Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 4 (2016), pp. 3327-3338

http://www.ripublication.com/gjpam.htm

A locally convex topology and an inner product[1]

José Aguayo

Departamento de Matemática,
Facultad de Ciencias Físicas y Matemáticas,
Universidad de Concepción, Chile.

Samuel Navarro

Departamento de Matemáticas y
Ciencias de la Computacion,
Facultad de Ciencia,
Universidad de Santiago, Chile.

Miguel Nova

Departamento de Matemática y
Física Aplicadas,
Facultad de Ingeniería,
Universidad de la Santísima, Concepción, Chile.

Abstract

Let $\mathbb{K}$ be a non-archimedean valued field whose residue class field is formally real, let l^∞ be the space of all bounded sequences of elements of $\mathbb{K}$ and let c_0 be the linear subspace of l^∞ of all null-convergence sequences. Under this condition, the supremun norm of c_0 is coming from an inner product and its dual is l^∞. In this paper the authors define a locally convex topology on c_0 such that the dual of this locally convex space is just itself. Many properties of such topology are study.

AMS subject classification:
Keywords: Non-Archimedean inner product, Strict topologies, Riemann-Lebesgue Property.

[1]This work was partially supported by Proyecto DIUC, No. 209.013.033-1.0. This work was partially supported by Proyecto DICYT, No.040833NH

1. Introduction and Notations

Throughout this paper $\mathbb{K}$ is a non-Archimedean valued field, complete with respect to the ultrametric induced by the nontrivial valuation $|\cdot|$.

Recall that the residue class field of $\mathbb{K}$ is the field

$$\Bbbk = B(0,1)/B^{-}(0,1),$$

where $B(0,1) = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$ and $B^{-}(0,1) = \{\lambda \in \mathbb{K} : |\lambda| < 1\}$.

Let E be a vector space over $\mathbb{K}$. By a non-Archimedean inner product we mean a map $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{K}$ which satisfies for all $a, b \in \mathbb{K}$ and $x, y, z \in E$

I.1 $x \neq 0 \Rightarrow \langle x, x \rangle \neq 0$;

I.2 $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$;

I.3 $|\langle x, y \rangle|^2 \leq |\langle x, x \rangle| \, |\langle y, y \rangle|$ (Cauchy-Schwarz type inequality)

A vector space E with $\langle \cdot, \cdot \rangle$ is called a non-Archimedean inner product space. If $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in E$, then $\langle \cdot, \cdot \rangle$ is called a symmetric inner product.

In what follows we omit "non-Archimedean" in "non-Archimedean inner product" in order to simplify the reading of this article.

We already know that each infinite-dimensional Banach space is linearly homeomorphic to c_0 (see [10], Th. 3.16, pp.70). Therefore, to study these kind of spaces is equivalent to study c_0, the space of all sequences in $\mathbb{K}$ which are convergent to 0. c_0 is a Banach space with the supremum norm $\|\cdot\|_\infty$. In c_0 there is a natural symmetric bilinear form $\langle \cdot, \cdot \rangle : c_0 \times c_0 \to \mathbb{K}$, defined by

$$\langle x, y \rangle = \sum_{n \in \mathbb{N}} x_n y_n. \tag{1.1}$$

It is easy to see that $|\langle x, y \rangle| \leq \|x\|_\infty \|y\|_\infty$. But, it may happen that $|\langle x, x \rangle| < \|x\|_\infty^2$ for some $x \in c_0$. In order to avoid the last strict inequality, we need an extra algebraic condition on $\mathbb{K}$.

Definition 1.1. A field F is formally real if for any finite subset $\{a_1, \ldots, a_n\}$ of F that satisfies $\sum_{i=1}^{n} a_i^2 = 0$ implies each $a_i = 0$.

According to this definition, $\mathbb{R}$ is formally real and $\mathbb{C}$ is not. Examples of non-archimedean fields which are not formally real are $\mathbb{Q}_p$ and $\mathbb{C}_p$; however, the Levi-Civita field given in [8] is formally real. If we take $\mathbb{R}$ instead of F in [10] Th.1.3, page 9, then we can have many examples of non-archimedean field whose respective "residual class fields" are formally real.

Now, in the non-Archimedean context, we have the following alternative condition for a formally real field:

Proposition 1.2. Let $\mathbb{K}$ be a non-archimedean valued field. Then, $\mathbb{k}$ is formally real if, and only if, for each finite subset $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $\mathbb{K}$,

$$\left|\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2\right| = \max\left\{\left|\lambda_1^2\right|, \left|\lambda_2^2\right|, \ldots, \left|\lambda_n^2\right|\right\}.$$

Proof. ($\Leftarrow$) Suppose that $\mathbb{k}$ is formally real. Then, by [6] , Corollary 6.3,

$$|\langle x, x\rangle + \langle y, y\rangle| = \max\{|\langle x, x\rangle|, \langle y, y\rangle\},$$

for $x, y \in c_0$. In particular, if

$$x = (\lambda_1, 0, \ldots) \ , \ y = (\lambda_2, 0, \ldots) \ \in c_0,$$

then

$$\left|\lambda_1^2 + \lambda_2^2\right| = \max\left\{|\lambda_1|^2, |\lambda_2|^2\right\}$$

The rest of the proof follows by induction
($\Rightarrow$) Let us suppose now that $\mathbb{k}$ is not formally real, that means, there exists a finite subset $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbb{K}$ so that

$$\overline{\lambda}_1^2 + \overline{\lambda}_2^2 + \cdots + \overline{\lambda}_n^2 = \overline{0}$$

and

$$\overline{\lambda}_i \neq \overline{0}. \tag{1.2}$$

Now, $\overline{\lambda}_1^2 + \overline{\lambda}_2^2 + \cdots + \overline{\lambda}_n^2 = \overline{0}$ implies

$$\left|\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2\right| < 1$$

and since

$$\left|\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2\right| = \max\left\{|\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_n|^2\right\}$$

we have

$$|\lambda_i| < 1$$

for each $i \in \{1, 2, \ldots, n\}$, which is equivalent to

$$\overline{\lambda}_i = \overline{0}$$

This is a contradiction to (2). ■

The following theorem was one of the main results proved in [6], (Th.6.1, p. 194):

Theorem 1.3. The symmetric bilinear form given in (1.1) is an inner product on c_0 which induces the original norm if and only if the residue class field $\mathbb{k}$ of $\mathbb{K}$ is formally real.

We already know that the dual of $(c_0, \|\cdot\|_\infty)$ is just l^∞. The main goal of this paper is to define a locally convex topology ν_0 on c_0 such that $(c_0, \nu_0)' = c_0$ and each closed subspace has a normal complemented, that is, if M is a v_0-closed subspace, then there exists a v_0-closed subspace N such that $c_0 = M \oplus N$ and $\langle m, n\rangle = 0$ for every $m \in M$ and $n \in N$. In other words, (c_0, v_0) is orthomodular with respect to this inner product.

Another concept that we will need to know is the following:

Definition 1.4. Let $\left(x^n\right)$ be a sequence of nonzero elements of c_0. Then, we will say that $\left(x^n\right)$ has the Riemann-Lebesgue Property (RLP in short) if for any $y \in c_0$ we have that

$$\left\langle x^n, y\right\rangle \to 0.$$

Of course, every base of c_0 have the RLP.

It is well-known that if X is a zero dimensional Hausdorff topological space, then there are some special locally convex topologies, so-called strict topologies, defined in $C_b(X, \mathbb{K})$, the space of all continuous and bounded functions from X into $\mathbb{K}$. One of them, which will be the topology that will be used in this paper, is constructed as follows (see [2] and [4]):

A function $v : X \to \mathbb{K}$ is said to be vanish at infinity if it is bounded and for each $\epsilon > 0$ there exists a compact K of X such that $\|v\|_{X\setminus K} < \epsilon$. Let us call $\mathcal{B}(X, \mathbb{K})$ the collection of all functions on X which are vanish at infinity. Given $v \in \mathcal{B}(X, \mathbb{K})$, we define the semi-norm

$$p_v(f) = \sup_{x\in X} |f(x)\, v(x)|\,; \text{ for } f \in C_b(X, \mathbb{K}).$$

The family $\{p_v\}_{v\in\mathcal{B}(X,\mathbb{K})}$ generates a locally convex topology which is denotes by β_0. This topology has many properties that was proved in [4] and we will only mention those that will be useful in this paper. Let us denote by τ_u the uniform topology on $C_b(X, \mathbb{K})$, that is, the topology generated by the supremum norm on $C_b(X, \mathbb{K})$, and by τ_c the compact-open topology on $C_b(X, \mathbb{K})$.

1. β_0 coincides with τ_c on each τ_u-bounded set.
2. $\tau_c \le \beta_0 \le \tau_u$
3. β_0 and τ_u have the same bounded sets.
4. A sequence (f_n) in $C_b(X, \mathbb{K})$ is β_0-convergent to f iff it is τ_u-bounded and $f_n \to f$ in τ_c.
5. If $C_0(X, \mathbb{K})$ denotes the space of those function $v \in \mathcal{B}(X, \mathbb{K})$ which are continuous and X is locally compact, then $C_0(X, \mathbb{K})$ is β_0-dense in $C_b(X, \mathbb{K})$.

Now, in order to describe the dual of $(C_b(X, \mathbb{K}), \beta_0)$, we need to introduce a space of measures. Let us denote by $S(X)$ the ring of clopen subsets of X; we will understand by a measures on X to any finitely-additive set-function $m : S(X) \to \mathbb{K}$ such that it is

$m(S(X))$ is bounded in $\mathbb{K}$. The space of all these functions will be denoted by $M(X)$ and it is normed by $|m| = |m|(X)$, where

$$|m|(A) = \sup\{|m(B)| : B \subset A;\ B \in S(X)\}$$

for any $A \in S(X)$. $M_t(X)$ will denote the subspace of all measures $m \in M(X)$ such that for any positive real number ϵ, there exists a compact subset K of X such that $|m|(X - K) < \epsilon$. It is well-known that $(C_b(X, \mathbb{K}), \beta_0)' = M_t(X)$ and for any $m \in M_t(X)$, each $f \in BC(X)$ is m-integrable. These two facts will be also used in this work.

2. Definition of a locally convex topology on c_0

We already know that the dual of $(c_0, \|\cdot\|)$ is l^∞, the space of all bounded sequences in $\mathbb{K}$. In this dual we can distinguish two kind of linear functionals: linear functional of the type $\langle \cdot, y\rangle : c_0 \to \mathbb{K}$, defined by $x \to \langle x, y\rangle$, for any $y \in c_0$, and linear functional of the type $f : c_0 \to \mathbb{K}$, defined by $x \to f(x) = \sum_{n\in\mathbb{N}} x_n a_n$, where $(a_n) \in l^\infty \setminus c_0$. The first ones are call Riesz functional and this collection forms a normed space that we will denote by

$$\mathfrak{R} = \left\{ f \in c_0' : f = \langle \cdot, y\rangle \text{ for some } y \in c_0 \right\}.$$

Clearly, $\mathfrak{R}$ is algebraically isomorphic to c_0. The next lemma will give us conditions for a continuous linear functional can be a Riesz functional. Here, e_n will denote the n^{th} canonical elements of c_0.

Lemma 2.1. A continuous linear functional f is a Riesz functional if and only if $\lim_{n\to\infty} f(e_n) = 0$. In other words, $e_n \to 0$ in the weak topology with respect to the duality $\langle c_0, \mathfrak{R}\rangle$.

Proof. ($\Rightarrow$) Let f be an element of $\mathfrak{R}$. Then, there exists $y \in c_0$ such that $f = \langle \cdot, y\rangle$. Now, by the fact that $\{e_n : n \in \mathbb{N}\}$ has the RLP, we have

$$\lim_{n\to\infty} f(e_n) = \lim_{n\to\infty} \langle e_n, y\rangle = 0.$$

($\Leftarrow$) On the other hand, if $\lim_{n\to\infty} f(e_n) = 0$, then $y = (f(e_n)) \in c_0$. We claim that $f = \langle \cdot, y\rangle$, in fact,

$$\begin{aligned} x \in c_0 &\Rightarrow x = \sum_{n=1}^{\infty} x_n e_n \\ &\Rightarrow f(x) = f\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_n f(e_n) = \langle x, y\rangle. \end{aligned}$$

■

Definition 2.2. A locally convex topology ν on c_0 is said to be admissible if $(c_0, \nu)' = \mathfrak{R}$.

The locally convex topology generated by the family of semi-norms $\{q_y(\cdot) = |\langle \cdot, y \rangle| : y \in c_0\}$ is admissible. It is clear that this topology is the weak topology $\sigma_{\mathfrak{R}}$ respect to the duality $\langle c_0, \mathfrak{R} \rangle$. By the fact that this topology is Hausdorff, every admissible topology is also Hausdorff.

Theorem 2.3. Let ν be an admissible topology. Then,

1. if $\mathbb{K}$ is spherically complete, then $e_n \overset{\nu}{\to} 0$.
2. if ν is of countable type, then $e_n \overset{\nu}{\to} 0$.

Proof. Both results follow from the fact that each weakly convergent sequence is convergent, since ν is strongly polar (see [9] Th.4.4 and Prop. 4.11 pages 197 and 200). ■

For each $y \in c_0$, we define

$$p_y(x) = \max\{|x_n y_n| : n \in \mathbb{N}\}.$$

It is clear that p_y is a semi-norm. Let us denote by ν_0 the locally convex topology generated by this family of semi-norms.

Lemma 2.4. For $y \in c_0$,

$$\lim_{n\to\infty} p_y(e_n) = 0.$$

Proof. This is immediately consequence of

$$p_y(e_n) = |y_n|.$$

■

Lemma 2.5. ν_0 is of countable type.

Proof. We need to prove that (c_0, p_y) is of countable type, for each $y \in c_0$. We claim that

$$c_0 = \overline{[\{e_1, e_2, \ldots\}]}^{p_y}.$$

In fact, let $x \in c_0$ and consider the following sequence $\left(\sum_{i=1}^{n} x_i e_i\right)$. Since

$$\begin{aligned} p_y\left(x - \sum_{i=1}^{n} x_i e_i\right) &= p_y(0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots) \\ &= \max\{|x_k y_k| : k \geq n+1\} \\ &\leq \|y\| \max\{|x_k| : k \geq n+1\} \overset{n\to\infty}{\to} 0 \end{aligned}$$

we conclude that $\left(\sum_{i=1}^{n} x_i e_i\right)$ is p_y-convergent to x, which is enough to prove the lemma.

■

Proposition 2.6. ν_0 is admissible.

Proof. Let us take $f \in (c_0, \nu_0)'$. Then, there exists a positive constant C and a finite collection $y^1, y^2, \ldots, y^k \in c_0$ such that

$$|f(x)| \leq C \max\left\{p_{y^l}(x) : l = 1, \ldots, k\right\}.$$

By Lemma 2.4,

$$p_{y^l}(e_n) \to 0,$$

which implies that

$$f(e_n) \to 0.$$

Thus, $f \in \mathfrak{R}$.
Let us take now $f \in \mathfrak{R}$ and prove that f is ν_0−continuous. Since $f \in \mathfrak{R}$, there exists $y \in c_0$ such that $f = \langle \cdot, y\rangle$. Consider the semi-norm

$$p_y(x) = \max\{|x_n y_n| : n \in \mathbb{N}\}.$$

We claim that

$$|f(x)| \leq p_y(x).$$

In fact,

$$|f(x)| = |\langle x, y\rangle| = \left|\sum_{n=1}^{\infty} x_n y_n\right| \leq \max\{|x_n y_n| : n \in \mathbb{N}\} = p_y(x).$$

Thus, f is ν_0-continuous. Therefore,

$$\mathfrak{R} = (c_0, \nu_0)'.$$

■

Theorem 2.7. ν_0 is the finest locally convex topology of countable type which is admissible.

Proof. Let ν be an admissible locally convex topology of countable type and let p be a ν-continuous semi-norm. By Th.2.3, $e_n \overset{\nu}{\to} 0$, and then

$$p(e_n) \to 0.$$

Choose $y = (y_n) \in c_0$ such that

$$|y_n| \geq p(e_n)$$

and consider the semi-norm $p_y(x) = \max\{|x_n y_n| : n \in \mathbb{N}\}$. Note that $\sum_{n=1}^{\infty} x_n e_n$ is p-convergent, since $p(x_n e_n) \to 0$. Then,

$$p(x) = p\left(\sum_{n=1}^{\infty} x_n e_n\right) \leq \max\{|x_n y_n| : n \in \mathbb{N}\} = p_y(x).$$

It is enough to show that

$$\nu \leq \nu_0.$$

■

3. ν_0 as a strict topology

Assuming that $\mathbb{N}$ has the discrete topology (which is a zero-dimensional topological space), we will have that $l^{\infty} = \{x = (x_n) : \|x\|_{\infty} < \infty\}$ is precisely $C_b(\mathbb{N}, \mathbb{K})$.

According to the first section, a function whose vanishes at infinity is a function $\eta : \mathbb{N} \to \mathbb{K}$ which satisfies that for a given $\epsilon > 0$, there exists a compact K in $\mathbb{N}$ such that

$$\|\eta(x)\|_{\mathbb{N}\setminus K} < \epsilon,$$

Now, compact sets in $\mathbb{N}$ are just the finite subsets, therefore the above condition is equivalent to say that

$$\lim_{n\to\infty} \eta(n) = 0,$$

that is, $\eta \in c_0$. Thus, we have proved that

$$B(K(X), \mathbb{K}) = \{\eta : \mathbb{N} \to \mathbb{K} : \eta \text{ vanish at infinity}\} = c_0$$

The first section tell us that the strict topology β_0 defined on $C_b(\mathbb{N}, \mathbb{K}) = l^{\infty}$ is the topology generated by the family of semi-norms $\{p_\eta\}_{\eta\in c_0}$. In this case, each p_η has the following form:

$$p_\eta((a_i)) = \sup_{i\in\mathbb{N}} |a_i \eta_i|$$

Also, the spaces $(l^{\infty}, \|\cdot\|_{\infty})$ and (l^{∞}, β_0) have the same bounded sets. Note that $C_0(\mathbb{N}, \mathbb{K})$ is just c_0 and hence c_0 is β_0-dense in l^{∞}. At the same time, the topology ν_0 defined in the second section is just the induced topology by β_0; hence we have the following corollary:

Corollary 3.1. The spaces (c_0, ν_0) and $(c_0, \|\cdot\|_{\infty})$ have the same bounded sets.

Proof. Let B be a ν_0-bounded set, hence

$$\forall y \in c_0 = B(K(X)) : \max_{x\in B} p_y(x) < \infty.$$

Therefore,

$$\forall \eta \in B\left(K\left(X\right)\right) = c_0 : \max_{x \in B} p_\eta\left(x\right) < \infty,$$

that is, B is β_0-bounded and then B is $\|\cdot\|_\infty$-bounded. ■

Remark 3.2. The Ascoli theorem type proved in [5] tells us that the compactoid sets of (c_0, ν_0) are precisely the bounded sets. In other words, (c_0, ν_0) is a semi-montel space.

Proposition 3.3. Let (x_n) be a bounded sequence in c_0. Then, (x_n) is ν_0-convergent to 0 if and only if (x_n) has the Riemann-Lebesgue Property.

Proof. Since the condition for a sequence must have the Riemann–Lebesgue Property is equivalent that this sequence convergent to 0 in the weak topology respect to the duality $\langle c_0, \mathfrak{R}\rangle$ and by the fact that bounded sets are compactoid, the proposition follows applying Th. 4.4 and Prop. 4.11 in [9]. ■

Remark 3.4. Let us mention some properties of the spaces $\left(l^\infty, \beta_0\right)$ and (c_0, ν_0) . Since $\mathbb{N}$ is a small set, is not a compact space, is a k-space, is a locally compact space and is an infinity set, we have

1. $\left(l^\infty, \beta_0\right)$ is not a barrelled space, Th. 7.2.14 in [9]
2. $\left(l^\infty, \beta_0\right)$ is not a reflexive space, Th. 7.5.10 in [9].
3. $\left(l^\infty, \beta_0\right)$ is a complete space, Th. 9 in [5] .
4. c_0 is β_0-dense in $\left(l^\infty, \beta_0\right)$ and then (c_0, ν_0) is not a complete space.

4. Duality

Now, by the fact that c_0 is β_0-dense in $\left(l^\infty, \beta_0\right)$, we have that

$$\mathfrak{R} = (c_0, \nu_0)' = \left(l^\infty, \beta_0\right)' = M_t\left(\mathbb{N}, \mathbb{K}\right).$$

Let us describe the elements of the space $M_t\left(\mathbb{N}\right)$. Here, the ring of subsets of N is $S\left(\mathbb{N}\right) = \mathcal{P}\left(\mathbb{N}\right)$ and if $m \in M_t\left(\mathbb{N}\right)$, then m for a given $\epsilon > 0$, there exists a compact set K of $\mathbb{N}$ such that $|m|\left(\mathbb{N} \setminus K\right) < \epsilon$. Since compact sets in $\mathbb{N}$ are finite, we have that $K = \{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}$ such that

$$|m|\left(\mathbb{N} \setminus \{n_1, n_2, \ldots, n_k\}\right) < \epsilon,$$

which this means that

$$|m|\left(\mathbb{N} \setminus \{n_1, n_2, \ldots, n_k\}\right) = \max\left\{|m\left(B\right)| : B \cap \{n_1, n_2, \ldots, n_k\} = \varnothing\right\} < \epsilon$$

On the other hand, $m \in M_t(\mathbb{N})$ is equivalent to say that for each sequence $\{G_n\}$ of subsets of $\mathbb{N}$ such that $G_n \cap G_k = \varnothing$ for $n \neq k$, $m(\cup G_n) = \sum_{n=1}^{\infty} m(G_n)$ which implies that $(m(G_n)) \in c_0$. Now, since each function $f \in l^{\infty}$ is m-integrable, we have

$$\int_{\mathbb{N}} f dm \in \mathbb{K}$$

But, by properties of the integral, we have

$$\int_{\mathbb{N}} f dm = \int_{\cup\{n\}} f dm = \sum_{n=1}^{\infty} \int_{\{n\}} f dm = \sum_{n=1}^{\infty} x_n m(\{n\}) = \langle f, (m(\{n\})) \rangle ,$$

and then we can define a linear homomorphism

$$\begin{aligned} \Phi : M_t(\mathbb{N}) &\to \mathfrak{R} \\ f &\to \Phi(m) = \langle \cdot, (m(\{n\})) \rangle \end{aligned}$$

We claim that Φ is an isomorphism, even more, it is a linear isometry. In fact, let $\langle \cdot, y \rangle$ be an element of $\mathfrak{R}$, for some $y = (y_n) \in c_0$. We define $m : \mathcal{P}(\mathbb{N}) \to \mathbb{K}$ by $m(A) = \sum_{i \in A} y_i$. Clearly, m is well-defined and is finite additive set-function. Moreover, since

$$|m(A)| = \left| \sum_{i \in A} y_i \right| \leq \max\{|y_i| : i \in A\} \leq \|y\|$$

we conclude that $m \in M(\mathbb{N})$. Now, for a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|y_n| < \epsilon$, for any $n \geq N$. Let us take $K = \{1, 2, \ldots, N-1\}$ which is compact in $\mathbb{N}$; hence

$$|m|(\mathbb{N} \setminus K) = \max\{|m(B)| : B \subset \mathbb{N} \setminus K\} < \epsilon,$$

that is, $m \in M_t(\mathbb{N})$. Now, take $m \in M_t(\mathbb{N})$; then

$$\begin{aligned} \|\Phi(m)\| &= |\langle \cdot, (m(\{n\})) \rangle| = \|(m(\{n\}))\| = \sup\{|m(\{n\})| : n \in \mathbb{N}\} \\ &\leq \sup\{|m(B)| : B \in \mathcal{P}(\mathbb{N})\} = |m|(\mathbb{N}) = |m| \end{aligned}$$

On the other hand, since $|m(B)| \leq \|y\|$, for all $B \in \mathcal{P}(\mathbb{N})$, we conclude that $\|\Phi(m)\| = |m|$.

5. ν_0-Closed subspaces

In this section we will study the ν_0-closed subspaces of c_0. Let us start with a Kernel of a ν_0-linear functional f. By sections 3, there exists $y \in c_0$ such that $f = \langle \cdot, y \rangle$ and then its Kernel $N(f)$ has the Riemann-Lebesgue Property. Conversely, if f is a continuous linear functional and its Kernel has the Riemann-Lebesgue Property, then there exists $y \in c_0$ such that $f = \langle \cdot, y \rangle$ (see [6]). From this facts, we have the following proposition.

Proposition 5.1. Let M be an one-codimensional and $\|\cdot\|$-closed subspaces of c_0. Then, M is ν_0-closed if and only if M has the Riemann–Lebesgue Property.

Proof. Let us denote by N its complemented of M, that is, $c_0 = M \oplus N$ and $N = [\{y\}]$, for some $y \in c_0$. Thus, there exists a unique pair $\{u, v\}$ in c_0 such that $x = u + v$, with $u \in M$ and $v \in N$. We define $f : c_0 \to \mathbb{K}$ by $f(x) = \alpha$, where $v = \alpha y$. Clearly, f is a linear functional whose $N(f) = M$. Now, if M is ν_0-closed, then f is ν_0-continuous which implies that $f = \langle \cdot, z \rangle$ for some $z \in c_0$. Now, invoking Th.5 in [1], we conclude that $N(f) = M$ has the Riemann–Lebesgue Property. Conversely, since M is $\|\cdot\|$-closed, we have that the above f is $\|\cdot\|$-continuous and its kernel $N(f)$ has the Riemann-Lebesgue Property; then by Cor. 9.2 in [6], we have that there exists $y \in c_0$ such that $f = \langle \cdot, y \rangle$ and conclude that $M = N(f)$ is ν_0-closed, since $f = \langle \cdot, y \rangle \in \mathfrak{R}$. ■

Corollary 5.2. Let M be a finite-codimensional and $\|\cdot\|$-closed subspaces of c_0. Then, M is ν_0-closed if and only if M has the Riemann–Lebesgue Property.

Proposition 5.3. Let M be a $\|\cdot\|$-closed subspace of c_0. Then, if M has the Riemann–Lebesgue Property, then M is ν_0-closed.

Proof. Since M has the Riemann-Lebesgue Property, there exists a normal complement N in c_0, that is, $c_0 = M \oplus N$ and $\langle x, y \rangle = 0$ for any $x \in M$ and $y \in N$ (see Cor. 3 in [1]). Under this condition we can sure the existence of a normal projection P such that $N(P) = N$ and $R(P) = M$. Now, by Th. 5 in [3], there exists an orthonormal basis $\{y_1, y_2, \ldots\}$ in N with the Riemann–Lebesgue Property such that

$$P = \sum_{n=1}^{\infty} \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n.$$

Now, each $f_n = \dfrac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle}$ is a Riesz functional and then is ν_0-continuous which implies that $N(f_n)$ is ν_0-closed. By the fact that $M = \cap_{n=1}^{\infty} N(f_n)$ we conclude that M is ν_0-closed. ■

Remark 5.4. If we $\mathcal{M} = \{M : M \leq c_0,\ M \text{ is } \|\cdot\| - \text{closed and } M \text{ has RLP}\}$ and $\mathcal{N} = \{M : M \leq c_0 \text{ and is } \nu_0 - \text{closed}\}$, then Prop. 5 tells us that $\mathcal{M} \subset \mathcal{N}$. On the other hand, Prop. 4 and Cor. 2 show that there exist ν_0-closed subspaces which have the RLP. The open question here is: $\mathcal{M} = \mathcal{N}$?

Our sincere thanks to the professors Wim Schikhof (RIP) and Lawrence Narici for the suggestions that they gave us to attack the problems in this paper.

References

[1] Aguayo, J. and Nova, M., (2007), "Non-archimedean Hilbert like Spaces", Bull. Belg. Math. So., Vol. 14, pp. 787–797.

[2] Aguayo, J., Navarro, S. and Nova, M., 2010, "Strict topologies on space of vector-valued continuous functions over non-archimedean field", Contemporary Mathematics, Vol. 508, pp. 1–12.

[3] Aguayo, J. and Nova, M., "Characterization of Compact and self-adjoint Operators on spaces with non-archimedean inner product", to appear in Contemporary Mathematics.

[4] Katsaras, A. K., 1984, "The strict topology in non-Archimedean vector-valued functions spaces", Proceedings A 87(2), pp. 189–201

[5] Martinez-Maurica, J and Navarro, S. , 1990, S., "P-adic Ascoli theorems", Rev. Mat. Univ. Complutense, Madrid 3, 19–27.

[6] Narici, L. and Beckenstein, E., 2005, "A non-Archimedean Inner Product", Contemporary Mathematics, vol. 384, 187–202.

[7] Perez-Garcia, C. and Schikhof, W., 2010, "Locally convex spaces over non-archimedean valued fields", Cambridge University Press.

[8] Shamseddine, K., 1999, "New Elements of Analysis on the Levi-Civita Field", Ph D Dissertation, Department of Mathematics and Department of Physics and Astronomy, Michigan State University.

[9] Schikhof, W., (1986), "Locally convex spaces over nonspherically complete valued fields I-II", Bull. Soc. Math. Belg. Sér. B 38, 187–224.

[10] van Rooij A. C. M., 1978, "Non-Archimedean Functional Analysis", M Dekker, New York.