{(n)} \delta^0 (U_i^{(n)})^p \geq -p(U_i^{(n)})^p \delta^0 U_i^{(n)} - p(p-1)h(\delta^0 U_i^{(n)})^2 U_i^{(n)} (U_{i-1}^{(n)})^{p-2},$$

we have

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + U_i^{(n)} \delta^0 J_i^{(n)} + (\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1}) J_i^{(n)}$$

$$\geq -\lambda U_i^{(n)} \delta^0 U_i^{(n)} ((U_i^{(n)})^{p-1} + p(p-1)h(\delta^0 U_i^{(n)})(U_{i-1}^{(n)})^{p-2}),$$

$$\delta_t J_0^{(n)} - \delta^2 J_0^{(n)} - p(U_0^{(n)})^{p-1} J_0^{(n)} \geq 0,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} - p(U_I^{(n)})^{p-1} J_I^{(n)} \geq 0.$$

From Lemma 2.8, $(U_i^{(n)})^{p-1} > -p(p-1)h(\delta^0 U_i^{(n)})(U_{i-1}^{(n)})^{p-2},\ 1 \leq i \leq I-1$, and using the fact that $-\lambda U_h^{(n)} \delta^0 U_h^{(n)} \geq 0$, we get

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + U_i^{(n)} \delta^0 J_i^{(n)} + (\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1}) J_i^{(n)} \geq 0,\ 1 \leq i \leq I-1,$$

$$\delta_t J_0^{(n)} - \frac{(2J_1^{(n)} - 2J_0^{(n)})}{h^2} - p(U_0^{(n)})^{p-1} J_0 \geq 0,$$

$$\delta_t J_I^{(n)} - \frac{(2J_{I-1}^{(n)} - 2J_I^{(n)})}{h^2} - p(U_I^{(n)})^{p-1} J_I^{(n)} \geq 0.$$

From (17), we observe that

$$J_i^{(0)} = \delta^2 \varphi_i - \varphi_i \delta^0 \varphi_i + \varphi_i^p - \lambda \varphi_i^p \geq 0, \quad 0 \leq i \leq I.$$

We deduce from Lemma 2.2 that $J_h^{(n)} \geq 0$ for $n \geq 0$, which implies that

$$\delta_t U_i^{(n)} \geq \lambda (U_i^{(n)})^p, \quad 0 \leq i \leq I, \tag{19}$$

which is equivalent to

$$U_i^{(n+1)} \geq U_i^{(n)} + \lambda \Delta t_n (U_i^{(n)})^p, \ 0 \leq i \leq I.$$

Therefore

$$U_i^{(n+1)} \geq U_i^{(n)} (1 + \lambda \Delta t_n (U_i^{(n)})^{p-1}), \ 0 \leq i \leq I, \tag{20}$$

which implies that

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty (1 + \lambda \Delta t_n \|U_h^{(n)}\|_\infty^{p-1}).$$

From Lemma 2.7, $\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty$. By induction, we obtain

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(0)}\|_\infty = \|\varphi_h\|_\infty.$$

Then, we have

$$\|U_h^{(n)}\|_\infty^{p-1} \geq \|\varphi_h\|_\infty^{p-1},$$

and with $\lambda \Delta t_n \|U_h^{(n)}\|_\infty^{p-1} \geq \tau'$, we arrive at

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty (1 + \tau').$$

By induction, we get

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(0)}\|_\infty (1 + \tau')^n, \ n \geq 0,$$

which leads us to

$$\|U_h^{(n)}\|_\infty \geq \|\varphi_h\|_\infty (1 + \tau')^n, \ n \geq 0.$$

Since the term on the right hand side of the above inequality tends to infinity as n approaches infinity, we conclude that $\|U_h^{(n)}\|_\infty$ tends to infinity. Now, let us estimate the numerical blow-up time. It is not hard to see that

$$\sum_{n=0}^{+\infty} \Delta t_n \leq \sum_{n=0}^{+\infty} \frac{\tau}{\|U_h^{(n)}\|_\infty^{p-1}} \leq \frac{\tau}{\|\varphi_h\|_\infty^{p-1}} \sum_{n=0}^{+\infty} \left(\frac{1}{(1+\tau')^{p-1}} \right)^n .$$

Using the fact that the series $\dfrac{\tau}{\|\varphi_h\|_\infty^{p-1}} \displaystyle\sum_{n=0}^{+\infty} \left(\frac{1}{(1+\tau')^{p-1}} \right)^n$ converges towards

$$\frac{\tau(1+\tau')^{p-1}}{\|\varphi_h\|_\infty^{p-1}((1+\tau')^{p-1}-1)},$$

we deduce that

$$T_h^{\Delta t} = \sum_{n=0}^{+\infty} \Delta t_n \leq \frac{\tau(1+\tau')^{p-1}}{\|\varphi_h\|_\infty^{p-1}((1+\tau')^{p-1}-1)},$$

and we conclude the proof. ■

Remark 3.2. Using Taylor's expansion, we get

$$(1+\tau')^{p-1} = 1 + (p-1)\tau' + o(\tau'),$$

which implies that

$$\frac{\tau}{(1+\tau')^{p-1}-1} = \frac{\tau}{\tau'} \frac{1}{(p-1+o(1))} \leq \frac{2\tau}{\tau'(p-1)}.$$

If we take $\tau = \dfrac{h^2}{2}$, we have

$$\frac{\tau'}{\tau} = \lambda \min\left\{ \frac{\|\varphi_h\|_\infty^{p-1}}{2}, 1 \right\},$$

and therefore

$$\frac{\tau}{\tau'} = \frac{1}{\lambda} \min\{2\|\varphi_h\|_\infty^{1-p}, 1\}.$$

Then

$$\frac{\tau}{(1+\tau')^{p-1}-1} \leq \frac{2\tau}{\tau'(p-1)} = \frac{2}{\lambda(p-1)} \min\{2\|\varphi_h\|_\infty^{1-p}, 1\}.$$

We conclude that $\dfrac{\tau}{(1+\tau')^{p-1}-1}$ is bounded.

Remark 3.3. From

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty(1+\tau'),$$

we get

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(q)}\|_\infty(1+\tau')^{n-q} \; for \; n \geq q,$$

which implies that

$$\sum_{n=q}^{+\infty} \Delta t_n \leq \frac{\tau}{\|U_h^{(q)}\|_\infty^{p-1}} \sum_{n=q}^{+\infty} \left(\frac{1}{(1+\tau')^{p-1}} \right)^{n-q}.$$

We deduce that

$$T_h^{\Delta t} - t_q \leq \frac{\tau}{\|U_h^{(q)}\|_\infty^{p-1}} \frac{(1+\tau')^{p-1}}{(1+\tau')^{p-1}-1} \;\text{ with }\; \Delta t_q = \sum_{j=0}^{q-1} \Delta t_j.$$

In the sequel, we take $\tau = \dfrac{h^2}{2}$.

4. Convergence of the numerical blow-up time

In this section, under some assumptions, we show that the discrete solution blows up in a finite time and its numerical blow-up time converges to the real one when the mesh size goes to zero. In order to obtain the convergence of the numerical blow-up time, we firstly prove the following theorem about the convergence of the discrete scheme.

Theorem 4.1. Assume that the continuous problem (1)–(3) has a solution $u \in C^{4,2}([0,1]\times[0,T])$ and the initial condition at (10) satisfies

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \; as \; h \to 0. \tag{21}$$

Then, for h sufficiently small, the discrete problem (7)–(10) has a solution $U_h^{(n)}$, $0 \leq n \leq J$, and we have the following relation

$$\max_{0\leq n\leq J} \|U_h^{(n)} - u_h(t_n)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \; as \; h \to 0. \tag{22}$$

where J is such that $\displaystyle\sum_{j=0}^{J-1} \Delta t_j \leq T$ and $t_n = \displaystyle\sum_{j=0}^{n-1} \Delta t_j$.

Proof. For each h, the discrete problem (7)–(10) has a solution $U_h^{(n)}$. Let $N \leq J$, the greatest value of n such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < 1 \; for \; n < N. \tag{23}$$

We know that $N \geq 1$ because of (21). Due to the fact $u \in C^{4,2}([0,1] \times [0,T])$, there exists a positive constant K such that $\|u\|_\infty \leq K$. Using the triangle inequality, we have

$$\|U_h^{(n)}\|_\infty \leq \|u_h(t_n)\|_\infty + \|U_h^{(n)} - u_h(t_n)\|_\infty \leq K + 1,\ n < N, \tag{24}$$

Since $u \in C^{4,2}([0,1] \times [0,T])$. Applying Taylor's expansion, we obtain

$$\begin{aligned}\delta_t u(x_i,t_n) - \delta^2 u(x_i,t_n) + u(x_i,t_n)\delta^0 u(x_i,t_n) - u^p(x_i,t_n) = \\ \frac{h^2}{6} u(x_i,t_n) u_{xxx}(\widetilde{x}_i,t_n) + \frac{\Delta t_n}{2} u_{tt}(x_i,\widetilde{t}_n),\ 1 \leq i \leq I-1,\end{aligned}$$

$$\delta_t u(x_0,t_n) - \delta^2 u(x_0,t_n) - u^p(x_0,t_n) = -\frac{h^2}{12} u_{xxxx}(\widetilde{x}_0,t_n) + \frac{\Delta t_n}{2} u_{tt}(x_0,\widetilde{t}_n),$$

$$\delta_t u(x_I,t_n) - \delta^2 u(x_I,t_n) - u^p(x_I,t_n) = -\frac{h^2}{12} u_{xxxx}(\widetilde{x}_I,t_n) + \frac{\Delta t_n}{2} u_{tt}(x_I,\widetilde{t}_n),$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization, for $n < N$,

$$\begin{aligned}\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} + u(x_i,t_n)\delta^0 e_i^{(n)} + (\delta^0 u(x_i,t_n) - p(\beta_i^{(n)})^{p-1}) e_i^{(n)} = \\ -\frac{h^2}{6} u(x_i,t_n) u_{xxx}(\widetilde{x}_i,t_n) - \frac{\Delta t_n}{2} u_{tt}(x_i,\widetilde{t}_n),\ 1 \leq i \leq I-1,\end{aligned}$$

$$\delta_t e_0^{(n)} - \frac{(2e_1^{(n)} - 2e_0^{(n)})}{h^2} - p(\beta_0^{(n)})^{p-1} e_0^{(n)} = \frac{h^2}{12} u_{xxxx}(\widetilde{x}_0,t_n) - \frac{\Delta t_n}{2} u_{tt}(x_0,\widetilde{t}_n),$$

$$\delta_t e_I^{(n)} - \frac{(2e_{I-1}^{(n)} - 2e_I^{(n)})}{h^2} - p(\beta_I^{(n)})^{p-1} e_I^{(n)} = \frac{h^2}{12} u_{xxxx}(\widetilde{x}_I,t_n) - \frac{\Delta t_n}{2} u_{tt}(x_I,\widetilde{t}_n),$$

where $\beta_i^{(n)}$ is intermediate value between $U_i^{(n)}$ and $u(x_i,t_n)$ for $i \in \{0,\ldots,I\}$. Since $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$, $u_{tt}(x,t)$ are bounded and $\Delta t_n = O(h^2)$, then there exists a positive constant $M > 0$ such that

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} + u(x_i,t_n)\delta^0 e_i^{(n)} \leq c_i^{(n)} e_i^{(n)} + Mh^2,\ 1 \leq i \leq I-1, \tag{25}$$

$$\delta_t e_0^{(n)} - \frac{(2e_1^{(n)} - 2e_0^{(n)})}{h^2} \leq c_0^{(n)} e_0^{(n)} + Mh^2, \tag{26}$$

$$\delta_t e_I^{(n)} - \frac{(2e_{I-1}^{(n)} - 2e_I^{(n)})}{h^2} \leq c_I^{(n)} e_I^{(n)} + Mh^2, \tag{27}$$

where

$$c_i^{(n)} = p(\beta_i^{(n)})^{p-1} - \delta^0 u(x_i, t_n),\ 1 \le i \le I-1,$$

$$c_0^{(n)} = p(\beta_0^{(n)})^{p-1},\ c_I^{(n)} = p(\beta_I^{(n)})^{p-1}.$$

Set $L = \max_{0\le i\le I}\{c_i^{(n)}\}$ and introduce the vector $W_h^{(n)}$ defined as follows

$$W_i^{(n)} = e^{(L+1)t_n}(\|\varphi_h - u_h(0)\| + Mh^2),\ 0 \le i \le I,\ n < N.$$

A straightforward computations reveals that

$$\delta_t W_i^{(n)} - \delta^2 W_i^{(n)} + u(x_i, t_n)\delta^0 W_i^{(n)} > c_i^{(n)} W_i^{(n)} + Mh^2,\ 1 \le i \le I-1, \tag{28}$$

$$\delta_t W_0^{(n)} - \frac{(2W_1^{(n)} - 2W_0^{(n)})}{h^2} > c_0^{(n)} W_0^{(n)} + Mh^2, \tag{29}$$

$$\delta_t W_I^{(n)} - \frac{(2W_{I-1}^{(n)} - 2W_I^{(n)})}{h^2} > c_I^{(n)} W_I^{(n)} + Mh^2, \tag{30}$$

$$W_i^{(0)} > e_i^{(0)},\quad 0 \le i \le I. \tag{31}$$

It follows from Lemma 2.3 that

$$W_i^{(n)} > e_i^{(n)},\quad 0 \le i \le I.$$

By the same way, we also prove that

$$W_i^{(n)} > -e_i^{(n)},\quad 0 \le i \le I,$$

which implies that

$$W_i^{(n)} > |e_i^{(n)}|,\quad 0 \le i \le I.$$

We deduce that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty \le e^{(L+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Mh^2),\ n < N. \tag{32}$$

Now, let us show that $N = J$. Suppose that $N < J$. If we replace n by N in (32), and taking into account the inequality (23), we obtain

$$1 \le \|U_h^{(N)} - u_h(t_N)\|_\infty \le e^{(L+1)T}(\|\varphi_h - u_h(0)\|_\infty + Mh^2). \tag{33}$$

Since $e^{(L+1)T}(\|\varphi_h - u_h(0)\|_\infty + Mh^2) \to 0$ as $h \to 0$, we deduce from (33) that $1 \leq 0$, which is impossible. Consequently $N = J$, and we conclude the proof. ■

Now, we are in a position to state the main theorem of this section.

Theorem 4.2. Suppose that the solution u of the continuous problem (1)–(3) blows up in a finite time T_b such that $u \in C^{4,2}([0,1] \times [0,T_b), \mathbb{R})$ and the initial condition at (10) satisfies

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0. \tag{34}$$

Under the assumptions of the Theorem 3.1, the discrete problem (7)–(10) has a solution $U_h^{(n)}$ which blows up in a finite time $T_h^{\Delta t}$ and the following relation holds

$$\lim_{h \to 0} T_h^{\Delta t} = T_b. \tag{35}$$

Proof. The Remark 3.2 allows us to say that $\dfrac{\tau}{(1+\tau')^{p-1} - 1}$ is bounded. Letting $0 < \varepsilon < \dfrac{T_b}{2}$. Then, there exists a constant $R > 0$ such that

$$\frac{\tau y^{1-p}}{(1+\tau')^{p-1} - 1} < \frac{\varepsilon}{2} \quad \text{for} \quad y \in [R, \infty). \tag{36}$$

Since u blows up at the time T_b. There exists $T_1 \in (T_b - \dfrac{\varepsilon}{2}, T_b)$ and $h_0(\varepsilon) > 0$ such that

$$\|u(., t_n)\|_\infty \geq 2R \quad \text{for} \quad t_n \in [T_1, T_b),\ h \leq h_0(\varepsilon).$$

Let $T_2 = \dfrac{T_1 + T_b}{2}$ and q be a positive integer such that

$$t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_1, T_2] \quad \text{for} \quad h \leq h_0(\varepsilon).$$

We have

$$0 < \|u_h(t_n)\|_\infty < \infty \quad \text{for} \quad n \leq q,\ h \leq h_0(\varepsilon).$$

It follows from Theorem 4.1 that the discrete problem (7)–(10) has a solution $U_h^{(n)}$, which verifies

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < R \quad \text{for} \quad n \leq q,\ h \leq h_0(\varepsilon),$$

which implies

$$\|U_h^{(n)}\|_\infty \geq \|u_h(t_n)\|_\infty - \|U_h^{(n)} - u_h(t_n)\|_\infty \geq R \quad \text{for} \quad h \leq h_0(\varepsilon).$$

From Theorem 3.1, $U_h^{(n)}$ blows up at the time $T_h^{\Delta t}$. It follows from Remark 3.3 and (36) that

$$|T_h^{\Delta_t} - t_q| \leq \frac{\tau(1+\tau')^{p-1}\|U_h^{(q)}\|_\infty^{1-p}}{(1+\tau')^{p-1}-1} < \frac{\varepsilon}{2},$$

because, we have $\|U_h^{(q)}\|_\infty \geq R$ for $h \leq h_0(\varepsilon)$. We deduce that for $h \leq h_0(\varepsilon)$,

$$|T_h^{\Delta_t} - T_b| \leq |T_h^{\Delta_t} - t_q| + |t_q - T_b| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which leads us to the desired result. ■

5. Numerical experiments

In this section, we present some numerical approximations to the blow-up time of the problem (1)–(3). We use the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} - U_i^{(n)}\frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} + (U_i^{(n)})^p,$$
$$1 \leq i \leq I-1, \quad t \in (0,T),$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + (U_I^{(n)})^p,$$

$$U_i^{(0)} > 0, \quad 0 \leq i \leq I,$$

where $n \geq 0$, $p \geq 2$, $\Delta t_n = \min(\frac{h^2}{2}, \tau\|U_h^{(n)}\|_\infty^{1-p})$ with $\tau \in (0,1)$.
Also we use the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} - U_i^{(n)}\frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} + (U_i^{(n)})^p,$$
$$1 \leq i \leq I-1, \ t \in (0,T),$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} + (U_I^{(n)})^p,$$

$$U_i^{(0)} > 0, \quad 0 \le i \le I,$$

where $n \ge 0$, $p \ge 2$, $\Delta t_n = \tau \|U_h^{(n)}\|_\infty^{1-p}$ with $\tau \in (0,1)$.

In the tables 1–10, in rows, we present the numerical blow-up times, numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024. The numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $\Delta t_n = |T^{n+1} - T^n| \le 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

First case: $U_i^{(0)} = \frac{1}{2}$, $p = 2$ and $\tau = \frac{h^2}{2}$.

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.003306	15788	-	-
32	2.000827	60265	-	-
64	2.000207	229656	2	1.99
128	2.000052	873150	9	1.99
256	2.000013	3310849	63	2.00
512	2.000003	12516533	464	2.00
1024	2.000001	47158825	3458	2.00

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

I	T^n	n	$CPU\ time$	s
16	2.003906	15631	-	-
32	2.000977	59637	-	-
64	2.000244	227142	2	2.00
128	2.000061	863093	12	2.00
256	2.000015	3270629	89	2.00
512	2.000004	12355655	672	2.00
1024	2.000001	46515309	5159	2.00

Second case: $U_i^{(0)} = 2$, $p = 2$ and $\tau = \dfrac{h^2}{2}$.

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.500977	15631	-	-
32	0.500244	59637	1	-
64	0.500061	227142	2	2.00
128	0.500015	863093	9	2.00
256	0.500004	3270629	62	2.00
512	0.500001	12355655	457	2.00
1024	0.500000	46515309	3407	2.00

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.500977	15631	-	-
32	0.500244	59637	-	-
64	0.500061	227142	2	2.00
128	0.500015	863093	12	2.00
256	0.500004	3270629	90	2.00
512	0.500001	12355655	674	2.00
1024	0.500000	46515309	5070	2.00

Third case: $U_i^{(0)} = 2$, $p = 4$ and $\tau = \dfrac{h^2}{2}$.

Table 5: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.041830	4872	-	-
32	0.041707	18527	-	-
64	0.041677	70305	-	2.00
128	0.041669	266066	4	2.00
256	0.041667	1003680	30	2.00
512	0.041667	3772429	223	2.00
1024	0.041667	14120612	1659	2.00

Table 6: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.041830	4872	-	-
32	0.041707	18527	1	-
64	0.041677	70305	1	2.00
128	0.041669	266066	8	2.00
256	0.041667	1003680	57	2.00
512	0.041667	3772429	422	2.00
1024	0.041667	14120612	3185	2.00

Fourth case: $U_i^{(0)} = \left(\dfrac{1}{2}\right)^{3-p} + (1-(ih)^2)^2$, $p = 2$ and $\tau = \dfrac{h^2}{2}$.

Table 7: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.861878	16031	-	-
32	0.860194	60831	-	-
64	0.859773	231919	1	2.00
128	0.859668	882201	9	2.00
256	0.859641	3347050	66	2.00
512	0.859635	12661342	488	2.00
1024	0.859633	47738061	3670	2.00

Table 8: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.861680	15961	-	-
32	0.860144	60832	-	-
64	0.859761	231919	2	2.00
128	0.859665	882201	13	2.00
256	0.859641	3347052	93	2.00
512	0.859635	12661342	699	2.00
1024	0.859633	47738061	5294	2.00

Fifth case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1-(ih)^2)^2$, $p = 4$ and $\tau = \frac{h^2}{2}$.

Table 9: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.012839	4693	-	-
32	0.012798	17825	-	-
64	0.012788	67544	-	1.98
128	0.012786	255190	3	1.99
256	0.012785	960804	18	1.99
512	0.012785	3603268	135	1.99
1024	0.012785	13452596	1002	1.99

Table 10: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.012841	4694	-	-
32	0.012799	17825	-	-
64	0.012788	67545	1	1.98
128	0.012786	255190	4	1.99
256	0.012785	960804	27	1.99
512	0.012785	3603268	197	1.99
1024	0.012785	13452596	1490	1.99

Remark 5.1. We observe that the blow-up phenomenon occurs faster for the large values of the initial data and the exponent p. In the case where the initial data is a constant, the solution of our problem blows up in a finite time for all $p \geq 2$, but slowly. This slowness is due to the absence of the turbulence effect, generated by the convection term. Therefore the blow-up only depends on the reaction term. When the initial data is not a constant, the convection term, head of turbulence, accelerates the blow-up created by the reaction term.

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 16$ and $p = 4$. In Figures 1, 2, 3 and 4, we can appreciate that the discrete solution blows up in a finite time when the initial data is a constant or no. In Figures 3 and 4, we see that the blow-up is faster when the initial data is not a constant. The Figures 5, 6, 7 and 8 show the effect of the convection term on the evolution of the solution. In Figures 9, 10, 11 and 12, we observe that the solution of our problem blows up in a finite time $t \simeq 0.04$ when the initial data is 2 and $t \simeq 0.01$ when the initial data is $\left(\frac{1}{2}\right)^{3-p} + (1-(ih)^2)^2$.

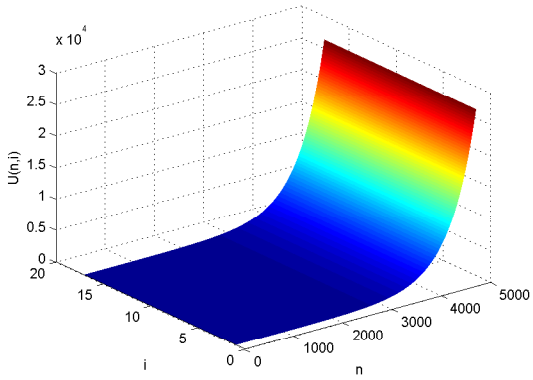

Figure 1: Evolution of the discrete solution (explicit scheme), $U_i^{(0)} = 2$, $p = 4$.

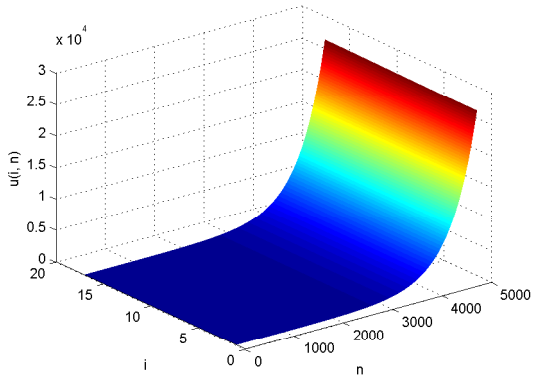

Figure 2: Evolution of the discrete solution (implicit scheme), $U_i^{(0)} = 2$, $p = 4$.

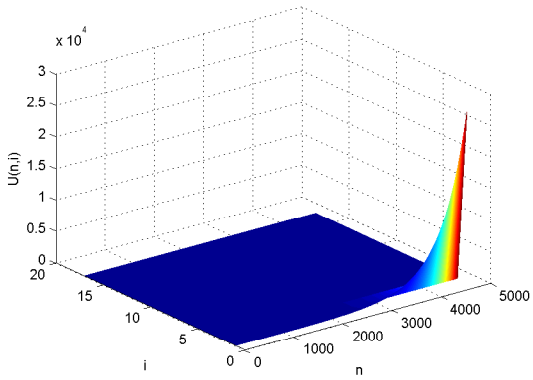

Figure 3: Evolution of the discrete solution (explicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1-(ih)^2)^2$, $p = 4$.

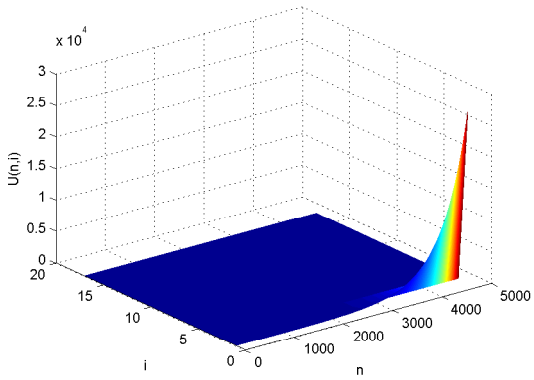

Figure 4: Evolution of the discrete solution (implicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1-(ih)^2)^2$, $p = 4$.

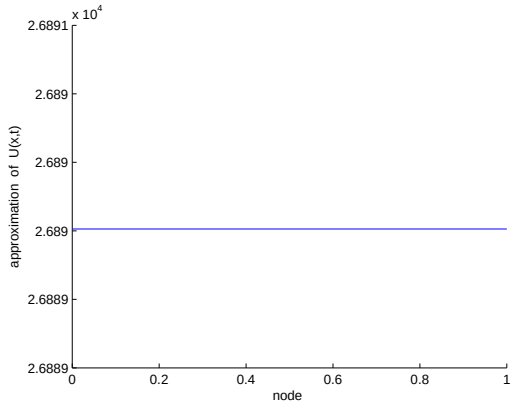

Figure 5: Evolution of U(x,t) according to the node (explicit scheme), $U_i^{(0)} = 2$, $p = 4$.

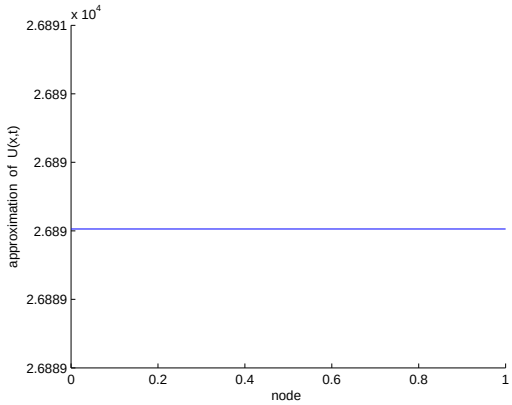

Figure 6: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = 2$, $p = 4$.

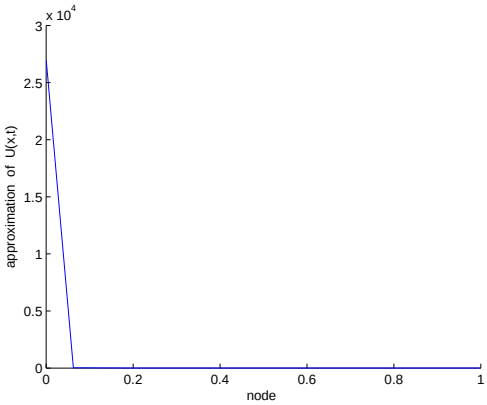

Figure 7: Evolution of U(x,t) according to the node (explicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1-(ih)^2)^2$, $p = 4$.

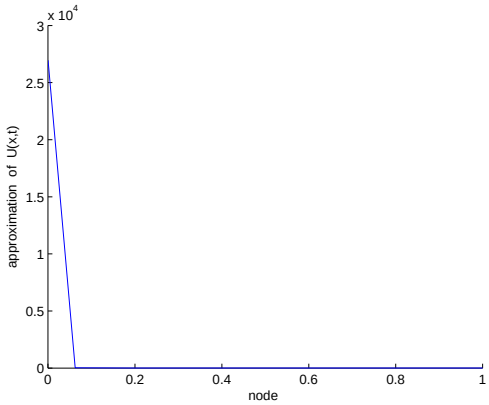

Figure 8: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1-(ih)^2)^2$, $p = 4$.

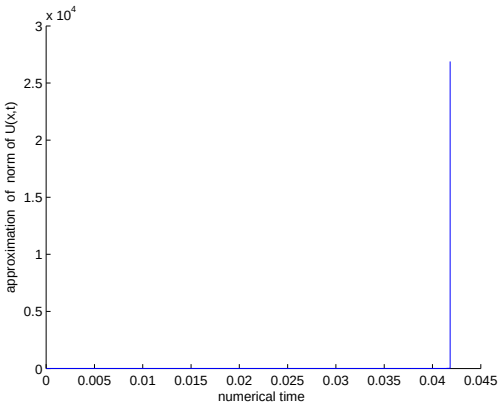

Figure 9: Evolution of norm of U(x,t) according to the time (explicit scheme), $U_i^{(0)} = 2, p = 4$.

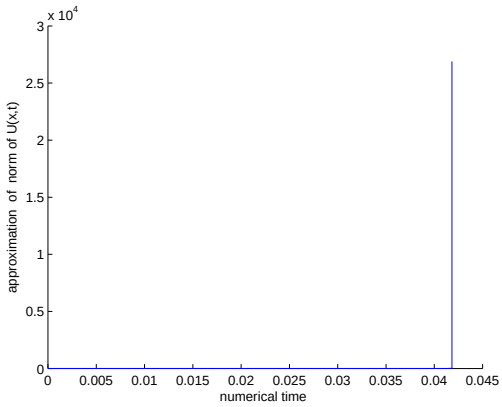

Figure 10: Evolution of norm of U(x,t) according to the time (implicit scheme), $U_i^{(0)} = 2, p = 4$.

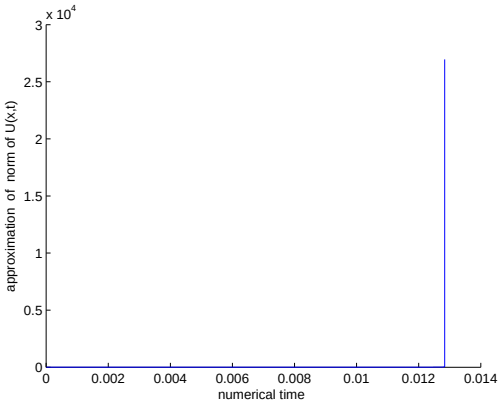

Figure 11: Evolution of norm of U(x,t) according to the time (explicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1-(ih)^2)^2, p = 4$.

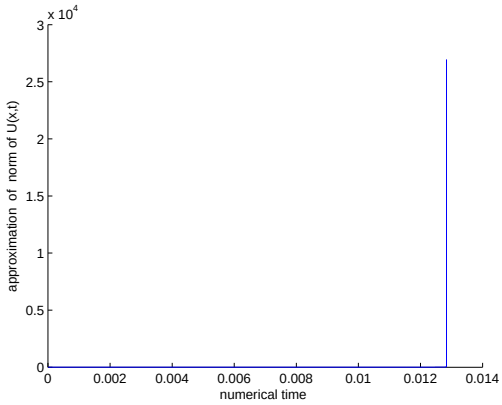

Figure 12: Evolution of norm of U(x,t) according to the time (implicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + (1-(ih)^2)^2, p = 4$.

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