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To subsonic flow around the wing profile with shock waves in supersonic zones. Equation for stream function.

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Abstract

We derive the equation for the stream function of planar transonic vortical flows of an ideal perfect gas. Vorticity arises at high subsonic flight velocity due to shock waves in the supersonic zones on an airfoil. Vortex is known to be constant along the stream line, therefore only the stream function equation should be used for precisely calculating the energy losses. However, as it was marked by L.Bers (see epigraph), there is no equation in the whole flow domain containing both the subsonic and supersonic sub-domains. Actually we derive two different equations with a polar singularity on the sonic line. To conjugate the neighbor solutions a regularity condition should be fulfilled.

Keywords: Ideal perfect gas, **s**tream function, transonic, first integral, shock wave, vorticity, supercritical wing.

"...If we shall attempt to exclude the velocity potential, then we shall meet difficulty in the connection with that that ρ *is two-valued function of the mass stream* $\rho V = \nabla \psi$

... There is no single equation of the second order to which the stream function would satisfy." L.Bers [1]

Introduction

Transonic aerodynamics [1] says: if the Mach number M_∞ of the flight upstream exceeds a critical value $M_{cr} < 1$, which is determined only by the form of the wing profile, then the local supersonic zones arise. Such zones are known to contain, as a

rule, shock waves. Length and intensity of shock waves and aerodynamic resistance sharply increases as $M_\infty \to 1$. Particularly, economic efficiency of the long-distance civil aviation depends on both the flight speed and form of the profile $\mathcal{P}$.
Therefore to lower the expenditure of the fuel, one should increase M_{cr} and coordinate this with the increase of M_∞. Designing a wing profile of high efficiency is possible in the only case when using the equation for the stream function.

Equation for the stream function

We study stationary flows of an ideal gas obeying the thermodynamic equations $p = \rho RT$, $E = c_V T$. As usual, $p,\ \rho,\ T,\ E$ are the pressure, density, temperature, internal energy. R is the gas constant and $c_V = const$ is specific heat at constant volume.
The Euler system for the Clapeiron gas

$$\nabla\cdot(\rho\boldsymbol{V}) = 0,\ \rho(\boldsymbol{V}\cdot\nabla)\boldsymbol{V} + \nabla p = 0,\ \rho\boldsymbol{V}\cdot\nabla(c_V T + |\boldsymbol{V}|^2/2) + \nabla\cdot(p\boldsymbol{V}) = 0$$

has two first integrals asserting that entropy and full enthalpy are constant along stream lines

$$i + |\boldsymbol{V}|^2/2 = i_0(\psi) = c_p T_0(\psi),\ S = p/\rho^\gamma = S(\psi)$$

Here $\boldsymbol{V}$ is the velocity vector, $i = c_p T = E + p/\rho$ is the enthalpy, S is the entropy, $\gamma = c_p / c_V$ is the isentropic exponent $(c_p = const,\ c_V = const,\ c_p - c_V = R)$, $\psi = \psi(x, y)$ is the stream function such that

$$\rho u = \psi_y,\ \rho v = -\psi_x,\ \boldsymbol{V} = u\boldsymbol{i} + v\boldsymbol{j}$$

It follows from the first integrals and the Clapeiron law that

$$\begin{gathered} T|_{V=0} = T_0(\psi),\ \rho|_{\boldsymbol{V}=0} = \rho_0(\psi) = \left(RT_0(\psi)/S(\psi)\right)^{1/(\gamma-1)} \\ p|_{\boldsymbol{V}=0} = p_0(\psi) = (RT_0)^{\gamma/(\gamma-1)}\left(S(\psi)\right)^{-1/(\gamma-1)} \\ i_0(\psi) = c_p T_0,\quad |\boldsymbol{V}| \le |\boldsymbol{V}|_{\max} = \sqrt{2c_p T_0} = const \end{gathered} \tag{1}$$

Denote by square brackets the breaks on shock waves. In concordance with the Rankin-Hugoniot conditions $[\rho_0(\psi)],\ [p_0(\psi)]$ are proportional to the curvature of the shock line, $[T_0(\psi)] = 0$.
We restrict oneself to the most important case, when the flow upstream is uniform. Then $T_0(\psi) \equiv const = T_0$.
By $a_{cr} = \sqrt{2\gamma RT_0/(\gamma+1)} = const$, $\lambda = \lambda(x, y) = |\boldsymbol{V}(x, y)|/a_{cr}$ denote the critical sound velocity and the velocity coefficient correspondingly. Formulas (1) allow to express $T(x, y),\ p(x, y),\ \rho(x, y)$ in the form

$$\begin{gathered} T(x, y) = T_0\tau(\lambda),\ \rho(x, y) = \rho_0(\psi)\varepsilon(\lambda),\ p(x, y) = p_0(\psi)\pi(\lambda) \\ T_0 = T|_{V=0},\ \rho_0(\psi) = \rho|_{V=0},\ p_0(\psi) = p|_{V=0} \end{gathered}$$

via so called *gas dynamical functions*

$\tau(\lambda)=1-\lambda^2(\gamma-1)/(\gamma+1),\ \varepsilon(\lambda)=[\tau(\lambda)]^{1/(\gamma-1)},\ \pi(\lambda)=[\tau(\lambda)]^{\gamma/(\gamma-1)}$

Denoting

$$q(\lambda)=\frac{\rho|V|}{(\rho|V|)\,|_{\lambda=1}}=\left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)}\lambda\varepsilon(\lambda)$$

we express $|\nabla\psi|$ in the form

$$|\nabla\psi|=\sqrt{\psi_x^2+\psi_y^2}=\left(\frac{2}{\gamma+1}\right)^{1(\gamma-1)}a_{cr}\rho_0(\psi)q(\lambda) \qquad (2)$$

Let $\Lambda_{1,2}(Q)=(q(\lambda))^{-1}$ be the two-valued inverse function determined on the segment [0,1] such that $\Lambda_1\in[0,1]$, $\Lambda_2\in[1,\sqrt{(\gamma+1/(\gamma-1)}]$. In correspondence with eq.(2) the velocity coefficient λ can be expressed in the domains of subsonic and supersonic velocities via $|\nabla\psi|,\psi$ as follows

$$\lambda=\Lambda_{1,2}(Q),\ Q=Q(x,y)=\frac{|\nabla\psi|}{A\rho_0(\psi)},\ A=a_{cr}\left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \qquad (3)$$

If there is the shock wave in the flow, the vorticity is determined by distribution of the velocity argument break along the shock wave arc. Let $\boldsymbol{k}$ be a unit vector orthogonal to the flow plane. Using the formula (see the text book [2])

$$\Omega=\boldsymbol{k}\cdot rot\boldsymbol{V}=\partial u/\partial y-\partial v/\partial x=-\rho\left(i_0'(\psi)-\frac{\gamma}{\gamma-1}p^{(\gamma-1)/\gamma}\theta'(\psi)\right),\ \theta=S^{1/\gamma}$$

and taking into account that $i_0'(\psi)=c_pT_0'(\psi)\equiv 0$, we obtain

$$\Omega=-\pi(\Lambda_{1,2})p_0'(\psi)=\left(\frac{\psi_y}{\rho}\right)_y+\left(\frac{\psi_x}{\rho}\right)_x=\frac{1}{\rho}\Delta\psi-\frac{1}{\rho^2}\left(\psi_y\rho_y+\psi_x\rho_x\right)\Rightarrow$$

$$\Rightarrow-\pi(\Lambda_{1,2})\varepsilon(\Lambda_{1,2})p_0'(\psi)\rho_0(\psi)=\Delta\psi-\nabla\psi\cdot\nabla\ln p_0(\psi)-\ \nabla\psi\cdot\nabla\ln\varepsilon(\Lambda_{1,2})\Rightarrow$$

$$\Rightarrow-\pi(\Lambda_{1,2})\varepsilon(\Lambda_{1,2})\frac{p_0'(\psi)p_0(\psi)}{RT_0}=\Delta\psi-|\nabla\psi|^2\frac{p_0'(\psi)}{p_0(\psi)}-\ \nabla\psi\cdot\nabla\ln\varepsilon(\Lambda_{1,2})$$

Let us express $\nabla\psi\cdot\nabla\ln\varepsilon(\Lambda_{1,2}(Q))$ via partial derivatives of the stream function using the formulas

$$\nabla Q = \frac{1}{A\rho_0(\psi)}\left(\nabla|\nabla\psi| - \frac{\rho_0'(\psi)}{\rho_0(\psi)}|\nabla\psi|\nabla\psi\right)$$

$$\nabla \ln \varepsilon(\Lambda_{1,2}) = -\frac{2\Lambda_{1,2}\Lambda_{1,2}'}{A(\gamma+1)\tau(\Lambda_{1,2})\rho_0(\psi)}\left(\nabla|\nabla\psi| - |\nabla\psi|\nabla\psi\frac{\rho_0'(\psi)}{\rho_0(\psi)}\right)$$

$$\nabla|\nabla\psi| = \nabla(\psi_x^2+\psi_y^2)^{1/2} = \frac{1}{|\nabla\psi|}\left[\left(\psi_x\psi_{xx}+\psi_y\psi_{xy}\right)\boldsymbol{i} + \left(\psi_x\psi_{xy}+\psi_y\psi_{yy}\right)\boldsymbol{j}\right]$$

$$\nabla\psi\cdot\nabla\ln\varepsilon(\Lambda_{1,2}) = -\frac{2}{\gamma+1}\frac{\Lambda_{1,2}\Lambda_{1,2}'}{A\tau(\Lambda_{1,2})\rho_0(\psi)|\nabla\psi|}\times$$

$$\times\left[(\psi_x^2\psi_{xx}+2\psi_x\psi_y\psi_{xy}+\psi_y^2\psi_{yy}) - \frac{\rho_0'(\psi)}{\rho_0(\psi)}|\nabla\psi|^4\right]$$

Thus we obtain finally the pair of the non-linear equations of second order describing sub- and supersonic flows

$$-\rho_0(\psi)p_0'(\psi)\tau^{\frac{\gamma+1}{\gamma-1}}(\Lambda_{1,2}) = \Delta\psi - \frac{p_0'(\psi)}{p_0(\psi)}|\nabla\psi|^2 +$$
$$+\frac{2}{\gamma+1}\frac{\Lambda_{1,2}\dot{\Lambda}_{1,2}}{A\tau(\Lambda_{1,2})\rho_0(\psi)|\nabla\psi|}\left[\psi_x^2\psi_{xx}+2\psi_x\psi_y\psi_{yx}+\psi_y^2\psi_{yy} - \frac{p_0'(\psi)}{p_0(\psi)}|\nabla\psi|^4\right] \tag{4}$$

In correspondence with eq.(3) equality $|\nabla\psi| = A\rho_0(\psi)$ takes place on the sonic line $\Lambda = 1$.

Monotone functions $\Lambda_{1,2}'(Q)$ are different in sub- and supersonic domains. We have $\Lambda_{1,2}(1) = 1,\ \Lambda_{1,2}'(Q)|_{Q\to 1\pm 0} \to \mp\infty$

As $Q(\Lambda)-1 \sim (\Lambda-1)^2$ at $\Lambda\to 1$, then $\Lambda_{1,2}'|_{\Lambda\to 1} \sim (\Lambda-1)^{-1}$. Let s be the sonic line length. In order to eq.(4) be twice continuously differentiable solution in the whole domain of the mixed sub- and supersonic flow, it is necessary that there exists a continuously differentiable function $L(s)$ such that

$$\frac{\psi_x^2\psi_{xx}+2\psi_x\psi_y\psi_{yx}+\psi_y^2\psi_{yy} - \left(\ln p_0(\psi)\right)'|\nabla\psi|^4}{(\Lambda_{1,2}-1)} \underset{\Lambda_{1,2}\to 1\pm 0}{\to} L(s) \tag{5}$$

It is easy to check that equality $\psi_x^2\psi_{xx}+2\psi_x\psi_y\psi_{yx}+\psi_y^2\psi_{yy} = 0$ takes place on the sonic line in each irrotational flow.

Eq.(4) is determined in the exterior of a profile. Asymptotic behavior of the vortical flow, which is subsonic outside of a finite circle containing the profile, is established in [3].The boundary condition on the profile is $\psi = 0$. Certainly, the condition Zhukovskii-Kutta-Chaplygin is supposed to be fulfilled. Existence and uniqueness of the boundary-value problem solution remain open.

Remark 1

Computing a *weakly* supercritical flow, **o**ne can simplify the problem. Starting from the irrotational flow we will increase step by step the vortical members in eq.(4). Calculating firstly the solution to the boundary-value problem for the velocity potential [1]

$$1-\frac{\gamma-1}{2}|\nabla\varphi|^2=\varphi_x^2\varphi_{xx}+2\varphi_x\varphi_y\varphi_{xy}+\varphi_y^2\varphi_{yy}$$

one can find the approximate position of the sonic line. The shock wave can be determined approximately as well, if considering it as the position of large velocity gradients. Then the solution can be corrected with using eq.(4).

Remark 2

The equation for axially symmetrical transonic vortical flow can be deduced by similar way.

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