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The form of solution of Dirichlet problem for the heat equation by using Elzaki transform

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Abstract

Every sufficient smooth solution of the wave equation satisfies the energy equation, and the energy equation is used for showing the uniqueness of the solution of Dirichlet problem for the heat equation. In this article, we have explored the form of solution of the energy equation and Dirichlet problem for the heat equation by using Elzaki transform.

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1. Introduction

The Dirichlet problem for the heat equation is related to energy equation, which is used for showing the uniqueness of the solution, and the form is as $u_t - ku_{xx} = f(x,t)$ with the conditions of $u(0,t) = g(t),\ u(l,t) = h(t)$ for $0 \le x \le l$. On the other hand, the energy equation $E(t)$ has the form of

$$E(t) = K(t) + V(t),$$

for $V(t)$ and $K(t)$ are the potential and the kinetic energy, respectively. Since $K(t) = \int_0^L \frac{1}{2}\rho_0 u_t^2\,dx$ and $V(t) = \int_0^L \frac{1}{2}T_0 u_x^2\,dx$, we can rewrite $E(t)$ as

$$E(t) = \int_0^L \frac{1}{2}[\rho_0 u_t^2 + T_0 u_x^2]dx.$$

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If we differentiate this equation with respect to t, and apply Green's formula, we have

$$\frac{d}{dt}\int_0^L \frac{1}{2}[\rho_0 u_t^2 + T_0 u_x^2]dx = \int_0^L \rho_0 q u_t dx + [T_0 u_x u_t]_0^L.$$

This is called the energy equation for the wave equation, which has the form of

$$\rho_0 u_{tt} = T_0 u_{xx} + \rho_0 q(x,t). \tag{1}$$

It is well-known fact that the small plane transverse vibrations of a perfectly flexible string are expressed by

$$u_{tt} = c^2 u_{xx},\ 0 < x < L,\ t > 0 \tag{2}$$

$$u(0,t) = 0,\ u(L,t) = 0$$

$$u(x,0) = f(x),\ u_t(x,0) = g(x)$$

where the constant c is given by $c = \sqrt{T_0/\rho_0}$, ρ_0 is the linear density of the string, and the T_0 is initial magnitude of a force [6]. If distributed external forces act on the string, the equation (2) changes to (1) because of the total external force [7].

On the other hand, the integral transforms method gives a somewhat reasonable tool for solving the nonhomogeneous ODEs and corresponding initial value problems. Although the existing integral transforms have many similarity, this kind of researches give a consideration to establish the exquisite theory with respect to general integral transform. Hence, we would like to use Elzaki transform [3,5,8] as a tool in this article. The transform is defined by

$$T(v) = v\int_0^\infty e^{-t/v} f(t)dt,$$

for $E[f(t)] = T(v)$, and the exact definition is as follows [3]; The Elzaki transform of the functions belonging to a class A, where $A = \{f(t) | \exists M, k_1, k_2 > 0$ such that $|f(t)| < Me^{|t|/k_j}$, if $t \in (-1)^j \times [0,\infty)\}$, and where $f(t)$ is denoted by $E[f(t)] = T(v)$ and defined as

$$T(v) = v^2\int_0^\infty f(vt)e^{-t}dt, \quad k_1, k_2 > 0,$$

or equivalently,

$$T(v) = v\int_0^\infty f(t)e^{-t/v}dt, \quad v \in (k_1, k_2).$$

The following results can be obtained from the definition and simple calculations.

a) $E[f'(t)] = T(v)/v - vf(0)$

b) $E[f''(t)] = T(v)/v^2 - f(0) - vf'(0)$

c) $E[tf'(t)] = v^2\frac{d}{dv}[T(v)/v - vf(0)] - v[T(v)/v - vf(0)]$

$d)$ $E[t^2 f'(t)] = v^4 \frac{d^2}{dv^2}[T(v)/v - vf(0)]$

$e)$ $E[tf''(t)] = v^2 \frac{d}{dv}[T(v)/v^2 - f(0) - vf'(0)] - v[T(v)/v^2 - f(0) - vf'(0)]$

$f)$ $E[t^2 f''(t)] = v^4 \frac{d^2}{dv^2}[T(v)/v^2 - f(0) - vf'(0)]$.

$g)$ $E\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{v}T(x,v) - vf(x,0)$

$h)$ $E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\partial f(x,0)}{\partial t}$

$i)$ $E\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d}{dx}[T(x,v)]$

$j)$ $E\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}[T(x,v)]$.

for $E(f(t)) = T(v)$ [5,8]. We can easily extend this result to the n-th partial derivative by using the induction.

Several researches have been pursued for energy equation [1, 4, 7, 9], and in this article, we have checked the form of solution of the energy equation and Dirichlet problem for the heat equation by using Elzaki transform.

2. The form of solution of Dirichlet problem for the heat equation by using Elzaki transform

In the following theorem, we would like to check the representation of a solution of energy equation by Elzaki transform.

Theorem 2.1. The solution $u(x,t)$ of the energy equation

$$\frac{d}{dt}\int_0^L \frac{1}{2}[\rho_0 u_t^2 + T_0 u_x^2]dx = [T_0 u_x u_t]_0^L + \int_0^L \rho_0 q u_t dx$$

can be represented by $u(x,t) = E^{-1}[U(x,v)]$ where,

$$U(x,v) = A(v)e^{\frac{1}{cv}x} - A(v)e^{-\frac{1}{cv}x} - e^{\frac{1}{cv}x}\frac{1}{2cv}\int e^{-\frac{1}{cv}x}\rho_0(f(x) + vg(x) + Q)\,dx$$
$$-e^{-\frac{1}{cv}x}\frac{1}{2cv}\int e^{\frac{1}{cv}x}\rho_0(f(x) + vg(x) + Q)\,dx$$

for $Q(x,v) = E[q(x,t)]$ and for a constant A.

Proof. The energy equation can be written as

$$\int_0^L \rho_0 u_t u_{tt}\,dx = \int_0^L T_0 u_{xx} u_t\,dx + \int_0^L \rho_0 q u_t\,dx$$

because of

$$u_x u_{xt} = \frac{\partial}{\partial t}(\frac{1}{2}T_0 u_x^2)$$

and

$$\int_0^L T_0 u_{xx} u_t \, dx = [T_0 u_x u_t]_0^L - \int_0^L T_0 u_x u_{tx} \, dx \text{ [8]}.$$

Dividing by u_t, we have

$$\int_0^L \rho_0 u_{tt} \, dx = \int_0^L T_0 u_{xx} \, dx + \int_0^L \rho_0 q \, dx.$$

Differentiating with respect to x, we obtain

$$\rho_0 u_{tt} = T_0 u_{xx} + \rho_0 q(x,t).$$

Let us write $U(x,v) = E[u(x,t)]$ and take Elzaki transform on both sides. Then we have

$$\rho_0[\frac{1}{v^2}U - f(x) - vg(x)] = T_0\frac{\partial^2 U}{\partial x^2} + \rho_0 Q(x,v)$$

for $Q[(x,v) = E[q(x,t)]$. Organizing the equality, we have

$$\frac{\rho_0}{v^2}U - T_0\frac{\partial^2 U}{\partial x^2} = \rho_0 f(x) + \rho_0 v g(x) + \rho_0 Q.$$

Since $c^2 = T_0/\rho_0$, a general solution $U_h(x,v)$ of the above ODE is

$$U_h(x,v) = A(v)e^{\frac{1}{cv}x} + B(v)e^{-\frac{1}{cv}x}$$

for constants A and B. Next let us find a particular solution U_p. By the simple calculation, the Wronskian of $e^{x/cv}$ and $e^{-x/cv}$ is expressed by $W = -\dfrac{1}{2cv}$ and so, we have

$$U_p(x,v) = -e^{\frac{1}{cv}x}\frac{1}{2cv}\int e^{-\frac{1}{cv}x}\rho_0(f(x) + vg(x) + Q)\,dx$$

$$-e^{-\frac{1}{cv}x}\frac{1}{2cv}\int e^{\frac{1}{cv}x}\rho_0(f(x) + vg(x) + Q)\,dx$$

for $Q[(x,v) = E[q(x,t)]$.

From $u(0,t) = 0$, we have $U(0,v) = E[u(0,t)] = E(0) = 0$. Hence

$$U(0,v) = A(v) + B(v) + \frac{1}{2cv}\int \rho_0(f(0) + vg(0) + Q(0,v))\,dx$$

$$-\frac{1}{2cv}\int \rho_0(f(0) + vg(0) + Q(0,v))\,dx = 0$$

for constants A and B. This implies $B(v) = -A(v)$. Thus we have

$$E[u(x,t)] = A(v)e^{\frac{1}{cv}x} - A(v)e^{-\frac{1}{cv}x} - e^{\frac{1}{cv}x}\frac{1}{2cv}\int e^{-\frac{1}{cv}x}\rho_0(f(x)+vg(x)+Q)\,dx$$

$$-e^{-\frac{1}{cv}x}\frac{1}{2cv}\int e^{\frac{1}{cv}x}\rho_0(f(x)+vg(x)+Q)\,dx$$

for $Q(x,v) = E[q(x,t)]$ and for a constant A.

Next, let us consider Dirichlet problem for the heat equation. This problem is related to energy integral, which is used for showing the uniqueness of the solution. The form is as the following;

$$u_t - ku_{xx} = f(x,t) \tag{3}$$

$$u(x,0) = \phi(x)$$

$$u(0,t) = g(t),\ u(l,t) = h(t)$$

for $0 \le x \le l$. ■

Theorem 2.2. The equation (3) has a gerneral solution of the form of $u(x,t) = E^{-1}[U(x,v)]$ where

$$\begin{aligned} U(x,v) = A(v))e^{\frac{1}{\sqrt{kv}}x} &+ B(v)e^{-\frac{1}{\sqrt{kv}}x} \\ &+ \frac{\sqrt{kv}}{2}e^{\frac{1}{\sqrt{kv}}x}\int e^{-\frac{1}{\sqrt{kv}}x}(-v\phi(x) - F(x,v))\,dx \\ &- \frac{\sqrt{kv}}{2}e^{-\frac{1}{\sqrt{kv}}x}\int e^{\frac{1}{\sqrt{kv}}x}(-v\phi(x) - F(x,v))\,dx \end{aligned}$$

for $U = T(v) = E[u(x,t)]$ and for $F(x,v) = E[f(x,t)]$.

Proof. Since $E[f'(t)] = T(v)/v - vf(0)$, taking Elzaki transform on both sides, we have

$$k\frac{\partial^2 U}{\partial x^2} - \frac{1}{v}U = -v\phi(x) - F(x,v).$$

First, let us find the homogeneous equation

$$k\frac{\partial^2 U}{\partial x^2} - \frac{1}{v}U = 0.$$

A general solution is

$$U_h(x,v) = A(v))e^{\frac{1}{\sqrt{kv}}x} + B(v)e^{-\frac{1}{\sqrt{kv}}x}$$

for A and B are arbitrary numbers.

Next, let us consider Lagrange's method to find a particular solution $U_p(x, v)$. This is called the method of variation of parameters. The Wronskian of bases of the corresponding homogeneous ODE is $W = -\frac{2}{\sqrt{kv}}$. Hence a particular solution U_p can be represented by

$$U_p(x, v) = -e^{\frac{1}{\sqrt{kv}}x} \int e^{-\frac{1}{\sqrt{kv}}x} \frac{(-v\phi(x) - F(x, v))}{-\frac{2}{\sqrt{kv}}}\, dx + e^{-\frac{1}{\sqrt{kv}}x} \int e^{\frac{1}{\sqrt{kv}}x} \frac{(-v\phi(x) - F(x, v))}{-\frac{2}{\sqrt{kv}}}\, dx$$

for $F(x, v) = E[f(x, t)]$. Organizing this equality, we have a general solution $U(x, v)$ as

$$U(x, v) = A(v))e^{\frac{1}{\sqrt{kv}}x} + B(v)e^{-\frac{1}{\sqrt{kv}}x} + \frac{\sqrt{kv}}{2} e^{\frac{1}{\sqrt{kv}}x} \int e^{-\frac{1}{\sqrt{kv}}x} (-v\phi(x) - F(x, v))\, dx - \frac{\sqrt{kv}}{2} e^{-\frac{1}{\sqrt{kv}}x} \int e^{\frac{1}{\sqrt{kv}}x} (-v\phi(x) - F(x, v))\, dx.$$

■

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