Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 4 (2016), pp. 3211–3224

http://www.ripublication.com/gjpam.htm

Kleinecke–Shirokov Theorem for Derivations on Banach–Jordan Pairs

H. Marhnine and C. Zarhouti

Centre de Formation des Enseignants (CRMEF),
Département de Mathématiques,
Av. My Abdelaziz, Souani, B.P. 3117 Tanger, Maroc.

Abstract

The aim of this paper consists in proving Kleinecke-Shirokov Theorem for derivations on Banach–Jordan pairs. Namely, the couples (D_+a, D_-b) *of a Banach–Jordan pair* $V = (V^+, V^-)$ are quasinilpotent for any derivation $D = (D_+, D_-)$ on V satisfying the conditions $D_+^2 a = 0$ and $D_-^2 b = 0$ generalizing by the way the famous Thomas's Theorem for derivations on Banach algebras.

AMS subject classification: 17C65, 46H40, 46H70.
Keywords: Banach–Jordan pair, Banach–Jordan algebra, derivation, primitive ideal, quasinilpotent couple.

1. Introduction

A very crucial topic in the Theory of derivations centres the question on local and global properties of these operators defined on some Banach (possibly nonassociative) algebras. One of the earliest results (1955) in the Theory of derivations is the famous Theorem of I. M. Singer and J. Wermer [15] asserting that every bounded derivation on a commutative Banach algebra has range in the radical. They conjectured later that this result remains true even if the boundedness assumption is dropped. It took more than thirty years before this classical conjecture was finally confirmed for commutative algebras by M. P. Thomas [16] . The fundamental work which started investigation into the elementary properties of derivations is independently due to D. C. Kleinecke [6] and F. V. Shirokov [14] who asserted that every commutator $[a, b]$ in a Banach algebra A satisfying $[a, [a, b]] = 0$ is quasinilpotent. This result may be stated in a general context, namely, Da is quasinilpotent whenever the second iteration of any derivation D on a Banach algebra A

annihilates the element a of A. The proof of this property seems to be easy in the case where D is a bounded derivation. However, if one drops the boundedness assumption of D, does the result hold to be true? This question remains open a long time ago before M. P. Thomas gave an affirmative answer [17] . But, whether or not this result remains true in the more general context of derivations on Banach-Jordan pairs is still an open question. In this context , the question is stated as follows: If $D = (D_+, D_-)$ is a derivation on a Banach-Jordan pair $V = (V^+, V^-)$ satisfying $D_+^2 a = 0$ and $D_-^2 b = 0$ for a couple $(a, b) \times V^+ \times V^-$, is the couple (D_+a, D_-b) quasinilpotent? In this paper, our aim consists in giving an affirmative answer to this question which generalize Thomas's result to the more general context of Banach-Jordan pairs. But, unlike Thomas's procedure based on representation theory, our approach consists in an intensive use of spectral theory in Banach-Jordan pairs recently developed in [5] . As it is pointed out in [17], the importance of this result lies in its connection with the famous Singer-Wermer conjecture in the noncommutative case which until now still defies solution.

2. Preliminaries

In this paper we shall deal with Jordan pairs and Jordan algebras over a commutative ring of scalars $\mathcal{R}$ of characteristic not two. The reader is referred to [7] for further details. However, we shall record in this section some notations and results.

Definition 2.1. A Jordan pair over a commutative ring $\mathcal{R}$ of characteristic not two is a pair of $\mathcal{R}$-modules $P = (P^+, P^-)$ endowed with a couple (Q^+, Q^-) of quadratic operators $Q^\sigma : P^\sigma \longrightarrow Hom_{\mathcal{R}}(P^{-\sigma}, P^\sigma)$ such that the following identities hold for all $(x, y) \in P^\sigma \times P^{-\sigma}$ $(\sigma = \pm)$

$$V^\sigma_{(x,y)} Q^\sigma_x = Q^\sigma_x V^{-\sigma}_{(y,x)}, \quad V^\sigma_{(Q^\sigma_x y, x)} = V^\sigma_{(x, Q^{-\sigma}_y x)},$$

where $V^\sigma_{(x,y)} z = Q^\sigma_{(x,z)} y = \{x, y, z\}_\sigma$, $Q^\sigma_{(x,z)} = Q^\sigma_{x+z} - Q^\sigma_x - Q^\sigma_z$ and $\{x, y, x\}_\sigma = 2Q^\sigma_x y$.

Every couple $(x, y) \in V^\sigma \times V^{-\sigma}$ gives rise to the so called *Bergman operator* defined by $B_{(x,y)} = Id_{V^\sigma} - V_{(x,y)} + Q_x Q_y$. Such operator is of great interest in Jordan theory.

An example of Jordan pairs over a field $\mathcal{K}$ of characteristic not two is given by taking $P^+ = M_{p,q}(R)$ and $P^- = M_{q,p}(R)$ $(p, q \in \mathbb{N}^*)$ the linear spaces of rectangular matrices with entries in an associative algebra R over the field $\mathcal{K}$. The multiplication in P is defined by:

$$Q^\sigma_u v = uvu \ \forall (u, v) \in P^\sigma \times P^{-\sigma},$$

the usual matrix product.

Definition 2.2. A Jordan pair $P = (P^+, P^-)$ is said to be normed provided the vector spaces P^+ and P^- are respectively endowed with norms $\|.\|_+$ and $\|.\|_-$ such that, for all $x, z \in V^\sigma$ and $y \in V^{-\sigma}$, we have

$$\left\| \{x, y, z\}_\sigma \right\|_\sigma \leq \|x\|_\sigma \, \|y\|_{-\sigma} \, \|z\|_\sigma .$$

If the norms $\|.\|_\sigma$ are complete then P is said to be a Banach-Jordan pair.

If there is no confusion, the triple products $\{x, y, z\}_\sigma$ and the norms $\|.\|_\sigma$ of P are merely denoted by $\{x, y, z\}$ and $\|.\|$.

A typical example of Banach-Jordan pairs is given by taking

$$P^+ = \mathcal{BL}(\mathcal{X}, \mathcal{Y}),\ P^- = \mathcal{BL}(\mathcal{Y}, \mathcal{X}),$$

the pair of linear bounded operators between real or complex Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with the multiplication $Q_u v = uvu$, the usual composition. The pair $P = (\mathcal{BL}(\mathcal{X}, \mathcal{Y}), \mathcal{BL}(\mathcal{Y}, \mathcal{X}))$ is frequently denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

A (linear) *Jordan algebra* is a vector space J endowed with a binary product $(a, b) \longmapsto ab$ satisfying the identities: $ab = ba$, and $a^2(ba) = (a^2b)a$. If a complete norm is defined on J and makes continuous its product ab, J is said to be a *Banach-Jordan algebra*. Jordan pairs are known by their intimate relationship with Jordan algebras. Indeed, any associative, alternative or Jordan algebra A gives rise to a Jordan pair (A, A) with a quadratic multiplication xyx or $U_x y$, with U denoting the usual U-operator of a Jordan algebra defined by $U_x y = 2x(xy) - x^2 y$.

Conversely, given a Jordan pair $V = \left(V^+, V^-\right)$ and an element $u \in V^{-\sigma}$, the vector space V^σ gives rise to a Jordan algebra by defining the U-operator $U_a = U_a^{(u)} = Q_a Q_u$, and the square $a^{(2,u)} = Q_a u$. This Jordan algebra, denoted by $V^{\sigma(u)}$, is called the *u-homotope* of V at u. If V is a linear Jordan pair, we just need to define the linear product in $V^{\sigma(u)}$ as follows: $a.b = \frac{1}{2} Q_{(a,b)} u = \frac{1}{2}\{a, u, b\}$. The left multiplication of the algebra $V^{\sigma(u)}$ is the linear operator $L_a^{(u)}$ such that, for all $x \in V^\sigma$, $L_a^{(u)} x = a.x = \frac{1}{2} V_{(a,u)} x$.

Let V be a normed Jordan pair, then $V^{\sigma(u)}$ is a normed Jordan algebra for the norm $|x| = \|x\|_\sigma \|u\|_{-\sigma}$. Moreover, $V^{\sigma(u)}$ is a Banach-Jordan algebra provided the norms of V are complete.

Primitive ideals and Jacobson radical. A Jordan pair $V = (V^+, V^-)$ is said to be *primitive* at $b \in V^{-\sigma}$ if there exists a proper inner ideal K of V^σ: $Q_K V^{-\sigma} \subset K$, such that:

i) K is a c-modular inner ideal of the homotope $V^{\sigma(b)}$ for some $c \in V^\sigma$,

ii) K complements the (σ)-parts of nonzero ideals: $I^\sigma + K = V^\sigma$ for any nonzero ideal $I = (I^+, I^-)$ of V.

For more details on modular inner ideals, we refer to [12] and [13].

An ideal P of a Jordan system (algebra or pair) V is called primitive if the factor system (algebra or pair) V/P is primitive.

Anquela and Cortés proved in [1] the following results:

- V is primitive at $b \in V^{-\sigma}$ if and only if the so called *local algebra* $V_b = V^{\sigma(b)}/Ker\, Q_b$ is a primitive Jordan algebra and V is strongly prime.

- If V is primitive at some $0 \neq b_0 \in V^{-\sigma}$ then so is V at every element $0 \neq b \in V^{\pm}$.

Further results on primitive Jordan pairs can be found in [1] .

The *core* of a maximal inner ideal M is the largest ideal contained in M. The core of M is a primitive ideal of V [1].

Following [7], the *Jacobson radical* of a Jordan pair V is defined as the ideal $Rad(V) = (Rad(V^+), Rad(V^-))$, where $Rad(V^\sigma)$ is the set of *properly quasi-invertible* elements of V^σ, that is, those elements which are quasi-invertible in all homotopes $V^{\sigma(u)}$, $u \in V^{-\sigma}$. The Jacobson radical of V is also the intersection of all its primitive ideals. A Jordan pair is said to be *semiprimitive* if $Rad(V) = 0$.

Capacity. A nondegenerate Jordan pair is said to have a *finite capacity* [10] if it satisfies, among other equivalent conditions, both chain conditions on principal inner ideals. For more details on finiteness conditions in Jordan pairs, we refer to [10].

If F is a subspace of a vector space E then we shall denote by $\pi_F : E \longrightarrow E/F$ the canonical map from E onto the quotient vector space E/F.

Let (a, b) be a couple of a Jordan pair V. The spectrum of (a, b) is defined as the set

$$Sp_V(a, b) = \left\{\lambda \in \mathbb{C}\colon \lambda - a \text{ is not invertible in } J^1\right\},$$

where $J^1 = \mathbb{C} \oplus J$ denote the unital hull of $J = V^{+(b)}$. Thus by definition, $0 \in Sp_V(a, b)$. From the basic properties of the quasi-inverse [7] , it follows that

$$\textbf{(1.3)} \quad 0 \neq \lambda \in Sp_V(a, b) \Leftrightarrow (\lambda^{-1}a, b) \text{ not quasi-invertible,}$$

$$0 \neq \lambda \in Sp_V(a, b) \Leftrightarrow B_{(\lambda^{-1}a,b)} \text{ not invertible.}$$

If V is a complex Banach-Jordan pair then, as for complex Banach algebras, $Sp_V(a, b)$ is a nonvoid compact subset of the complex field $\mathbb{C}$.

Lemma 2.3. Let V be a Jordan pair. Then, for every $(a, b) \in V^+ \times V^-$, $Sp_V(a, b) = \cup_{P\ primitive} Sp_{V/P}(\pi_P a, \pi_P b) = Sp_{V/Rad(V)}(\pi_{Rad(V)} a, \pi_{Rad(V)} b)$.

Proof. It is clear that the inclusion $\cup_{P\ primitive} Sp_{V/P}(\pi_P a, \pi_P b) \subset Sp_V(a, b)$ is easily obtained from $\left[9,\ \text{Prop. 13.b}\right]$. For the reverse inclusion take

$$\alpha \notin \cup_{P\ primitive} Sp_V(\pi_P a, \pi_P b),$$

and show that $\alpha \notin Sp_V(a, b)$. Suppose on the contrary that $\alpha \in Sp_V(a, b)$. The set $B_{(\alpha^{-1}a,b)} V^\sigma$ is an inner ideal of V different from V^σ. Thus $B_{(\alpha^{-1}a,b)} V^\sigma$ is contained in a maximal inner ideal M of V^σ. Consider the core $P^\sigma = Core(M^\sigma)$ of M. This is a primitive ideal of V for which we have so $\alpha \notin Sp_{V/P}(\pi_P a, \pi_P b)$. By 1.3, this means that the Bergman operator $B_{(\alpha^{-1}\pi_P a, \pi_P b)}$ is invertible. Consequently, we do have

$$\pi_P V^\sigma = B_{(\alpha^{-1}\pi_P a, \pi_P b)} \pi_P V^\sigma = \pi_P B_{(\alpha^{-1}a,b)} V^\sigma \subseteq \pi_P M.$$

This proves that $M = V^\sigma$ which is clearly a contradiction.

For the second equality $Sp_V(a,b) = Sp_{V/Rad(V)}(\pi_{Rad(V)}a, \pi_{Rad(V)}b)$, the inclusion $Sp_{V/Rad(V)}(\pi_{Rad(V)}a, \pi_{Rad(V)}b) \subseteq Sp_V(a,b)$ holds by $\left[9, \text{ Prop. 1.3.b}\right]$. For the reverse inclusion, take $\alpha \notin Sp_{V/Rad(V)}(\pi_{Rad(V)}a, \pi_{Rad(V)}b)$. Then, by 1.3, the couple $(\alpha^{-1}\pi_{Rad(V^+)}a, \pi_{Rad(V^-)}b)$ is quasi-invertible in $V/Rad(V)$. But since $Rad(V)$ is a quasi-invertible ideal of V, $(\alpha^{-1}a, b)$ is quasi-invertible in V by [7, Lemma. 4.3] and so $\alpha \notin Sp_V(a,b)$ by 1.3. ■

Remark 2.4. The equality $Sp_V(a,b) = Sp_{V/I}(\pi_I a, \pi_I b)$ holds for any quasi-invertible ideal $I = (I^+, I^-)$ of the Jordan pair V.

3. Bounded derivations

Definition 3.1. Let $V = (V^+, V^-)$ be a Jordan pair over an arbitrary ring of scalars Φ. Following [7] , a pair of linear mappings $D = (D_+, D_-) : D_\sigma : V^\sigma \longrightarrow V^\sigma$, is called a derivation on V if the condition

$$D_\sigma(Q_x y) = \{D_\sigma(x), y, x\} + Q_x D_{-\sigma}(y)$$

holds for every $x \in V^\sigma,\ y \in V^{-\sigma}$ and $\sigma \in \{+,-\}$. If $\frac{1}{2} \in \Phi$, this is equivalent to say that the identity,

$$D_\sigma(\{x,y,z\}) = \{D_\sigma(x), y, z\} + \{x, D_{-\sigma}(y), z\} + \{x, y, D_\sigma(z)\},$$

holds for every $x, z \in V^\sigma,\ y \in V^{-\sigma}$ and $\sigma = \pm$.

An example of derivations is given by defining the linear operators $D = (D_+, D_-)$ on the Banach-Jordan pair $\mathcal{B}(\mathcal{X}, \mathcal{Y}) = (\mathcal{BL}(\mathcal{X}, \mathcal{Y}), \mathcal{BL}(\mathcal{Y}, \mathcal{X}))$ by

$$D_+(x) = xa - bx,\ D_-(y) = yb - ay, \text{ for all } (x,y) \in \mathcal{BL}(\mathcal{X}, \mathcal{Y}) \times \mathcal{BL}(\mathcal{Y}, \mathcal{X})$$

for suitable linear operators a and b in the Banach algebras $\mathcal{BL}(\mathcal{X})$ and $\mathcal{BL}(\mathcal{Y})$ of all bounded operators defined on the Banach spaces X and Y. Let us note that the structure of linear derivations on the Banach pair $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is given in [18, Th.2.3.1] whereas the structure of non-linear (additive) derivations on the same pair is established in [11, Th. 4.4 and 4.5].

A tedious computation enables to establish the following *Leibnitz formula* for a derivation $D = (D_+, D_-)$ on a Jordan pair V. For any positive integer n,

$$\textbf{(2.2)} \quad D_\sigma^n(\{x,y,z\}) = \Sigma_{i+j+k=n} \frac{n!}{i!j!k!} \left\{ D_\sigma^i(x), D_{-\sigma}^j y, D_\sigma^k z \right\}$$
$$\text{for all } x, z \in V^\sigma,\ y \in V^{-\sigma}.$$

Let us note that the main result in this section holds to be true under the assumption that the components D_+ and D_- are bounded. However, we do mention that technics used here in the proof are entirely different from those given in the proof of the main result in this paper where the boundeness of D_+ and D_- is redundant as it will be seen later.

Recall that the spectral radius of an element x of a Banach algebra A is defined by the positive real number $\rho(x) = \lim \left\|x^n\right\|^{\frac{1}{n}} = \max\{|\alpha| : \alpha \in Sp_A(x)\}$. The spectral radius of $(a, b) \in V$ is defined in the same way as the positive real number

$$\rho(a, b) = \lim \left\|a^{(n,b)}\right\|^{\frac{1}{n}} = \max\{|\alpha| : \alpha \in Sp_V(a, b)\},$$

where $a^{(n,b)}$ is the n^{th} power of a in the the Jordan algebra $V^{+(b)}$ for which we have $a^{(n+1,b)} = Q_a b^{(n,a)}$. By an induction process together with the first identity in 1.1, we can show that the following formula holds to be true

$$\textbf{(2.3)} \qquad a^{(n+1,b)} = \frac{1}{2^n} V^n_{(a,b)} a.$$

The couple (a, b) is said to be *quasinilpotent* if $\rho(a, b) = 0$, equivalently, $Sp_V(a, b) = \{0\}$.

For more detailed study of spectral theory and holomorphic functional calculus in Banach–Jordan pairs the reader is referred to [5] .

The following lemma is of live interest in the sequel.

Lemma 3.2. Let $D = (D_+, D_-)$ be a derivation on a Jordan pair V and (a, b) be a couple of $V^+ \times V^-$ such that $D^2_+ a = 0$ and $D^2_- b = 0$, then

i) For all $n \in \mathbb{N}$, $D^{2n+2}_+ V^n_{(a,b)} a = 0$. Accordingly, for all positive integers h, k such that $k \geq 2h + 2$, we have $D^k_+ V^h_{(a,b)} a = 0$.

ii) For all $n \in \mathbb{N}$, $D^{2n+1}_+ V^n_{(a,b)} a = (2n + 1)! V^n_{(D_+a, D_-b)} D_+ a$.

Proof.

i) We proceed by induction. For $n = 0$, this is trivial with the notation $V^0_{(a,b)} = Id_{V^+}$. Suppose that $D^{2n+2}_+ V^n_{(a,b)} a = 0$. Then, for $n + 1$, we have

$$\begin{aligned}
D^{2(n+1)+2}_+ V^{n+1}_{(a,b)} a &= D^{2n+4}_+ \left\{a, b, V^n_{(a,b)} a\right\} \\
&= \Sigma_{i+j+k=2n+4} \frac{(2n+4)!}{i!j!k!} \left\{D^i_+ a, D^j_- b, D^k_+ V^n_{(a,b)} a\right\} \\
&= \Sigma^{k \geq 2n+2}_{i+j+k=2n+4} \frac{(2n+4)!}{i!j!k!} \left\{D^i_+ a, D^j_- b, D^k_+ V^n_{(a,b)} a\right\} \\
&= \frac{(2n+4)!}{1!1!(2n+2)!} \left\{D_+ a, D_- b, D^{2n+2}_+ V^n_{(a,b)} a\right\} \\
&= 0,
\end{aligned}$$

since all others terms vanish under the hypothesis $D^2_+ a = 0$ and $D^2_- b = 0$ or either the induction assumption.

ii) As in i), we proceed by induction. For $n = 0$, this is trivial since $V^0_{(a,b)} = Id_{V^+}$. Suppose that the equality holds for n. Then, for $n + 1$, we have

$$\begin{aligned}
D_+^{2n+3} V_{(a,b)}^{n+1} a &= D_+^{2n+3} \left\{a, b, V_{(a,b)}^n a\right\} \\
&= \Sigma_{i+j+k=2n+3} \frac{(2n+3)!}{i!j!k!} \left\{D_+^i a, D_-^j b, D_+^k V_{(a,b)}^n a\right\} \\
&= \Sigma_{i+j+k=2n+3}^{k \geq 2n+1} \frac{(2n+3)!}{i!j!k!} \left\{D_+^i a, D_-^j b, D_+^k V_{(a,b)}^n a\right\} \\
&= \frac{(2n+3)!}{1!1!(2n+1)!} \left\{D_+ a, D_- b, D_+^{2n+1} V_{(a,b)}^n a\right\} \\
&= (2n+3)(2n+2) \left\{D_+ a, D_- b, D_+^{2n+1} V_{(a,b)}^n a\right\},
\end{aligned}$$

because all terms as

$$\left\{D_+^i a, D_-^j b, D_+^k V_{(a,b)}^n a\right\}$$

such that $2 \leq i$ or $2 \leq j$ vanish under the conditions $D_+^2 a = 0$ and $D_-^2 b = 0$ and only the term

$$\left\{D_+ a, D_- b, D_+^{2n+1} V_{(a,b)}^n a\right\}$$

survives in the sum. Let us note that terms

$$\left\{D_+^i a, D_-^j b, D_+^k V_{(a,b)}^n a\right\}$$

for which $i = j = 0$ and $k = 2n + 3$ or either $(i, j) \in \{(0, 1), (1, 0)\}$ and $k = 2n+2$, they vanish by i). Therefore, using the induction hypothesis, we have

$$\begin{aligned}
D_+^{2n+3} V_{(a,b)}^{n+1} a &= (2n+3)(2n+2) \left\{D_+ a, D_- b, D_+^{2n+1} V_{(a,b)}^n a\right\} \\
&= (2n+3)(2n+2) \left\{D_+ a, D_- b, (2n+1)! V_{(D_+a, D_-b)}^n D_+ a\right\} \\
&= (2n+3)! V_{(D_+a, D_-b)} V_{(D_+a, D_-b)}^n D_+ a \\
&= (2n+3)! V_{(D_+a, D_-b)}^{n+1} D_+ a.
\end{aligned}$$

Finally, for all $n \in \mathbb{N}$, $D_+^{2n+1} V_{(a,b)}^n a = (2n+1)! V_{(D_+a, D_-b)}^n D_+ a$ as required. ∎

Theorem 3.3. Let $D = (D_+, D_-)$ be a bounded derivation on a Banach–Jordan pair V and (a, b) be a couple of $V^+ \times V^-$. If $D_+^2 a = 0$ and $D_-^2 b = 0$, then (D_+a, D_-b) is quasinilpotent.

Proof. Consider the Banach-Jordan algebra $V^{\sigma(u)}$ the u-homotope of V at $u = D_-b$ and the multiplication algebra $\mathcal{M}(V^{+(u)})$ of $V^{+(u)}$). For all $a \in V^+$, the left multiplication operator $L_a = \frac{1}{2}V_{(a,u)}$ satisfies, for all $x \in V^+$,

$$\begin{aligned}
\left[D_+, L_a\right]x &= D_+L_a x - L_a D_+ x \\
&= \frac{1}{2}(D_+\{a, u, x\} - \{a, u, D_+x\}) \\
&= \frac{1}{2}(D_+\{a, D_-b, x\} - \{a, D_-b, D_+x\}) \\
&= \frac{1}{2}(\{D_+a, D_-b, x\} + \left\{a, D_-^2 b, x\right\} + \{a, D_-b, D_+x\} - \{a, D_-b, D_+x\}) \\
&= \frac{1}{2}V_{(D_+a, D_-b)}x \\
&= L_{D_+a}x.
\end{aligned}$$

This shows that $\left[D_+, L_a\right] = L_{D_+a}$. Hence, the subalgebra of those elements S in the multiplication algebra $\mathcal{M}(V^{+(u)})$, for which $D_+S - SD_+$ lies in $\mathcal{M}(V^{+(u)})$, coincides with $\mathcal{M}(V^{+(u)})$ since it contains all left multiplication operators on $V^{+(u)}$. Therefore, we can define a derivation δ on $\mathcal{M}(V^{+(u)})$ by

$$\delta(S) = \left[D_+, S\right] = D_+S - SD_+,$$

for all $S \in \mathcal{M}(V^{+(u)})$. Since this derivation is bounded, it can be extended to a derivation, also denoted by δ, on $\overline{\mathcal{M}(V^{+(u)})}$ the completion of $\mathcal{M}(V^{+(u)})$ with respect to the operator-norm of $\mathcal{M}(V^{+(u)})$. Such extension is clearly unique. Now, we compute to show that

$$\delta(V_{(D_+a,b)} + V_{(a,D_-b)}) = 2V_{(D_+a,D_-b)},$$

and then we have

$$\delta^2(V_{(D_+a,b)} + V_{(a,D_-b)}) = 0.$$

Now, Kleinecke-Shirokov Theorem [6] , [14] applies to the Banach algebra $\overline{\mathcal{M}(V^{+(u)})}$ to have the quasinilpotency of the operator $V_{(D_+a,D_-b)}$ in $\mathcal{M}(V^{+(u)})$. That is $\rho(V_{(D_+a,D_-b)}) = 0$. This enables to obtain, together with 2.3,

$$\begin{aligned}
\lim \left\| D_+a^{(n+1,D_-b)} \right\|^{\frac{1}{n+1}} &= \lim \left\| \frac{1}{2^n} V^n_{(D_+a,D_-b)} D_+a \right\|^{\frac{1}{n+1}} \\
&\leq \lim \frac{1}{2^{\frac{n}{n+1}}} \left\| V^n_{(D_+a,D_-b)} \right\|^{\frac{1}{n+1}} \|D_+a\|^{\frac{1}{n+1}} \\
&\leq 0,
\end{aligned}$$

Since $\lim \left\| V^n_{(D_+a,D_-b)} \right\|^{\frac{1}{n+1}} = \rho(V_{(D_+a,D_-b)}) = 0$, as it is just pointed out above. This proves that $\rho((D_+a, D_-b) = 0$ which completes the proof. ■

Example 3.4. Consider de Banach-Jordan pair $\mathcal{B}(\mathcal{X}, \mathcal{Y}) = (\mathcal{BL}(\mathcal{X}, \mathcal{Y}), \mathcal{BL}(\mathcal{Y}, \mathcal{X}))$ and a fixed bounded linear operators a and b in the Banach algebras $\mathcal{BL}(\mathcal{X})$ and $\mathcal{BL}(\mathcal{Y})$. Let u be an element in $\mathcal{BL}(\mathcal{X}, \mathcal{Y})$. As in [18, Cor.4.2.2], the operator u is said to be *intertwining* with the couple (a, b) if

$$ua = bu.$$

Let (u, v) be two operators in $(\mathcal{BL}(\mathcal{X}, \mathcal{Y}) \times \mathcal{BL}(\mathcal{Y}, \mathcal{X}))$ such that $ua - bu$ and $bv - va$ are intertwining with (a, b). We claim that the couple $(ua - bu, bv - va)$ is quasinilpotent. Indeed, by defining the bounded derivation $D = (D_+, D_-)$ on the Banach-Jordan pair $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ =($\mathcal{BL}(\mathcal{X}, \mathcal{Y})$,$\mathcal{BL}(\mathcal{Y},\mathcal{X})$) by

$$D_+x = xa - bx,\ \ D_-y = by - ya \text{ for all } (x, y) \in (\mathcal{BL}(\mathcal{X}, \mathcal{Y}) \times \mathcal{BL}(\mathcal{Y}, \mathcal{X})).$$

We compute to check that $D_+^2 u = 0$ and $D_-^2 v = 0$. Therefore, by Theorem 2.5, $(D_+u, D_-v) = (ua - bu, bv - va)$ is quasinilpotent.

The quasinilpotency of (D_+a, D_-b), quoted in Theorem 2.5, may be obtained from an alternative assumption, namely, $D_\sigma Rad(V^\sigma) \subseteq Rad(V^\sigma)$ as it is shown in the following result.

Corollary 3.5. Let $D = (D_+, D_-)$ be a derivation on a Banach–Jordan pair V leaving invariant $Rad(V)$: $D_\sigma Rad(V^\sigma) \subseteq Rad(V^\sigma)$ and (a, b) be a couple of $V^+ \times V^-$. If $D_+^2 a = 0$and $D_-^2 b = 0$, then (D_+a, D_-b) is quasinilpotent.

Proof. Consider the quotient pair $V/Rad(V)$. This is a Banach-Jordan pair because, by [4, A.5.2], $Rad(V^\sigma)$ is closed in V^σ. On the other hand, since $D_\sigma Rad(V^\sigma) \subseteq Rad(V^\sigma)$, $D = (D_+, D_-)$ induces a derivation $\delta = (\delta_+, \delta_-)$ on the quotient pair $V/Rad(V)$ defined by

$$\delta_\sigma(x + Rad(V^\sigma)) = D_\sigma x + Rad(V^\sigma).$$

Moreover, δ_σ is bounded by [3, Th. 3.9] since $V/Rad(V)$ is semiprimitive. Now, since $\delta_+^2(a + Rad(V^+)) = \delta_-^2(b + Rad(V^-)) = 0$, it follows by Theorem 2.5 that

$$Sp_{V/Rad(V)}(\pi_{Rad(V)}a, \pi_{Rad(V)}b) = 0.$$

But

$$Sp_{V/Rad(V)}(\pi_{Rad(V)}a, \pi_{Rad(V)}b) = Sp_V(D_+a, D_-b)$$

by Lemma 1.4. ∎

4. The main result

The following technical result of Functional Analysis, frequently used in the automatic continuity Theory, will be needed in the proof of the result of this paper. This proof is in fact devided into the following steps.

Lemma 4.1. [18] Let S be a possibly unbounded linear operator on a Banach space X and let M be a closed linear subspace of X such that $\pi_M S^n$ is bounded for all positive integer n. Then, there exists a strictly positive real number λ such that $\left\|\pi_M S^n\right\| \leq \lambda^n$, for all n.

Before going on the proof of the main result, recall that if $P = (P^+, P^-)$ is a primitive ideal of a Banach-Jordan pair $V = (V^+, V^-)$, then, by [4, A.5.2], P^σ is closed in the Banach space V^σ. Consequently, V/P is also a Banach-Jordan pair.

Proposition 4.2. Let $D = (D_+, D_-)$ be a possibly unbounded derivation on a Banach–Jordan pair V and $P = (P^+, P^-)$ be a primitive ideal of V. Suppose that $P$is invariant under D : $D_\sigma P^\sigma \subseteq P^\sigma$. Then, for any couple (a, b) of $V^+ \times V^-$satisfying $D_+^2 a = 0$ and $D_-^2 b = 0$, $(\pi_P D_+ a, \pi_P D_- b)$ is quasinilpotent.

Proof. Since the ideal P is primitive, it is closed and the quotient V/P is a primitive Banach-Jordan pair. On the other hand, since P is invariant under D, the operator D drops to a derivation $\delta = (\delta_+, \delta_-)$ on the quotient pair V/P defined by

$$\delta_\sigma(x + P^\sigma) = D_\sigma(x) + P^\sigma.$$

It follows that δ_σ is bounded by means of $\left[3,\ \text{Prop. } 3.7\right]$. Moreover, δ satisfies

$$\pi_P D_\sigma^n = \delta_\sigma^n \pi_P \quad \forall n \in \mathbb{N}.$$

It follows that $\pi_P D_\sigma^n$ is bounded. Now, by Lemma 2.4 ii), we have the equality

$$D_+^{2n+1} V_{(a,b)}^n a = (2n+1)!\, V_{(D_+a, D_-b)}^n D_+ a,$$

which is equivalent to

$$V_{(D_+a, D_-b)}^n D_+ a = (2n+1)!^{-1} D_+^{2n+1} V_{(a,b)}^n a.$$

This enables to obtain, using 2.3,

$$\begin{aligned}
\left\|\pi_P (D_+a)^{(n+1, D_-b)}\right\|^{\frac{1}{n+1}} &= \left\|\frac{1}{2^n} \pi_P V_{(D_+a, D_-b)}^n D_+ a\right\|^{\frac{1}{n+1}} \\
&= (2n+1)!^{\frac{-1}{n+1}} \left\|\frac{1}{2^n} \pi_P D_+^{2n+1} V_{(a,b)}^n a\right\|^{\frac{1}{n+1}} \\
&\leq (2n+1)!^{\frac{-1}{n+1}} \left\|\pi_P D_+^{2n+1}\right\|^{\frac{1}{n+1}} \left\|\frac{1}{2^n} V_{(a,b)}^n a\right\|^{\frac{1}{n+1}}.
\end{aligned}$$

Since $\pi_P D_\sigma^n$ is bounded, by Lemma 3.1, there exists a strictly positive real number λ such that

$$\left\|\pi_P D_+^{2n+1}\right\| \leq \lambda^{2n+1}, \text{ for all } n \in \mathbb{N}..$$

This allows to obtain

$$\begin{aligned}\left\|\pi_P(D_+a)^{(n+1,D_-b)}\right\|^{\frac{1}{n+1}} &\leq (2n+1)!^{\frac{-1}{n+1}}\lambda^{\frac{2n+1}{n+1}}\left\|\frac{1}{2^n}V^n_{(a,b)}a\right\|^{\frac{1}{n+1}}\\ &\leq (2n+1)!^{\frac{-1}{n+1}}\lambda^{\frac{2n+1}{n+1}}\left\|a^{(n+1,b)}\right\|^{\frac{1}{n+1}}.\end{aligned}$$

The latter term in the previous inequality tends to zero when n tends to ∞. It follows that $\lim\left\|\pi_P(D_+a)^{(n+1,D_-b)}\right\|^{\frac{1}{n+1}} = 0$, that is $\rho(\pi_P D_+a, \pi_P D_-b) = 0$, which completes the proof. ■

Before stating the main Theorem in this paper, we recall a result established in [2, Th. 2.12] on the invariance of primitive ideals of a Banach-Jordan pair V under a derivation D defined on V.

Lemma 4.3. [2, Th. 2.12] Let $D = (D_+, D_-)$ be a possibly unbounded derivation on a Banach-Jordan pair V. Then $DP \subseteq P$ for all primitive ideals $P = (P^+, P^-)$ of V except possibly finitely many primitive ideals whose quotients are Banach–Jordan pairs of finite capacity.

Theorem 4.4. L et $D = (D_+, D_-)$ be a possibly unbounded derivation on a Banach–Jordan pair V. Then, for any couple (a, b) of $V^+ \times V^-$ satisfying $D^2_+a = 0$ and $D^2_-b = 0$, (D_+a, D_-b) is quasinilpotent.

Proof. It suffices to prove that $Sp_V(D_+a, D_-b)$ is reduced to $\{0\}$. As a first step, we start by showing that $Sp_V(D_+a, D_-b)$ is finite. By Lemma 3.3, D leaves invariant all primitive ideals of V except possibly finitely many primitive ideals say $P_1, \dots, P_n$ which provide necessarily quotient Banach-Jordan pairs V/P_k of finite capacity. Therefore, by Proposition 3.2, for any primitive ideal P not lying in the set $\{P_1, \dots, P_n\}$, we have

$$Sp_{V/P}(\pi_P D_+a, \pi_P D_-b) = \{0\}.$$

On the other hand, by [8] and [9], for all $k \in \{1, \dots, n\}$, $Sp_{V/P_k}(\pi_{P_k}D_+a, \pi_{P_k}D_-b)$ if finite. In virtue of Lemma 1.4, it follows that

$$Sp_V(D_+a, D_-b) = \cup_{k=1}^{k=n} Sp_{V/P_k}(\pi_{P_k}D_+a, \pi_{P_k}D_-b).$$

This proves that $Sp_V(D_+a, D_-b)$ is finite.

Assume actually that $Sp_V(D_+a, D_-b)$ is not reduced to $\{0\}$ and take $0 \neq \alpha \in Sp_V(D_+a, D_-b)$. Then, by 1.3, the Bergman operator $B_{(\alpha^{-1}D_+a, D_-b)}$ is not invertible. The set $B_{(\alpha^{-1}D_+a, D_-b)}V^+$ is an inner ideal of V which is different from V^+. We claim that $B_{(\alpha^{-1}D_+a, D_-b)}V^+$ is invariant under the operator D_+. Indeed, for all $u \in V^+$, we have taking into account the conditions $D_+^2 a = 0$ *and* $D_-^2 b = 0$,

$$\begin{aligned} D_+ B_{(\alpha^{-1}D_+a, D_-b)} u &= D_+(u - V_{(\alpha^{-1}D_+a,D_-b)}u + Q_{\alpha^{-1}D_+a} Q_{D_-b} u) \\ &= D_+u - D_+\left\{\alpha^{-1}D_+a, D_-b, u\right\} + D_+ Q_{\alpha^{-1}D_+a} Q_{D_-b} u \\ &= D_+u - \left\{\alpha^{-1}D_+a, D_-b, D_+u\right\} + Q_{\alpha^{-1}D_+a} D_- Q_{D_-b} u \\ &= D_+u - V_{(\alpha^{-1}D_+a,D_-b)} D_+u + Q_{\alpha^{-1}D_+a} Q_{D_-b} D_+u \\ &= B_{(\alpha^{-1}D_+a,D_-b)} D_+u. \end{aligned}$$

Since $B_{(\alpha^{-1}D_+a,D_-b)} D_+u$ belongs to $B_{(\alpha^{-1}D_+a,D_-b)} V^+$, this proves our claim:

$$D_+ B_{(\alpha^{-1}D_+a,D_-b)} V^+ \subseteq B_{(\alpha^{-1}D_+a,D_-b)} V^+.$$

By means of the main Theorem in [12] together with Zorn's Lemma applied to the inductive set $\left\{I^+ \text{ inner ideal of } V^+ : \ D_+ I^+ \subseteq I^+\right\}$, there exists a maximal inner ideal $M^+ \subset V^+$, also invariant under D^+, such that $B_{(\alpha^{-1}D_+a,D_-b)} V^+ \subseteq M^+$. By the symmetry of the argument, applied to the inner ideal $B_{(\alpha^{-1}D_-b,D_+a)} V^-$, there exists a maximal inner ideal $M^- \subset V^-$, also invariant under D^-, such that $B_{(\alpha^{-1}D_-b,D_+a)} V^- \subseteq M^-$. Let $P = (P^+, P^-) = Core(M)$ be the core of $M = (M^+, M^-)$ that is the largest ideal of V contained in M. The ideal P is primitive and also invariant under the derivation D. Indeed, note first that an easy computation enables to check that $DP + P$ is an ideal of V. Moreover, P^σ satisfies $DP^\sigma + P^\sigma \subseteq M^\sigma$. Since, by definition, $P = (P^+, P^-) = Core(M)$ is the largest ideal of V contained in M, we have $DP^\sigma + P^\sigma \subseteq P^\sigma$. This proves that $DP \subseteq P$. Now, in virtue of Proposition 3.2,

$$Sp_{V/P}(\pi_P D_+a, \pi_P D_-b) = \{0\}.$$

This shows that $\alpha \notin Sp_{V/P}(\pi_P D_+a, \pi_P D_-b)$. Equivalently, $B_{(\alpha^{-1}\pi_P D_+a, \pi_P D_-b)}$ is invertible. Therefore, we obtain,

$$\pi_P V^+ = B_{(\alpha^{-1}\pi_P D_+a,\pi_P D_-b)} \pi_P V^+ = \pi_P B_{(\alpha^{-1}D_+a,D_-b)} V^+ \subseteq \pi_P M^+.$$

Hence, as in the proof of Lemma 1.4, we deduce that $M^+ = V^+$, which is a contradiction. Finally, we obtain that $Sp_V(D_+a, D_-b) = \{0\}$ as required. ■

Actually, Theorem 3.4 may be used to derive a result on derivations of Banach-Jordan algebras together with Thomas's brilliant result on derivations of Banach algebras [17].

Corollary 4.5. Let D be a (possibly unbounded) derivation on a Banach–Jordan algebra J. Then, for any element a of J satisfying $D^2 a = 0$, Da is quasinilpotent.

Proof. Consider the unital hull $J' = \mathbb{C} \oplus J$ of the Jordan algebra J. It is also a Banach-Jordan algebra with respect to the norm $\|\alpha + x\| = |\alpha| + \|x\|$. Moreover, D extends to a derivation, also denoted by D, on J' such that

$$D(\alpha + x) = Dx, \quad \text{for all } \alpha \in \mathbb{C} \text{ and } x \in J.$$

It is known that $V = (J^{'}, J^{'})$ is a Banach-Jordan pair for the product $Q_a b = 2a(ab) - a^2 b$ and $\delta = (D, D)$ is a derivation on V. Now, since, $D^2 a = D^2 1 = 0$, then by Theorem 3.4, the couple $(a, 1)$ is quasinilpotent in V. That is $Sp_V(a, 1) = Sp(a, J^{'(1}) = Sp(a, J^{'}) = \{0\}$. ■

Corollary 4.6. [**Thomas's Theorem,** 17] Let D be a (possibly unbounded) derivation on a Banach algebra A. Then, for any element a of A satisfying $D^2 a = 0$, Da is quasinilpotent. Accordingly, if D is a Jordan derivation on A: $D(x \circ y) = D(x) \circ y + x \circ D(y)$ with $x \circ y = \frac{1}{2}(xy + yx)$,satisfying $D^2 a = 0$ for some element $a \in A$, then Da is quasinilpotent.

Acknowledgements

The authors wish to express their gratitude to Professor A. Fernandez Lopez for his valuable remarks and for his drawing their attention to Montaner's works on modular inner ideals of Jordan pairs [12] , [13].

References

[1] J. Anquela and T. Cortes, Primitive Jordan pairs and triple systems, *J. Algebra* **184**, (1996) 632–678.

[2] N. Boudi, H. Marhnine and C. Zarhouti, Additive derivations on Banach-Jordan pairs, *Comm. in Algebra*, Vol. 32, No. **9**, (2004), pp. 3609–3625.

[3] A. Fernandez Lopez, H. Marhnine and C. Zarhouti, Derivations on Banach-Jordan pairs, *Quart. J. Math.*, **52** (2001), 269–283.

[4] G. Hessenberger, Riesz and Fredholm-Theorie in Banach-Jordan systemen, Universität Innsbruck, 1994.

[5] G. Hessenberger, Banach-Jordan-Paare (Diplomarbeit, Universität Innsbruck, 1992).

[6] D. C. Kleinecke, On Operator Commutators, *Proc. Amer. Math. Soc.*, **8** (1957), 535–536.

[7] O. Loos, Jordan Pairs, Lecture Notes in Mathematics 460, Springer-Verlag, Berlin, 1975.

[8] O. Loos, On the socle of a Jordan pair, *collect. Math.*, **40** (1989), 109–125.

[9] O. Loos, Properly Algebraic and Spectrum finite Ideals in Jordan Systems, *Math. Proc. Camb. Phil. Soc.*, **114** (1993), 149–161.

[10] O. Loos, Finiteness conditions on Jordan pairs, *Math. Z.*, 206 (1991), 577–587.

[11] H. Marhnine and C. Zarhouti, Structure of additive derivations on some class of Jordan-Banach pairs, *Comptes rendus de la première Rencontre maroco-andalouse*

sur les Algèbres et leurs Applications, Université Abdelmalek Assaadi, Tetouan, (2001), pp. 65–79.

[12] F. Montaner, Maximal Modular Inner Ideals in Jordan Systems, *Comm. in Algebra*, Vol. 31, No. **2** (2003), 697–749.

[13] F. Montaner, A Note on Moduli of Inner Ideals in Jordan Systems, *Comm. in Algebra*, Vol. 30 (**1**) (2002), 411–423.

[14] F. V. Shirokov, Proof of a conjecture of Kaplansky, *Uspekhi. Mat. Nauk.*, **11** (1956), 167–168.

[15] I. M. Singer and J. Wermer, Derivations on normed algebras, *Math. Ann.*, **129** (Soc. 955), 260–264.

[16] M. P. Thomas, The image of a derivation is contained in the radical, *Annals of Math.*, **128** (1988), 435–460.

[17] M. P. Thomas, Primitive Ideals and Derivations on noncommutative Banach algebras, *Pacific. J. Math.*, **159**, (1993) 139–152.

[18] M. V. Velasco and A. R. Villena, Derivations on Banach pairs, *Rocky Mountain J. Math.*, (**28**) (1998), 1153–1187.