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Transformation of the Navier-Stokes Equations in Curvilinear Coordinate Systems with Maple

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Abstract

In this research, we study the general form of the Navier-Stokes equations in arbitrary curvilinear coordinate system which it is the governing equation of fluid flow. We often found that some engineering or applied sciences problems are in complex geometries domain such as irregular or infinite which it is not easy to solve their problems. Most Researchers will be transform into a domain that is easy to calculate, such as rectangle domain for the flow in two dimensions or rectangular shape domain for the flow in three dimensions. This makes it easy to calculate the basic parameters of the problem, such as speed, pressure and so on. Therefore, the main purpose of this work discusses the detailed transformation of the Navier-Stokes equations in curvilinear coordinate systems. In addition, we write the code in symbolic computation for the transformation by Maple Software. The Maple software was used to perform calculate all operators of scalar, vector and tensor quantities in the Navier-Stokes equations. The program for transformation the Navier-Stokes equations in orthogonal curvilinear coordinate systems was obtained.

AMS subject classification:
Keywords: Transformation, Navier-Stokes equations, Curvilinear coordinate systems.

1. Introduction

A general form of the Navier-Stokes equations in a fixed curvilinear coordinate system were investigated long ago using coordinate transformation. We can find the standard form of these equations and their derivation in tensor calculus textbooks [1]–[3]. However, its have not been used widely in numerical simulations, because of the calculation of the covariant derivatives in curvilinear coordinate systems is generally very complicate and spent time to much for calculation. Most researchers have utilized particular form of the Navier-Stokes equations when deal with flows in complex geometries via mapping into regular domain. Many researchers compute on cylindrical coordinate system for solving the problem of blood flow in the arteries or choose Spherical coordinate system for calculate primitive variables in the problem of fluid flow through circular object. Ogawa and Ishiguro [4] proposed a new method for computing flow fields with arbitrarily moving boundaries. Under the concept of Lie derivatives the field equations in general moving coordinates. They derived the temporal derivative of tensor vectors by considering the infinitesimal geometric motion of the curvilinear coordinates. Haoxiang Luo and Thomas R. Bewley [5] presented the contravariant form of the Navier-Stokes equations in time-dependent curvilinear coordinate systems. They derived the temporal derivative of tensor vectors by using an alternative approach and the quotient rule of tensor analysis. Surattana Sungnul [6] presented the Navier-Stokes equation in cylindrical bipolar coordinate system because of this coordinate can be transform infinite domain into finite domain in the problem of incompressible fluid flow over two rotating circular cylinders., etc. In this work, we introduced main definitions of tensor-vector calculus related with curvilinear coordinate system. We then present detailed transformation of the Navier-Stokes equation into orthogonal curvilinear coordinate system. The complete form of the Navier-Stokes equations with respect covariant, contravariant and physical components of velocity vector are presented. The program in Maple software for transformation the Navier-Stokes equations in curvilinear coordinate systems are obtained.

2. Definition of Operations in the Curvilinear Coordinate Systems

Let $\Omega \subset \Re^3$ be an open set. A one-to-one and reciprocal continuously differentiable mapping from Ω to subset of $\Re^3$ is called a coordinate system [1]–[3]. This mapping is defined by the formula

$$\vec{x} \longrightarrow (X^1(\vec{x}), X^2(\vec{x}), X^3(\vec{x})) \tag{1}$$

The value of functions $X^i(\vec{x})$ are called coordinates (curvilinear coordinates) of the point $\vec{x}$.

2.1. Basis and Cobasis

Let a point $\vec{x} \in \Omega$ be fixed. At this point there are vectors

$$\begin{aligned} \vec{e}_i &= \frac{\partial \vec{x}}{\partial X^i}, \qquad i = 1, 2, 3 \\ \vec{e}^i &= \frac{\partial X^i}{\partial \vec{x}} = \nabla X^i, \qquad i = 1, 2, 3 \end{aligned} \tag{2}$$

which form a basis, $\vec{e}_i$, and cobasis, $\vec{e}^{\,i}$, in $\Re^3$. These basis are called a coordinate basis and cobasis of the coordinate system X at the point $\vec{x}$. A vector $\vec{v}$ can be decomposed along either basis or cobasis vectors $\vec{v} = v_i \vec{e}^{\,i} = v^i \vec{e}_i$. The components v_i and v^i are called covariant and contravariant components of vector $\vec{v}$ respectively. A coordinate system is called orthogonal if its basis is orthogonal,

$$\vec{e}_i \cdot \vec{e}_j = 0, \qquad \vec{e}^{\,i} \cdot \vec{e}^{\,j} = 0, \qquad (i \neq j). \tag{3}$$

2.2. Fundamental Tensor

The fundamental tensor and its inverse are defined by

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j, \qquad g^{ij} = \vec{e}^{\,i} \cdot \vec{e}^{\,j}. \tag{4}$$

Coordinates of the fundamental tensor g with respect to an orthogonal coordinate system are

$$(g_{ij}) = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}, \qquad (g^{ij}) = \begin{bmatrix} g^{11} & 0 & 0 \\ 0 & g^{22} & 0 \\ 0 & 0 & g^{33} \end{bmatrix}$$

where

$$g_{ij} = |\vec{e}_i|^2 = \left| \frac{\partial \vec{x}}{\partial X^i} \right|^2, \quad g^{ij} = |\vec{e}^{\,i}|^2 = \left| \nabla X^i \right|^2, \quad i = 1, 2, 3$$

and $|g| = \left[\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)\right]^2$.

Derivatives of the basis and cobasis vectors with respect to curvilinear coordinates can be represented in terms of the basis $\{\vec{e}_i\}$ and cobasis $\{\vec{e}^{\,i}\}$ respectively,

$$\frac{\partial \vec{e}_i}{\partial X^j} = \Gamma^s_{ij} \vec{e}_s, \quad \frac{\partial \vec{e}^{\,i}}{\partial X^j} = -\Gamma^i_{js} \vec{e}^{\,s}, \tag{5}$$

where the coefficients Γ^s_{ij} are called Christoffel's symbols of second order. The Christoffel symbols are related to the derivatives of the fundamental tensor

$$\Gamma^l_{ij} = \frac{1}{2} \left(\frac{\partial g_{is}}{\partial X^j} + \frac{\partial g_{js}}{\partial X^i} - \frac{\partial g_{ij}}{\partial X^s} \right) g^{ls}. \tag{6}$$

Covariant derivatives are expressed in terms of partial derivatives with respect to corresponding coordinates, Christoffel's symbols and components of a tensor. The simplest are covariant derivatives of scalar field, F, which coincide with usual partial derivative

$$F_{,i} = \frac{\partial F}{\partial X^i}.$$

The covariant derivatives of the covariant and contravariant components of a second order tensor Φ are

$$\Phi_{ij,l} = \frac{\partial \Phi_{ij}}{\partial X^l} - \Gamma^s_{li} \Phi_{sj} - \Gamma^s_{lj} \Phi_{is}, \tag{7}$$

$$\Phi^{ij}_{,l} = \frac{\partial \Phi^{ij}}{\partial X^l} + \Gamma^i_{ls}\Phi^{sj} + \Gamma^j_{ls}\Phi^{is}. \tag{8}$$

Similarly, the covariant derivatives of the mixed components are

$$\Phi^{\cdot j}_{i\cdot,l} = \frac{\partial \Phi^{\cdot j}_{i\cdot}}{\partial X^l} - \Gamma^s_{li}\Phi^{\cdot j}_{s\cdot} + \Gamma^j_{ls}\Phi^{\cdot s}_{i\cdot}, \tag{9}$$

$$\Phi^{j\cdot}_{\cdot i,l} = \frac{\partial \Phi^{j\cdot}_{\cdot i}}{\partial X^l} - \Gamma^s_{li}\Phi^{j\cdot}_{\cdot s} + \Gamma^j_{ls}\Phi^{s\cdot}_{\cdot i}. \tag{10}$$

In the above equations and everywhere below, a comma with an index in a subscript denotes covariant differentiation. A derivative of the vector field, $\vec{v}$, is the second order tensor which is denoted by the symbol, $\left(\frac{\partial \vec{v}}{\partial \vec{x}}\right)$. Covariant and mixed coordinates of the $\left(\frac{\partial \vec{v}}{\partial \vec{x}}\right)$ are

$$\left(\frac{\partial \vec{v}}{\partial \vec{x}}\right)_{ij} = \frac{\partial v_i}{\partial X^j} - \Gamma^s_{ij}v_s = v_{i,j}$$

$$\left(\frac{\partial \vec{v}}{\partial \vec{x}}\right)^i_{,j} = \frac{\partial v^i}{\partial X^j} - \Gamma^i_{js}v_s = v^i_{,j}.$$

2.3. Divergence of a vector field

The divergence of a vector field, $\vec{v}$, is a scalar. Divergence can be expressed in terms of the covariant derivatives of the contravariant components of vector field, $\vec{v}$,

$$\begin{aligned}\mathbf{div}\vec{v} = v^i_{,i} &= \frac{\partial v^i}{\partial X^i} + \Gamma^i_{is}v^s \\ &= \frac{\partial v^i}{\partial X^i} + \frac{v^i}{\sqrt{|g|}}\frac{\partial\sqrt{|g|}}{\partial X^i} \\ &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial X^i}\left(\sqrt{|g|}v^i\right).\end{aligned} \tag{11}$$

2.4. Gradient of the scalar function

The gradient of the scalar function F is defined by

$$\nabla F = \frac{\partial F}{\partial \vec{x}} = \frac{\partial F}{\partial X^i}\vec{e}^i. \tag{12}$$

Covariant components of the gradient vector are

$$(\nabla F)_i = F_{,i} = \frac{\partial F}{\partial X^i},$$

and contravariant components of the gradient vector are

$$(\nabla F)^i = g^{is}(\nabla F)_s.$$

2.5. Laplace operator of the scalar function

The Laplace operator of the scalar function F is defined by

$$\begin{aligned}\Delta F = \mathbf{div}(\nabla F) &= \left((\nabla F)^i\right)_{,i} \\ &= \left(g^{is}(\nabla F)_s\right)_{,i} \\ &= g^{is}\left(\frac{\partial^2 F}{\partial X^s \partial X^i} - \Gamma^{\alpha}_{is}\frac{\partial F}{\partial X^\alpha}\right).\end{aligned}$$

We can rewrite this formula in the following form

$$\Delta F = \mathbf{div}(\nabla F) = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial X^i}\left(\sqrt{|g|}g^{is}\frac{\partial F}{\partial X^s}\right). \tag{13}$$

2.6. Curl of a vector field

A curl of a vector field, $\vec{v}$, is the vector field. In a right-handed basis $\varepsilon_{123} = \vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3) = \sqrt{|g|}$, the contravariant components of $\mathbf{curl}\vec{v}$ are

$$\begin{aligned}(\mathbf{curl}\vec{v})^1 &= \frac{1}{\sqrt{|g|}}\left(\frac{\partial v_3}{\partial X^2} - \frac{\partial v_2}{\partial X^3}\right), \\ (\mathbf{curl}\vec{v})^2 &= \frac{1}{\sqrt{|g|}}\left(\frac{\partial v_1}{\partial X^3} - \frac{\partial v_3}{\partial X^1}\right), \\ (\mathbf{curl}\vec{v})^3 &= \frac{1}{\sqrt{|g|}}\left(\frac{\partial v_2}{\partial X^1} - \frac{\partial v_1}{\partial X^2}\right),\end{aligned} \tag{14}$$

2.7. Divergence of a tensor field

Divergence of a tensor field is a vector. The s-th contravariant component of this vector is

$$(\mathbf{div}P)^s = P^{sj}_{,j} = \mathbf{div}(\bar{P}^s) + \Gamma^s_{j\alpha}P^{j\alpha}, \tag{15}$$

where $\bar{P}^s = \left(P^{s1}, P^{s2}, P^{s3}\right)$ is s-th row of a matrix which represents second order tensor P and

$$\mathbf{div}(\bar{P}^s) = \frac{\partial P^{sj}}{\partial X^j} + \Gamma^j_{j\alpha}P^{s\alpha}.$$

2.8. Laplace operator of vector field

The Laplace operator of a vector field is a vector field and the contravariant components of this vector are

$$(\Delta\vec{v})^l = (\Delta v^l) + 2g^{ij}\Gamma^l_{is}\frac{\partial v^s}{\partial X^j} + g^{ij}\left(\frac{\partial \Gamma^l_{is}}{\partial X^j} - \Gamma^\alpha_{ji}\Gamma^l_{\alpha s} + \Gamma^l_{j\alpha}\Gamma^\alpha_{is}\right)v^s, \tag{16}$$

where

$$(\Delta v^l) = g^{ij}\left[\frac{\partial^2 v^l}{\partial X^j \partial X^i} - \Gamma^s_{ji}\frac{\partial v^l}{\partial X^s}\right]$$

is the Laplace operator of scalar function v^l.

2.9. Covariant and contravariant components of the acceleration of a vector field

The covariant and contravariant components of the acceleration of a vector field, $\vec{v}$, are

$$\left(\frac{d\vec{v}}{dt}\right)_i = \frac{\partial v_i}{\partial t} + v^s \frac{\partial v_i}{\partial X^s} - \Gamma^j_{is} v^s v_j, \tag{17}$$

$$\left(\frac{d\vec{v}}{dt}\right)^i = \frac{\partial v^i}{\partial t} + v^s \frac{\partial v^i}{\partial X^s} - \Gamma^i_{js} v^j v^s. \tag{18}$$

If vectors of coordinate bases and cobases are not normed, then components of tensors have different numerical values in different baseds even if directions of basis vectors coincides. numerical values of tensor components divided by the length of corresponding basis or cobasis vectors, which define these components are called physical components of the tensor. For example, if $u_i = \vec{u}\vec{e}_i$ are covariant coordinates of a vector $\vec{u}$, then the physical components are

$$\tilde{u}_i = \frac{u_i}{|\vec{e}_i|},$$

the covariant components of a tensor, T_{ij}, related with the physical components as

$$\tilde{T}_{ij} = \frac{T_{ij}}{|\vec{e}_i||\vec{e}_j|}.$$

3. The Navier-Stokes Equations

Fluid obey the general laws of continuum mechanics: conservation of mass, linear momentum and energy. In the Eulerian representation of the flow, we represent the density $\rho(\vec{x}, t)$ as a function of the position $\vec{x}$ and time t. The conservation of mass is expressed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \mathbf{div}(\rho\vec{v}) = 0. \tag{19}$$

If we assume that the fluid is incompressible and homogeneous, then the density is constant in space and time: $\rho(\vec{x}, t) = \rho_0$. Then the continuity (19) becomes

$$\mathbf{div}\vec{v} = 0. \tag{20}$$

The momentum equation, which is base on Newton's second law, represents the balance between various forces acting on a fluid element. The forces are

1) The force due to rate of change of momentum, generally referred to as the inertia force
2) Body forces, f : such as buoyancy force, magnetic force, and electrostatic force
3) Pressure force
4) Viscous forces (causing shear stress).

For a fluid element under equilibrium,

$$\text{Inertia force} + \text{body force} + \text{pressure force} + \text{viscous force} = 0.$$

or we can write

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) + f = -\nabla p + \mu\Delta\vec{v}, \tag{21}$$

where μ is called the coefficient of dynamic viscosity. Equations (20) and (21) are governing equation the motion of a viscous incompressible fluid, they are called "the Navier-Stokes equations for viscous incompressible flow". In case of the motion of viscous incompressible fluid flow which has no body force can be described by the Navier-Stokes equations in vector form (22)

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} &= -\frac{1}{\rho}\nabla p + \nu\Delta\vec{v}, \\ \mathbf{div}\vec{v} &= 0, \end{aligned} \tag{22}$$

where $\vec{v}$ is the velocity vector of fluid, p is the pressure of fluid, ρ is the density of fluid and $\nu = \dfrac{\mu}{\rho}$ is the kinematics viscosity.

4. Main Results

We remind main definitions of vector-tensor operation in the curvilinear coordinate system and the Navier-Stokes equations for incompressible fluid flow in previous section. Equation (22) represents invariant form of the Navier-Stokes equations in a curvilinear coordinate system. In index form equation (22) have the following contravariant form (23)

$$\begin{aligned} \frac{\partial v^i}{\partial t} + v^s\frac{\partial v^i}{\partial X^s} + \Gamma^i_{js}v^j v^s &= -\frac{1}{\rho}g^{is}(\nabla p)_s + \nu(\Delta\vec{v})^i, \\ v^i_{,i} &= 0, \end{aligned} \tag{23}$$

and the covariant form 24

$$\begin{aligned} \frac{\partial v_i}{\partial t} + v^s\frac{\partial v_i}{\partial X^s} - \Gamma^j_{is}v^s v_j &= -\frac{1}{\rho}(\nabla p)_i + \nu(\Delta\vec{v})_i, \\ v_{i,i} &= 0. \end{aligned} \tag{24}$$

We can write out the Navier-Stokes equations with respect to the contravariant and covariant components by using definitions and formulae from previous section. We write the code in Maple software to perform calculation for the transformation of the Navier-Stokes equations into the curvilinear coordinate systemse with respect to physical component.

Let $\vec{e}_i$ be a fixed right-handed orthonormal Cartesian basis in $\Re^3$. A vector $\vec{x} \in \Re^3$ has components $\vec{x}(x, y, z)$; $(x = \vec{x} \cdot \vec{e}_1, y = \vec{x} \cdot \vec{e}_2, z = \vec{x} \cdot \vec{e}_3)$ in this basis. Curvilinear coordinates are transformation from $\Re^3$ to subset of $\Re^3$

$$X(\vec{x}) = \left(X^1(x, y, z), X^2(x, y, z), X^3(x, y, z)\right). \tag{25}$$

4.1. The Navier-Stokes equations in cylindrical coordinate system

In the cylindrical coordinate, we use notation

$$r = X^1(x, y, z), \quad \theta = X^2(x, y, z), \quad z = X^3(x, y, z). \tag{26}$$

The cylindrical coordinate system can be defined by inverse transformation X^{-1} :

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \tag{27}$$

where $r > 0, 0 \le \theta \le 2\pi, z \in (-\infty, \infty)$.

The Navier-Stokes equations in cylindrical coordinate system with respect to physical component are following

$$\begin{aligned}&\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\\ &\quad = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right\},\end{aligned} \tag{28}$$

$$\begin{aligned}&\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\\ &\quad = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right\},\end{aligned} \tag{29}$$

$$\begin{aligned}&\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\\ &\quad = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right\},\end{aligned} \tag{30}$$

$$\frac{1}{r}\left(\frac{\partial(rv_r)}{\partial r} + \frac{\partial(v_\theta)}{\partial \theta}\right) + \frac{\partial v_z}{\partial z} = 0. \tag{31}$$

Equations (28)–(30) represent the conservation of linear momentum equations in r, θ and$z$ directions, respectively. Equation (31) represents the continuity equation.

4.2. The Navier-Stokes equations in spherical coordinate system

In the spherical coordinate, we use notation

$$r = X^1(x, y, z), \quad \phi = X^2(x, y, z), \quad \theta = X^3(x, y, z). \tag{32}$$

The spherical coordinate system can be defined by inverse transformation X^{-1} :

$$x = r \sin\phi \cos\theta, \quad y = r \sin\phi \sin\theta, \quad z = r \cos\phi, \tag{33}$$

where $r > 0, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$.

The Navier-Stokes equations in spherical coordinate system with respect to physical component are following

$$\begin{aligned}
&\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_r}{\partial \phi} + \frac{v_\theta}{r \sin\phi} \frac{\partial v_r}{\partial \theta} - \frac{1}{r}\left(v_\phi^2 + v_\theta^2\right) \\
&\quad = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\nu}{r^2}\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_r}{\partial r}\right) + \frac{\partial^2 v_r}{\partial \phi^2} + \cot\phi \frac{\partial v_r}{\partial \phi} + \\
&\qquad + \frac{1}{\sin^2\phi} \frac{\partial^2 v_r}{\partial \theta^2} - 2\left(\frac{\partial v_\phi}{\partial \phi} + \frac{1}{\sin\phi} \frac{\partial v_\theta}{\partial \theta} + v_r + \cot\phi\, v_\theta\right)\},
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\theta}{r \sin\phi} \frac{\partial v_\phi}{\partial \theta} + \frac{1}{r}\left(v_\phi v_r - \cot\phi\, v_\theta^2\right) = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi} + \\
&\quad + \frac{\nu}{r^2}\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right) + \frac{1}{r \sin\phi} \frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial v_\phi}{\partial \phi}\right) + \frac{1}{r \sin^2\phi} \frac{\partial^2 v_\phi}{\partial \theta^2} + \\
&\qquad +2\left(\frac{\partial v_\phi}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \phi} - \frac{\cos\phi}{r \sin^2\phi} \frac{\partial v_\theta}{\partial \theta} - \frac{1}{2r \sin^2\phi} v_\phi\right)\},
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\theta}{\partial \phi} + \frac{v_\theta}{r \sin\phi} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r}\left(v_\theta v_r + \cot\phi\, v_\theta v_\phi\right) = -\frac{1}{\rho r \sin\phi} \frac{\partial p}{\partial \theta} + \\
&\quad + \frac{\nu}{r^2}\{r \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right) + \frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial}{\partial \phi}\left(\frac{v_\theta}{\sin\phi}\right)\right) + \frac{1}{\sin^2\phi} \frac{\partial^2 v_\theta}{\partial \theta^2} + \\
&\quad +2\left(r^2 \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \cos\phi \frac{\partial}{\partial \phi}\left(\frac{v_\theta}{\sin\phi}\right) + \frac{1}{\sin\phi} \frac{\partial v_r}{\partial \theta} + \csc\phi \cot\phi \frac{\partial v_\phi}{\partial \theta}\right)\},
\end{aligned} \tag{36}$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{\sin\phi} \frac{\partial}{\partial \phi}(\sin\phi\, v_\phi) + \frac{1}{\sin\phi} \frac{\partial v_\theta}{\partial \theta} = 0. \tag{37}$$

Equations (34)–(36) represent the conservation of linear momentum equations in r, ϕ and θ directions, respectively. Equation (37) represents the continuity equation.

4.3. The Navier-Stokes equations in cylindrical bipolar coordinate system

In the cylindrical bipolar coordinate, we use notation

$$\xi = X^1(x, y, z), \quad \eta = X^2(x, y, z), \quad z = X^3(x, y, z). \tag{38}$$

The cylindrical bipolar coordinate system can be defined by inverse transformation X^{-1} :

$$x = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}, \quad y = \frac{a \sin \xi}{\cosh \eta - \cos \xi}, \quad z = z, \tag{39}$$

where $0 \le \xi \le 2\pi$, $\eta \in (-\infty, \infty)$, $z \in (-\infty, \infty)$ and a is characteristic length in the cylindrical bipolar coordinate system which is positive.

The Navier-Stokes equations in cylindrical bipolar system with respect to physical component are following

$$\frac{\partial v_\xi}{\partial t} + \frac{1}{h}\left(v_\xi \frac{\partial v_\xi}{\partial \xi} + v_\eta \frac{\partial v_\xi}{\partial \eta}\right) + v_z \frac{\partial v_\xi}{\partial z} - \frac{1}{a}\left(\sinh \eta (v_\xi v_\eta) - \sin \xi (v_\eta)^2\right) = -\frac{1}{h}\frac{1}{\rho}\frac{\partial p}{\partial \xi}$$
$$+\frac{\nu}{h}\left\{\frac{1}{h}\left(\frac{\partial^2 v_\xi}{\partial \xi^2} + \frac{\partial^2 v_\xi}{\partial \eta^2}\right) - \frac{2}{a}\left(\sinh \eta \frac{\partial v_\eta}{\partial \xi} - \sin \xi \frac{\partial v_\eta}{\partial \eta}\right) - \left(\frac{\cosh \eta + \cos \xi}{a}\right) v_\xi\right\}, \tag{40}$$

$$\frac{\partial v_\eta}{\partial t} + \frac{1}{h}\left(v_\xi \frac{\partial v_\eta}{\partial \xi} + v_\eta \frac{\partial v_\eta}{\partial \eta}\right) + v_z \frac{\partial v_\eta}{\partial z} + \frac{1}{a}\left(\sinh \eta (v_\xi)^2 - \sin \xi (v_\xi v_\eta)\right) = -\frac{1}{h}\frac{1}{\rho}\frac{\partial p}{\partial \eta}$$
$$+\frac{\nu}{h}\left\{\frac{1}{h}\left(\frac{\partial^2 v_\eta}{\partial \xi^2} + \frac{\partial^2 v_\eta}{\partial \eta^2}\right) + \frac{2}{a}\left(\sinh \eta \frac{\partial v_\xi}{\partial \xi} - \sin \xi \frac{\partial v_\xi}{\partial \eta}\right) - \left(\frac{\cosh \eta + \cos \xi}{a}\right) v_\eta\right\}, \tag{41}$$

$$\begin{aligned}\frac{\partial v_z}{\partial t} + \frac{1}{h}\left(v_\xi \frac{\partial v_z}{\partial \xi} + v_\eta \frac{\partial v_z}{\partial \eta}\right) + v_z \frac{\partial v_z}{\partial z} \\ = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left\{\frac{1}{h^2}\left(\frac{\partial^2 v_z}{\partial \xi^2} + \frac{\partial^2 v_z}{\partial \eta^2}\right) + \frac{\partial^2 v_\xi}{\partial z^2}\right\},\end{aligned} \tag{42}$$

$$\frac{1}{h^2}\left\{\frac{\partial (h v_\xi)}{\partial \xi} + \frac{\partial (h v_\eta)}{\partial \eta}\right\} + \frac{\partial v_z}{\partial z} = 0, \tag{43}$$

where $h = \dfrac{a}{\cosh \eta - \cos \xi}$. Equations (40)-(42) represent the conservation of linear momentum equation in ξ, η and z directions, respectively. Equation (43) represents the continuity equation.

Similarly, we can write out the Navier-Stokes Equations in arbitrary curvilinear coordinates systems.

5. Conclusion

We have presented the complete form of the Navier-Stokes equations with respect covariant, contravariant and physical components of velocity vector. In addition, we obtained the code Maple program for transformation of the Navier-Stokes equations in curvilinear coordinate systems. In generalized, it is not difficult to transform the Navier-Stokes Equations to another curvilinear coordinate system.

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