Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 4 (2016), pp. 3357-3366

http://www.ripublication.com/gjpam.htm

On Almost Strongly Prime Submodules

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Abstract

The purpose of this paper is to present some characterizations of n-almost prime submodules; and introduce almost strongly prime submodules as a new generalization of prime submodules and study their properties. It is shown that this notion of almost strongly prime submodules inherits most of the essential properties of the usual notion of a prime ideal.

AMS subject classification: 13C13, 13C05, 13A15.
Keywords: Prime submodule, Almost prime submodule, Strongly prime submodule, Almost strongly prime submodule.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary. For a submodule N of M, let $(N : M)$ denote the set of all elements $r \in R$ such that $rM \subseteq N$. An R-module M is called a multiplication R-module if for each submodule N of M, there is an ideal I of R such that $N = IM$. Several authors have extended the notion of prime ideals to modules [2], [3]. Almost prime ideals were introduced by S. M. Bhatwadekar and P. K. Sharma [6], and were later studied by D. D. Anderson and M. Bataineh [9]. An integral domain R with quotient field K is called a pseudo-valuation domain if whenever a prime ideal P contains the product xy of two elements of K, we have $x \in P$ or $y \in P$. Such a prime ideal P is called a strongly prime ideal. (see [4], [5], [7]). A proper submodule N of an R-module M is called an almost prime submodule of M if whenever $r \in R$ and $m \in M$ such that $rm \in N - (N : M)N$, then either $m \in N$ or $r \in (N : M)$, see [8]. If we consider R to be an integral domain with quotient field K, $T = R - \{0\}$ and M is a unitary torsion-free R-module, then a prime submodule P of M is called strongly prime if for any $y \in K$ and $x \in M_T$, $yx \in P$ gives $x \in P$ or $y \in (P : M)$, where $M_T = \{\frac{a}{t} \mid a \in M, t \in T\}$ (see [1, p. 399]). In the first section of this paper, we generalize the notion of an almost prime ideal to an almost prime submodule by classifying all modules in which every submodule can be written as a product of almost prime submodules. In the second section, we introduce the notion of almost strongly prime submodules and obtain some basic results.

2. Almost Prime Submodules

Following [11], an ideal P is called an almost prime ideal, whenever $xy \in P - P^2$, then either $x \in P$ or $y \in P$. A proper submodule N of an R-module M is called a weakly prime submodule if whenever $r \in R$ and $m \in M$ such that $0 \neq rm \in N$, then either $m \in N$ or $r \in (N : M)$. Also N is called an n-almost prime submodule of M if whenever $r \in R$ and $m \in M$ such that $rm \in N - (N : M)^n N$, then either $m \in N$ or $r \in (N : M)$ (see [8]).

Proposition 2.1. Let N be an $(n + 1)$-almost prime submodule of an R-module M. Then N is an n-almost prime submodule.

Proof. Let $r \in R$, $m \in M$ with $rm \in N - (N : M)^n N$. Since $N - (N : M)^n N \subseteq N - (N : M)^{n+1} N$ then $rm \in N - (N : M)^{n+1} N$; and since N is an $(n + 1)$-almost prime submodule, then $r \in (N : M)$ or $m \in N$. ■

The following proposition shows the relation between n-almost prime submodules and weakly prime submodules.

Proposition 2.2. Let M be an R-module and N be a submodule of M. Then N is an n-almost prime submodule of M if and only if $N/(N : M)^n N$ is a weakly prime submodule in $M/(N : M)^n N$.

Proof. $(\Rightarrow)$ Let $r \in R$ and $m \in M$ such that $\bar{0} \neq r(m + (N : M)^n N) \in N/(N : M)^n N$ then $rm \in N - (N : M)^n N$. So $r \in (N : M)$ or $m \in N$. Therefore $r \in (N/(N : M)^n N : M/(N : M)^n N)$ or $m + (N : M)^n N \in N/(N : M)^n N$. Hence $N/(N : M)^n N$ is a weakly prime submodule.

($\Leftarrow$) Suppose that $N/(N:M)^nN$ is a weakly prime in $M/(N:M)^nN$. Let $r\in$R and $m\in M$ with $rm\in N-(N:M)^nN$. Then $\bar{0}\neq r(m+(N:M)^nN)\in N/(N:M)^nN$. Therefore $r\in(N/(N:M)^nN:M/(N:M)^nN)$ or $m+(N:M)^nN\in N/(N:M)^nN$. Hence $r\in(N:M)$ or $m\in N$. ■

The following theorem characterizes n-almost prime submodules of multiplication modules.

Theorem 2.3. Let M be a multiplication R-module and N be a proper submodule of M. Then N is n-almost prime in M if and only if for any ideal A of R and submodule K of M with $AK\subseteq N-(N:M)^nN$, then $A\subseteq(N:M)$ or $K\subseteq N$.

Proof. ($\Rightarrow$) Let A be any ideal of R and K be a submodule of M with $AK\subseteq N$, $A\nsubseteq(N:M)$ and $K\nsubseteq N$. Let $a\in A-(N:M)^n$, then $aK\subseteq N$ so $K\subseteq(N:_M\langle a\rangle)$. As N is n-almost prime and $K\nsubseteq N$, then by [6, Prop. 2.2] we have $K\subseteq((N:M)^nN:_M\langle a\rangle)$. Therefore $aK\subseteq(N:M)^nN$. Now suppose that $a\in A\bigcap(N:M)^n$ and $b\in(K:M)$. If $b\in(N:M)^n$, then $ab\in(N:M)^{2n}\subseteq((N:M)^nN:M)$ and so $abM\subseteq(N:M)^nN$. Since b is arbitrary, then $a(K:M)M\subseteq(N:M)^nN$. Therefore $aK\subseteq(N:M)^nN$. If $b\in(K:M)-(N:M)^n$, then $bM\subseteq K$ and $bAM\subseteq AK\subseteq N$. Therefore $AM\subseteq(N:_M\langle b\rangle)$ with $AM\nsubseteq N$ (since $A\nsubseteq(N:M)$). Therefore by [6, Prop. 2.2], $AM\subseteq((N:M)^nN:_M\langle b\rangle)$ so $bAM\subseteq(N:M)^nN$. Since b is arbitrary, then $aK=a(K:M)M\subseteq(N:M)^nN$.

($\Leftarrow$) Let $r\in R$ and $m\in M$ with $rm\in N-(N:M)^nN$. Then $\langle r\rangle\langle m\rangle\subseteq N-(N:M)^nN$. Therefore $\langle r\rangle\subseteq(N:M)$ or $\langle m\rangle\subseteq N$. Hence $r\in(N:M)$ or $m\in N$. ■

Proposition 2.4. Let M be a finitely generated faithful multiplication R-module and N a submodule of M. Then the following are equivalent.

(i) N is an n-almost prime submodule of M.

(ii) $(N:M)$ is an $(n+1)$-almost prime ideal of R.

Proof. $(i)\Rightarrow(ii)$ Let $a,b\in R$ with $ab\in(N:M)-(N:M)^{n+1}$, then $abM\subseteq N-(N:M)^nN$. If $abM\subseteq(N:M)^nN$. Then $ab\in((N:M)^nN:M)$. Therefore $ab\in((N:M)^nN:M)=(N:M)^n(N:M)=(N:M)^{n+1}$, which is a contradiction. So $abM\subseteq N-(N:M)^nN$. Since N is n-almost prime, then $a\in(N:M)$ or $bM\subseteq N$. Hence $a\in(N:M)$ or $b\in(N:M)$.

$(ii)\Rightarrow(i)$ Let $r\in R$ and $m\in M$ such that $rm\in N-(N:M)^nN$. Then $r(\langle m\rangle:M)\subseteq(\langle rm\rangle:M)\subseteq(N:M)$ and $r(\langle m\rangle:M)\nsubseteq(N:M)^{n+1}$, for if $r(\langle m\rangle:M)\subseteq(N:M)^{n+1}$, then $r(\langle m\rangle:M)M\subseteq(N:M)^{n+1}M$. Therefore $r\langle m\rangle\subseteq(N:M)^nN$, so $rm\in(N:M)^nN$ which is a contradiction. Now $r(\langle m\rangle:M)\subseteq(N:M)-(N:M)^{n+1}$. Since, $(N:M)$ is an $(n+1)$-almost prime ideal of R, then $r\in(N:M)$ or

$(\langle m\rangle : M) \subseteq (N : M)$ and so $r \in (N : M)$ or $\langle m\rangle = (\langle m\rangle : M)M \subseteq (N : M)M = N$. Hence $r \in (N : M)$ or $m \in N$. ■

Recall that if M is an R-module, L and N are two submodules of M, then there are ideals I and J of R such that $N = IM$ and $L = JM$. Define the multiplication of N and L by $(IJ)M$. If $m_1, m_2 \in M$, then there are ideals I_1 and I_2 such that $Rm_1 = I_1M$ and $Rm_2 = I_2M$, where the multiplication of m_1 and m_2 is defined by $m_1m_2 = (I_1I_2)M$.

Following [11], a subset $S \subseteq R$ is called almost multiplicatively closed if $a, b \in S$ implies $ab \in S$ or $ab \in (R - S)^2$. Now we will extend this definition to submodules.

Definition 2.5. Let M be a multiplication R-module and S be a subset of M. Then S is called an almost multiplicatively closed subset of M if $a, b \in S$, then $ab \subseteq S$ or $ab \subseteq (M - S : M)(M - S)$.

Proposition 2.6. Let M be a multiplication R-module and N a submodule of M. Then N is an almost prime submodule of M if and only if $M - N$ is an almost multiplicatively closed subset of M.

Proof. ($\Rightarrow$) Let N be an almost prime submodule of M and $a, b \in M - N$. We need to show that $ab \subseteq (M - N)$ or $ab \subseteq (N : M)N$. So assume that $ab \nsubseteq (N : M)N$. If $ab \subseteq N$, then $ab \subseteq N - (N : M)N$. Since N is almost prime, then $a \in N$ or $b \in N$, which is a contradiction. Therefore $ab \nsubseteq N$. Hence $ab \in M - N$.

($\Leftarrow$) Suppose that $M - N$ is an almost multiplicatively closed subset of M and to the contrary that N is not an almost prime submodule of M. Let $a, b \in M$ be such that $ab \subseteq N - (N : M)N$ but $a \notin N$ and $b \notin N$. Now we have $a, b \in M - N$ and $ab \nsubseteq (N : M)N$. Since $M - N$ is an almost multiplicatively closed subset of M then $ab \subseteq (M - N)$, so $ab \nsubseteq N$, a contradiction. ■

A well known result of Krull states that if S is a multiplicatively closed subset of R and I is an ideal of R maximal with respect to $I \cap S = \emptyset$, then I is prime. The following proposition is a generalization to modules.

Proposition 2.7. Let M be a multiplication R-module, L be a submodule of M, and S be a multiplicatively closed subset of M such that $L \cap S = \emptyset$. Then there is a submodule N of M maximal with respect to the properties $L \subseteq N$ and $N \cap S = \emptyset$. Furthermore, N is a prime submodule of M.

Proof. The proof is similar to [14, Theorem 5]. ■

Theorem 2.8. Let M, L be any two R-modules. Then A is an almost prime submodule of M if and only if $A \times L$ is an almost prime submodule of $M \times L$.

Proof. ($\Rightarrow$) Suppose that A is an almost prime submodule of M. Let $r \in R$, $(a, b) \in M \times L$ with $r(a, b) \in A \times L - (A \times L : M \times L)A \times L$. Now $r(a, b) \in A \times L - (A \times L : M \times L)A \times L = (A - (A : M)A) \times L$. Hence $ra \in A - (A : M)A$ and $rb \in L$. Since

A is an almost prime submodule of M, then we will have either $r \in (A : M)$ or $a \in A$; therefore, $(a, b) \in A \times L$ or $r \in (A \times L : M \times L)$, where $(A \times L : M \times L) = (A : M)$.

($\Leftarrow$) Suppose that $A \times L$ is almost prime of $M \times L$. Let $r \in R$, $a \in M$ with $ra \in A - (A : M)A$. Then $r(a, 0) \in (A - (A : M)) \times L$ implies $r(a, 0) \in A \times L - (A \times L : M \times L)A \times L$. Since, $A \times L$ is an almost prime submodule of $M \times L$, then $(a, 0) \in A \times L$ or $r \in (A \times L : M \times L)$. It follows that $a \in A$ or $r \in (A : M)$. ■

Lemma 2.9. Let M and T be two R-modules, N be a submodule of M and L is a submodule of T. If $N \times L$ is an almost prime submodule of $M \times T$, then N is an almost prime submodule of M and L is an almost prime submodule of T.

Proof. Let $a \in M$ and $r \in R$ with $ra \in N - (N : M)N$, and $r(a, 0) = (ra, 0) \in (N - (N \times L : M \times T)N) \times L \subseteq N \times L - (N \times L : M \times T)N \times L$. Since $N \times L$ is an almost prime, then $(a, 0) \in N \times L$ or $r \in (N \times L : M \times T) \subseteq (N : M)$. Hence $a \in N$ or $r \in (N : M)$. Similarly, we can show that L is an almost prime submodule of T. ■

Note that, in general, the converse of the above lemma is not true; for example, take $R = Z$, $M = Z$, $T = Z$, $N = 2Z$ and $L = 3Z$, then N and L are almost primes with $3 \notin (N \times L : M \times T)$ and $(2, 4) \notin (N \times L)$, but $3(2, 4) \in N \times L - (N \times L : M \times T)N \times L$.

The next two theorems classify all finitely generated faithful multiplication modules, for which every submodule is an almost prime submodule. For that we need to recall the following proposition and lemma (see [11, Proposition 2.23] and [11, Lemma 2.26]).

Proposition 2.10. Let (R, M) be a quasilocal ring. Suppose I is an ideal of R with $M^2 \subseteq I \subseteq M$. Then I is almost prime if and only if $M^2 = I^2$.

Lemma 2.11. Let (R, M) be a quasilocal ring with every proper principal ideal almost prime. Then $M^2 = 0$.

Theorem 2.12. Let T be a finitely generated faithful multiplication R-module, (T, L) be a local R-module and N be a submodule of T. If $(L : T)L \subseteq N \subseteq L$, then N is almost prime if and only if $(N : T)N = (L : T)L$.

Proof. ($\Rightarrow$) Let $(L : T)L \subseteq N \subseteq L$. Since T is a multiplication R-module and L is a maximal submodule of T, then $L = IT$; where I is a maximal ideal in R. Now $(IT : T) = I$ and $(L : T)L \subseteq N \subseteq L$ imply $(L : T)^2 \subseteq (N : T)$. So $I^2 \subseteq (N : T) \subseteq I$. Since N is almost prime, then by [8, Theorem 3.5] we have $(N : T)$ is almost prime. Therefore by Proposition 10, we have $I^2 = (N : T)^2$, so $I^2T = (N : T)^2T$. Hence $(L : T)L = (N : T)N$.

($\Leftarrow$) Let $(N : T)N = (L : T)L$. Since T is a multiplication R-module and L is a maximal submodule of T, then $L = (L : T)T$; where $(L : T)$ is a maximal in R and $N = (N : T)T$. Now $(L : T)L \subseteq N \subseteq L$ implies $(L : T)^2T \subseteq (N : T)T \subseteq (L : T)T$. Also, since multiplication modules are cancelation modules, then $(L : T)^2 \subseteq (N : T) \subseteq$

$(L : T)$ and $(N : T)N = (L : T)L$. So $(N : T)^2 = (L : T)^2$. Now $(L : T)$ is a maximal ideal in R, $(L : T)^2 \subseteq (N : T) \subseteq (L : T)$ and $(N : T)^2 = (L : T)^2$. So by Proposition 10, $(N : T)$ is almost prime. Hence $N = (N : T)T$ is also an almost prime submodule of T, by [8, Theorem 3.5]. ■

Theorem 2.13. Let T be a finitely generated faithful multiplication R-module and $(T : M)$ be a local R-module. Then every proper submodule of T is an almost prime submodule if and only if $M^2 = 0$.

Proof. ($\Rightarrow$) Since T is a multiplication R-module and M is a maximal submodule of T, then $M = IT$, for some maximal ideal I in R. Let A be a proper ideal in R. Then there is a submodule N of T such that $A = (N : T)$. If we consider the module $N = AT$, then $(N : T) = (AT : T) = A$. Now let Q be a proper principal ideal in R, then $Q = (Y : T)$ for some submodule Y of T, and since every proper submodule is an almost prime submodule, then Y is an almost prime submodule. Therefore by [8, Theorem 3.5] we have $Q = (Y : T)$ is an almost prime ideal in R. Hence (R, I) is a quasilocal ring in which every principal proper ideal is almost prime. So $I^2 = 0$ by Lemma 11. It follows that $M^2 = I^2T = 0$.

($\Leftarrow$) Let N be a proper submodule of T. Since every weakly prime submodule is an almost prime submodule, it is enough to show that N is a weakly prime submodule of T. Let $r \in R$, $m \in T$ with $0 \neq rm \in N$. Now we show that $r \in (N : T)$ or $m \in N$. Since, $rm \neq 0$, then $rm \notin M^2$ and $rm \in M - M^2$. But every maximal submodule is an almost prime submodule. So $r \in (M : T)$ or $m \in M$, where $(M : T)$ is a unique maximal ideal of R (the uniqueness of $(M : T)$ is guaranteed by the uniqueness of M). Now suppose that $r \notin (M : T)$. Then r is a unit in R. Therefore $m = r^{-1}(rm) \in N$. So suppose $m \notin M$ and $m \in N$. Then $Rm = T$ and hence $r \in (N : T)$, since $rm \in N$. ■

3. Almost Strongly Prime Submodules

Let R be an integral domain with quotient field K and M be a torsion-free R-module. For any submodule N of M, suppose $y = \frac{r}{s} \in K$ and $x = \frac{a}{t} \in M_T$. We say $yx \in N$ if there exists $n \in N$ such that $ra = stn$, where $T = R - \{0\}$ and $M_T = \{\frac{a}{t} \mid a \in M, t \in T\}$. It is clear that this is a well-defined operation (see [1]).

Following [1], a prime submodule P of M is called strongly prime if for any $y \in K$ and $x \in M_T$, $yx \in P$ gives $x \in P$ or $y \in (P : M)$.

Definition 3.1. Let R be an integral domain with quotient field K and M be a torsion-free R-module. A submodule N of M is called almost strongly prime if for any $y \in K$ and $x \in M_T$ with $yx \in N - (N : M)N$, then $y \in (N : M)$ or $x \in N$.

Example 3.2. Let $R = \mathbb{R}$, $M = \mathbb{R}$ and $N = \mathbb{Q}$. Then N is an almost strongly prime submodule of M.

It is clear that every almost strongly prime submodule is almost prime, but the converse is not necessarily true. Consider the Z-module $M = Z$ and the submodule $N = 2Z$, then clearly, N is almost prime, but not almost strongly prime.

Following [12], an R-submodule N of M_T is called a fractional submodule of M if there exists $r \in T$ such that $rN \subseteq M$.

Corollary 3.3. Let N be a proper submodule of M. Then N is an almost strongly prime submodule if and only if for any $y \in K$ and any fractional submodule L of M, $yL \subseteq N$ but $yL \nsubseteq (N : M)N$, then $L \subseteq N$ or $y \in (N : M)$.

Theorem 3.4. Let N be a prime submodule of M. For the following statements, we have $(i) \Leftrightarrow (ii)$, $(iii) \Leftrightarrow (iv)$, $(iv) \Rightarrow (i)$.

(i) N is an almost strongly prime submodule of M.

(ii) For any fractional ideal I of R and any fractional submodule L of M, $IL \subseteq N - (N : M)N$ gives $L \subseteq N$ or $I \subseteq (N : M)$.

(iii) $N - (N : M)N$ is comparable to each cyclic fractional submodule of M.

(iv) $N - (N : M)N$ is comparable to each fractional submodule of M.

Proof. $(i) \Leftrightarrow (ii)$ Follows from Theorem 12.
$(iii) \Rightarrow (iv)$ Let L be a fractional submodule of M such that $L \nsubseteq N - (N : M)N$. Then there exist $x \in L - (N - (N : M)N)$ for which $Rx \nsubseteq N - (N : M)N$, where Rx is the cyclic fractional submodule of M generated by x. Hence $N - (N : M)N \subseteq Rx \subseteq L$.
$(iv) \Rightarrow (iii)$ is clear.
$(iv) \Rightarrow (i)$ Suppose that for $y \in K$ and $x \in M_T$, $yx \in N - (N : M)N$ but $y \notin (N : M)$ and $x \notin N$. Since $x \notin N$, then by (iv) we have $N - (N : M)N \subseteq Rx$, $yx \in N - (N : M)N \subseteq Rx$, implies $yx \in Rx$. So $x \in R$. On the other hand, since $y \notin (N : M)$ by (iv) $N - (N : M) \subseteq yM$, $yx \in yM$; therefore, $x \in M$. Now for $x \in M$ and $y \in R$ with $yx \in N - (N : M)N$; where N is a prime submodule of M, we have $x \in N$ or $y \in (N : M)$, which is a contradiction. Thus N is an almost strongly prime submodule of M. ■

Note that as in Theorem 17, in general, $(i) \nRightarrow (iii)$. For example, take $R = \mathbb{Q}$, $M = \mathbb{Q} \oplus \mathbb{Q}$ and $N = \mathbb{Q} \oplus (0)$ which is a strongly prime submodule. Let $x = (0, 4) \in M$ and consider the fractional submodule Rx of M. Then $Rx \nsubseteq N$ and $N \nsubseteq Rx$.

Notice that our next results are on multiplication modules and since modules are assumed to be torsion-free, then for any multiplication module M and ideal I we have $(IM : M) = I$. Notice that our next results are on multiplication modules and since modules are assumed to be torsion-free, then for any multiplication module M and ideal I we have $(IM : M) = I$.

Lemma 3.5. Let M be a multiplication R-module, P be an almost strongly prime submodule of M. Then $(P : M)$ is an almost strongly prime ideal of R.

Proof. Let $a, b \in K$ with $ab \in (P : M) - (P : M)^2$. Then $abM \subseteq (P : M)M - (P : M)^2M$, because if $abM \subseteq (P : M)^2M$, then by $ab \in ((P : M)^2M : M) = (P : M)^2(M : M) = (P : M)^2R = (P : M)^2$ which is a contradiction, see [8, Lemma 3.4]. Since M is a multiplication R-module, then $abM \subseteq P - (P : M)P$. Now by Corollary 17, we have $a \in (P : M)$ or $bM \subseteq P$. If $bM \subseteq P$, then $b \in (P : M)$, and $a \in (P : M)$ or $b \in (P : M)$. Hence $(P : M)$ is an almost strongly prime ideal of R. ∎

Definition 3.6. Let M be an R-module and P be an ideal of a ring R. Define $T_P(M) = \{m \in M \mid (1 - p)m = 0, \text{ for some } p \in P\}$. If $M = T_P(M)$, then M is called P-torsion R-module, and if there exist $m \in M, q \in P$ such that $(1 - q)M \subseteq Rm$, then M is called P-cyclic R-module [13].

Theorem 3.7. Let M be an R-module. Then M is a multiplication R-module if and only if for every maximal ideal P of R, M is P-cyclic or P-torsion R-module. See [13].

Proposition 3.8. Let P be an almost prime ideal of R, and M be a faithful multiplication R-module. Then P is an almost strongly prime ideal if and only if PM is an almost strongly prime submodule of M.

Proof. ($\Leftarrow$) Suppose that PM is an almost strongly prime submodule, then by Lemma 19, $P = (PM : M)$ is an almost strongly prime ideal.

($\Rightarrow$) Let $y = \frac{r}{s} \in K$ and $x = \frac{a}{t} \in M_T$ such that $yx \in PM - (PM : M)PM$. We show that $y \in (PM : M) = P$ or $x \in PM$. If $y \notin P$, put $A = \{b \in R \mid bx \in PM\}$. A is an ideal of R. If $A = R$, then $x \in PM$. So suppose $A \neq R$. Then there exists a maximal ideal Q of R such that $A \subseteq Q$. Since $M \neq T_Q(M)$, then M is Q-cyclic by the previous theorem. So there exist $m \in M$ and $q \in Q$ such that $(1 - q)M \subseteq Rm$. Therefore $(1 - q)[PM - P^2M] \subseteq Pm - P^2m$. Now $yx \in PM - (PM : M)PM$ and hence $(1-q)yx \in Pm - P^2m$. Therefore by [1] there exists $v \in P$ such that $(1-q)ra = stvm$ and $u \in R$ such that $(1 - q)a = um$ which implies that $(1 - q)ra = rum = stvm$. Therefore $ru = stv$. Again by [1], we have $\frac{r}{s} \cdot \frac{u}{t} = v \in P - P^2$. For if $v \in P^2$, then $(1 - q)yx \in P^2m$ which is a contradiction. Now, $\frac{r}{s} \cdot \frac{u}{t} = v \in P - P^2$ and $y = \frac{r}{s} \notin P$, so $\frac{u}{t} \in P$ and $(1 - q)a = um$. Then $(1 - q)\frac{a}{t} = \frac{u}{t}m \in PM$. Hence $(1 - q) \in A \subseteq Q$ is a contradiction. So $A = R$ and hence $x \in PM$, and PM is an almost strongly prime submodule. ∎

The next Proposition gives the relationship between almost prime and almost strongly prime submodules.

Proposition 3.9. Let P be an almost prime submodule of M, then P is an almost strongly prime if and only if for any $y \in K$, $y^{-1}P \subseteq P$ or $y \in (P : M)$.

Proof. ($\Rightarrow$) Recall that if $x \in (P : M)$ then M is an ideal of R and P is an ideal of M.

So we will consider two cases.
case (1): Assume that M is an ideal of R and P an ideal of M. Let $0 \neq y \in K$, $y^{-1}P \nsubseteq P$ and $y \notin (P : M) = P$. For P is not an almost strongly prime ideal, let $x \in M_T$ and $yx \in P - P^2$. Now, since $y^{-1}P \nsubseteq P$ then there exists $x_\circ \in P$ such that $y^{-1}x_\circ \notin P$ and hence $yx + x_\circ \in P - P^2$ and $y(x + y^{-1}x_\circ) \in P - P^2$. But $y \notin P = (P : M)$ and $x + y^{-1}x_\circ \notin P$. Hence, P is not almost strongly prime.
Case (ii): Assume that M is an R-module but not an ideal of R. Suppose that P is an almost strongly prime submodule of M, $y \in K - (P : M)$, $x \in P$ and $x = y^{-1}yx \in P$. If $y^{-1}yx \in (P : M)P$, then $y^{-1}y \in (P : M)$. So $1 \in (P : M)$ and $M \subseteq P$, contradiction. So $y^{-1}yx \in P - (P : M)P$. Since P is an almost strongly prime, then $y^{-1}x \in P$. Hence $y^{-1}P \subseteq P$.

($\Leftarrow$) Suppose for any $y \in K$, $y^{-1}P \subseteq P$ or $y \in (P : M)$. Let $y \in K$ and $x \in M_T$ with $yx \in P - (P : M)P$. If $y^{-1}P \subseteq P$, then $y^{-1}(yx) \in P$, $x \in P$. Otherwise, $y \in (P : M)$. So P is an almost strongly prime submodule of M. ■

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