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Some Properties of Fuzzy S-posets

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Abstract

Given a set S, a fuzzy subset of S (or a fuzzy set in S) is, by definition, an arbitrary mapping $f : S \to [0, 1]$, where $[0, 1]$ is the usual interval of real numbers. If the set S bears some structure, one may distinguish some fuzzy subsets of S in terms of that additional structure. This important concept of a fuzzy set was first introduced by Zadeh. Fuzzy groups have been first considered by Rosenfeld, fuzzy semigroups by Kuroki. A theory of fuzzy sets on ordered groupoids and ordered semigroups can be developed. Some results on ordered groupoids-semigroups have been already given by the same authors in [N. Kehayopulu, M. Tsingelis, Fuzzy sets in ordered groupoids, *Semigroup Forum*, **65** (2002), 128–132, N. Kehayopulu, M. Tsingelis, The embedding of an ordered groupoid into a *poe*-groupoid in terms of fuzzy sets, *Inform. Sci.*, **152** (2003), 231–236], where S has been endowed with the structure of an ordered semigroup and defined "fuzzy" analogous for several notions that have been proved to be useful in the theory of ordered semigroups. Here we are going to investigate some properties of fuzzy S-posets. We first make a S-poset from the fuzzy subposets of a S-poset A. Then we use this tool to give a characterization for fuzzy S-posets. Also we introduce the notion of generated fuzzy S-posets by a fuzzy subposet of a S-poset and give a characterization for the fuzzy actions. Then we define the notion of indecomposable fuzzy S-poset and find some results.

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1. Preliminaries

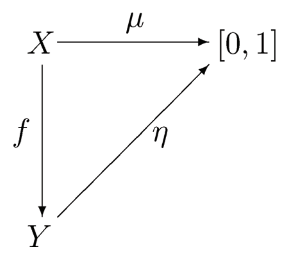
The application of fuzzy technology in information precessing is already important and it will certainly increase in importance in the future. Our aim is to promote research and the development of fuzzy technology by studying the fuzzy ordered monoids and fuzzy S-posets. Some results on ordered groupoids-semigroups have been already given by the same authors in [2] and [3]. The goal is to explain new methodological developments in fuzzy ordered monoids and fuzzy S-posets which will also be of growing importance in the future. This paper can be a bridge passing from the theory of ordered monoids and S-posets to the theory of fuzzy ordered monoids and fuzzy S-posets respectively. Rosenfeld [5] was the first who considered the case when S is a groupoid. He gave the definition of a fuzzy subgroupoid and the fuzzy left (right, two-sided) ideal of S and justified these definitions by showing that a (conventional) subset A of a groupoid S is a (conventional) subgroupoid or a left (right, two-sided) ideal of S, if the characteristic function

$$f_A : S \to [0, 1] \mid x \to f_A(x) := \begin{cases} 1 \ if \ x \in A \\ 0 \ if \ x \notin A \end{cases}$$

is, respectively, a fuzzy subgroupoid or a fuzzy left (right, two-sided) ideal of S. Actions of a semigroup (monoid or pomonoid) S on a set (poset) A have always been interest to mathematicians, specially to computer scientists and logicians. For background material on algebraic structures and some other terminologies of this concepts, we refer the reader to [1, 4] and the references contained therein. Here we are going to study some of fuzzy properties of this structures. First we recall that a partially order set (or poset) is a set X together with a partial order $\leq$ on X and denoted by $(X, \leq)$.

Definition 1.1. A poset $(X, \leq)$ together with a function $\mu : X \to [0, 1]$ is called a *fuzzy poset* if $x \leq y$ implies $\mu(x) \leq \mu(y)$, and is denoted by $X^{(\mu)}$. We call X the underlying poset.

A fuzzy function from $X^{(\mu)}$ to $Y^{(\eta)}$, written as $f : X^{(\mu)} \to Y^{(\eta)}$, is an ordinary function $f : X \to Y$, such that in diagram:



$\mu \le \eta f$ (that is $\mu(x) \le \eta f(x)$, for every $x \in X$).

Clearly fuzzy posets together with fuzzy functions between them form a category. We denote this category by **FPoset**.

The set of all fuzzy posets with a fixed underlying poset X is called the set of fuzzy subposets of X and is denoted by **FSubposet** X.

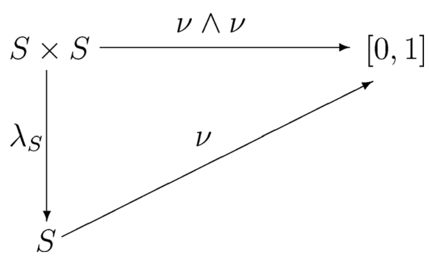
Recall that a *partially ordered monoid* (or pomonoid) is a monoid S together with a partial order $\le$ on S, such that if $s, s', u \in S$ and $s \le s'$, then $su \le s'u$ and $us \le us'$.

Definition 1.2. A pomonoid S together a function $\nu : S \to [0, 1]$ is called a *fuzzy pomonoid* if:

(i) $\nu(1) = 1$;

(ii) $x \le y$ implies $\nu(x) \le \nu(y)$, for every $x, y \in S$;

(iii) $\nu(s) \wedge \nu(r) \le \nu(rs)$, for every $r, s \in S$, that is the following diagram is a fuzzy triangle:

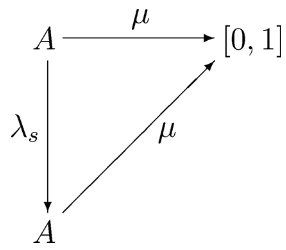


which $\lambda_S((s, r)) = sr$, for every $s, r \in S$.

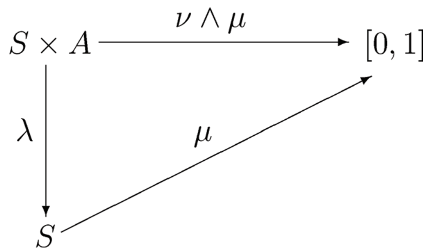
For a pomonoid S a poset A is called a *left S-poset* if A equipped with a map $S \times A \to A$, $(s, a) \rightsquigarrow sa$ such that (i) the action is monotonic in each of the variables (that is $a \le a'$ implies $sa \le sa'$, and $s \le s'$ implies $sa \le s'a$, for every $a, a' \in A$ and $s, s' \in S$), (ii) $(ss')a = s(s'a)$, for every $s, s' \in S$ and $a \in A$. The notation ${}_SA$ will often be used to denote a left S-poset. The S-poset maps (or morphisms) are a monotone map between S-posets which preserves action. We denote the category of all left S-posets, with S-poset maps between them, by **S-Pos** (see [1]).

Definition 1.3. Let $S^{(\nu)}$ be a fuzzy pomonoid and $A^{(\mu)}$ be a fuzzy poset such that A is a S-poset. Then $A^{(\mu)}$ is called:

(i) A fuzzy S-poset if $\mu(a) \le \mu(sa)$, for every $s \in S$ and $a \in A$. That is, for every $s \in S$, the following triangle is fuzzy triangle:



(ii) A fuzzy $S^{(\nu)}$-poset, if $\nu(s) \wedge \mu(a) \le \mu(sa)$, for every $s \in S$ and $a \in A$. That is the following triangle is fuzzy triangle:



Note that every fuzzy pomonoid $S^{(\nu)}$ is naturally a fuzzy $S^{(\nu)}$-poset.

A morphism between fuzzy $S^{(\nu)}$-posets, also called a $S^{(\nu)}$-map. The set of all $S^{(\nu)}$-posets with a fixed poset A is called *fuzzy S-subposets* and denoted by $\mathbf{S}^{(\nu)}$**-FSubposet** A, and the category of all fuzzy $S^{(\nu)}$-posets is denoted by $S^{(\nu)}$**-FPoset**.

Here we give an example of fuzzy S-poset.

Example 1.4. Let S be a pomonoid and $\mathbb{N}$ be the set of all non-negative integers. Then $\mathbb{N}$ is a left S-poset, under the action $sn = n$, for every $n \in \mathbb{N}$ and $s \in S$. We define $\mu : \mathbb{N} \to [0, 1]$ by

$$\mu(x) = \begin{cases} \dfrac{1}{3} & \text{if } n \neq 0 \\ \\ 0 & \text{if } n = 0. \end{cases}$$

Clearly μ is a fuzzy subposet of $\mathbb{N}$. Also since $\mu(sn) = \mu(n) \ge \mu(n)$, for every $s \in S$ and $n \in \mathbb{N}$, thus $\mathbb{N}^{(\mu)}$ is a fuzzy S-poset.

2. Fuzzy Subposets of a S-poset as a Fuzzy S-poset

In this section we make an S-action from the fuzzy subposets of a S-poset A and give a characterization of fuzzy S-posets.

Lemma 2.1. Let S be a commutative pomonoid and A be a S-poset. Then fuzzy subposets of A form a fuzzy S-poset.

Proof. For each fuzzy S-subposet $A^{(\mu)}$ and each $m \in S$ we define:

$$m\mu : A \longrightarrow [0, 1]$$

$$(m\mu)(a) := \mu(ma).$$

First we note that $m\mu$ is a fuzzy S-poset, because

$$(m\mu)(na) = \mu(m(na)) = \mu(n(ma)) \geq \mu(ma) = m\mu(a).$$

Now we check the S-poset properties. At first $(1_S\mu)(a) = \mu(1_S a) = \mu(a)$, and so $1_S\mu = \mu$. Also

$$\begin{aligned} ((m_1m_2)\mu)(a) &= \mu((m_1m_2)a) = \mu((m_2m_1)a) = \mu(m_2(m_1a)) \\ &= (m_2\mu)(m_1a) = (m_1(m_2\mu))(a) \end{aligned}$$

for every $a \in A$, and so $(m_1m_2)\mu = m_1(m_2\mu)$. Now let $m_1 \leq m_2$, for $m_1, m_2 \in S$. Then $m_1a \leq m_2a$, for every $a \in A$, and so $\mu(m_1a) \leq \mu(m_2a)$. Thus $(m_1\mu)(a) \leq (m_2\mu)(a)$, that is $m_1\mu \leq m_2\mu$. Also if $\nu \leq \mu$, which ν and μ are fuzzy subposets of A, then $(m\nu)(a) = \nu(ma) \leq \mu(ma) = (m\mu)(a)$, for every $a \in A$ and $m \in S$, that is $m\nu \leq m\mu$. Therefore fuzzy subposets of A form a fuzzy S-poset, as required. ■

Theorem 2.2. Let S be a pomonoid and $\mu : A \to [0, 1]$ be a fuzzy subposet. Then $A^{(\mu)}$ is a fuzzy S-poset if and only if $m\mu \geq \mu$, for every $m \in S$.

Proof. Necessity. Let $A^{(\mu)}$ is a fuzzy S-poset. Then $\mu(ma) \geq \mu(a)$, for every $m \in S$ and $a \in A$. Thus $(m\mu)(a) \geq \mu(a)$, and so $m\mu \geq \mu$, as required.

Sufficiency. Let $A^{(\mu)}$ be a fuzzy subposet. To prove we must show that $\mu(ma) \geq \mu(a)$, for every $m \in S$ and $a \in A$. By assumption $m\mu \geq \mu$, for every $m \in S$. Thus $(m\mu)(a) \geq \mu(a)$, for every $a \in A$, and so $\mu(ma) \geq \mu(a)$, as required. ■

3. Cyclic Fuzzy S-posets

In this section we define a generated fuzzy S-poset by a fuzzy subposet of an action and then we characterize the generated fuzzy S-actions by the action introduced in Lemma 2.1. Then we define the cyclic fuzzy S-posets, which are a useful class of fuzzy S-posets. Also we show that every fuzzy S-poset is made of a class of cyclic fuzzy S-posets.

Lemma 3.1. Let S be a pomonoid. Intersection and union of fuzzy S-posets of a S-poset A is a fuzzy S-poset.

Proof. Let $\left\{A^{(\mu_i)}\right\}_{i\in I}$ be a family of fuzzy S-posets. Then $(\cup_{i\in I}\mu_i)(ma) = \vee_{i\in I}\mu_i(ma) \geq \vee_{i\in I}\mu_i(a) = (\cup_{i\in I}\mu_i)(a)$ and $(\cap_{i\in I}\mu_i)(ma) = \wedge_{i\in I}\mu_i(ma) \geq \wedge_{i\in I}\mu_i(a) = (\cap_{i\in I}\mu_i)(a)$. Therefore $(\cup_{i\in I}\mu_i)(ma) \geq (\cup_{i\in I}\mu_i)(a)$ and $(\cap_{i\in I}\mu_i)(ma) \geq (\cap_{i\in I}\mu_i)(a)$. ■

Definition 3.2. For a fuzzy poset $X^{(\mu)}$ and $\alpha \in [0,1]$, $X^{(\mu)}_{\alpha} := \{x \in X \mid \mu(x) \geq \alpha\}$ is called the *α-cut* of the fuzzy poset $X^{(\mu)}$.

Now we show that the intersection and union of a family of α-cuts of a fuzzy poset form a α-cut.

Theorem 3.3. Let S be a pomonoid, $\mu : A \to [0,1]$ be a fuzzy S-poset and $\{\mu_i\}_{i \in I \subseteq [0,1]}$ be a family of i-cuts of μ. Then $\cup_{i \in I}\mu_i$ and $\cap_{i \in I}\mu_i$ are fuzzy S-posets of the form α-cut.

Proof. By Lemma 3.1, it is enough to show that $\cup_{i \in I}\mu_i = \mu_{\vee_{i \in I}}$ and $\cap_{i \in I}\mu_i = \mu_{\wedge_{i \in I}}$. But since $(\cup_{i \in I}\mu_i)(a) = \vee_{i \in I}\mu_i(a) \geq i$, for every $i \in I$ and $a \in A$, thus $\vee_{i \in I}\mu_i(a) \geq \vee_{i \in I} i$. Similarly $(\cap_{i \in I}\mu_i)(a) = \wedge_{i \in I}\mu_i(a) \geq \wedge_{i \in I} i$. So $\cup_{i \in I}\mu_i$ and $\cap_{i \in I}\mu_i$ are fuzzy S-posets of the form α-cut, as required. ■

Definition 3.4. Let S be a pomonoid, $\mu : A \to [0,1]$ be a fuzzy S-poset. Take $< \mu >$ to be $\cap\{\nu : A \to [0,1] \mid \mu \leq \nu$ and ν is a fuzzy S-poset$\}$. The fuzzy S-poset $< \mu >$ is called the *generated fuzzy S-poset by μ.*

Theorem 3.5. Let S be a commutative pomonoid and $A^{(\mu)}$ be a fuzzy S-poset. Then $< \mu > = \cup_{m \in B} m\mu$, which $B := \{s \in S \mid s \leq 1\}$.

Proof. First we prove that $\cup_{m \in B} m\mu$ is a fuzzy S-poset. To do we must show that $(\cup_{m \in B} m\mu)(a) \leq (\cup_{m \in B} m\mu)(na)$, for every $n \in S$ and $a \in A$. By Lemma 2.1, $m\mu$ is a fuzzy S-poset, for every $m \in S$. Thus $(\cup_{m \in B} m\mu)(na) = \vee_{m \in B}(m\mu)(na) \geq \vee_{m \in B}(m\mu)(a) = (\cup_{m \in B} m\mu)(a)$. Also, by Theorem 2.2, $m\mu \geq \mu$, for every $m \in S$. Thus $\cup_{m \in B} m\mu = \vee_{m \in B} m\mu \geq m\mu \geq \mu$, that is $\mu \leq \cup_{m \in B} m\mu$. Now let $A^{(\mu)}$ be a fuzzy S-poset such that $\mu \leq \nu$. Since $m \leq 1$, for every $m \in B$, thus $m\mu \leq 1\mu = \mu \leq \nu$, and so $\cup_{m \in B} m\mu = \vee_{m \in B} m\mu \leq \nu$. Therefore, $< \mu > = \cup_{m \in B} m\mu$. ■

Theorem 3.6. Let S be a pomonoid, $A^{(\mu)}$ and $\left\{A^{(\mu_i)}\right\}_{i \in I}$ be a family of fuzzy S-posets. Then the following statements are satisfied: Let S be a pomonoid, $A^{(\mu)}$ and $\left\{A^{(\mu_i)}\right\}_{i \in I}$ be a family of fuzzy S-posets. Then the following statements are satisfied:

(i) $<< \mu >> = < \mu >$;

(ii) $< \cup_{i \in I}\mu_i > = \cup_{i \in I} < \mu_i >$.

Proof.

(i) It is obvious, by definition of generated fuzzy S-poset.

(ii) By Theorem 3.5, we have

$$\begin{aligned}
< \cup_{i\in I}\mu_i > (a) &= \Big(\cup_{m\in B}\ (m(\cup_{i\in I}\mu_i))\Big)(a) = \vee_{m\in B}\Big((m(\cup_{i\in I}\mu_i))(a)\Big) \\
&= \vee_{m\in B}\Big((\cup_{i\in I}\mu_i)(ma)\Big) = \vee_{m\in B}\Big(\ \vee_{i\in I}\ \mu_i(ma)\Big) \\
&= \vee_{i\in I}\Big(\ \vee_{m\in B}\ \mu_i(ma)\Big) = \vee_{i\in I}\Big(\ \vee_{m\in B}\ (m\mu_i)(a)\Big) \\
&= \vee_{i\in I}\Big((\cup_{m\in B}m\mu_i)(a)\Big) = \vee_{i\in I}\Big(\ < \mu_i > (a)\Big) \\
&= \Big(\cup_{i\in I} < \mu_i > \Big)(a).
\end{aligned}$$

Therefore $< \cup_{i\in I}\mu_i >= \cup_{i\in I} < \mu_i >$. ■

Definition 3.7. Let S be a pomonoid, A be a left S-poset, $\alpha \in [0, 1]$, and $x \in A$. Then by cyclic fuzzy S-poset $< x_\alpha >$, we mean $< x_\alpha > (a) := \begin{cases} \alpha & \text{if } \alpha \in Sx \\ 0 & \text{otherwise} \end{cases}$, for every $a \in A$.

From this definition we have the following corollary.

Corollary 3.8. Let S be a pomonoid, $A^{(\mu)}$ be a fuzzy S-poset and $x \in A$. Then we have $< x_{\mu(x)} >\leq \mu$.

In the following theorem we describe a fuzzy S-poset by cyclic fuzzy S-posets.

Theorem 3.9. Let S be a pomonoid and $A^{(\mu)}$ be a fuzzy S-poset. Then $\mu = \cup_{x\in A} < x_{\mu(x)} >$.

Proof. We have:

$$\begin{aligned}
\Big(\cup_{x\in A} < x_{\mu(x)} > \Big)(a) &= \vee_{x\in A}\Big(\ < x_{\mu(x)} > (a)\Big) \\
&= \vee_{x\in A}\{\mu(x) \mid a = nx,\ \text{ for some } n \in S\}.
\end{aligned}$$

Since $1_S a = a$, thus $\mu(a) \leq \vee_{x\in A}\Big(\ < x_{\mu(x)} > (a)\Big)$. Also $< x_{\mu(x)} > (a) \leq \mu(a)$, for every $a \in A$, by Corollary 3.8. Thus $\vee_{x\in A}\Big(\ < x_{\mu(x)} > (a)\Big) \leq \mu(a)$. Therefore

$$\mu(a) \leq \vee_{x\in A}\Big(\ < x_{\mu(x)} > (a)\Big) \leq \mu(a),$$

that is $\mu(a) = \vee_{x\in A}\Big(\ < x_{\mu(x)} > (a)\Big) = \Big(\cup_{x\in A} < x_{\mu(x)} > \Big)(a)$, for every $a \in A$. Thus $\mu = \cup_{x\in A} < x_{\mu(x)} >$. ■

Theorem 3.10. Let S be a pomonoid, $B = \{m \in S;\ m \leqslant 1\}$ and $< x_\alpha >$ be a cyclic fuzzy S-poset of A. Then $m < x_\alpha >=< x_\alpha >$, for every $m \in B$.

Proof. Let $m \in B$. Then $(m < x_\alpha >)(a) =< x_\alpha > (ma) = \begin{cases} \alpha & \text{if } ma \in Sx \\ 0 & \text{otherwise} \end{cases}$.
Also $< x_\alpha > (a) = \begin{cases} \alpha & \text{if } a \in Sx \\ 0 & \text{otherwise} \end{cases}$. Now let $a \in Sx$. Then $ma \in Sx$, and so $< x_\alpha > (ma) =< x_\alpha > (a)$, that is $(m < x_\alpha >)(a) =< x_\alpha > (a)$. If $a \notin Sx$, then $ma \leq a$, and so $< x_\alpha > (ma) \leq< x_\alpha > (a) = 0$. Therefore $< x_\alpha > (ma) = 0$, that is $(m < x_\alpha >)(a) =< x_\alpha > (a)$. Thus $m < x_\alpha >=< x_\alpha >$, as required. ■

4. Decomposable and Indecomposable Fuzzy S-poset

Here we give a definition of indecomposable fuzzy S-poset and show that the cyclic fuzzy S-poset are indecomposable.

Definition 4.1. Let S be a pomonoid. A fuzzy S-poset $\mu \neq 0$ of A is called *decomposable* whenever there exist two fuzzy S-poset $\nu, \eta \neq 0$ of A such that $\nu, \eta \leq \mu$, $\nu \vee \eta = \mu$ and $\nu \wedge \eta = 0$, otherwise μ is called *indecomposable*.

Theorem 4.2. Let S be a commutative pomonoid. Then every cyclic fuzzy S-poset $< x_i >$ of A, $i \in [0, 1]$, is indecomposable.

Proof. Let $< x_i >$ be decomposable. Then there are fuzzy S-posets ν and η of A such that $\nu, \eta \leq< x_i >$, $\nu \vee \eta =< x_i >$ and $\nu \wedge \eta = 0$. Therefore $\eta(a) \wedge \nu(a) = 0$ and $\eta(a) \vee \nu(a) = \begin{cases} i & \text{if } a = mx, \text{ for some } m \in S \\ 0 & \text{otherwise} \end{cases}$, for every $a \in A$. Now let $m_0 \in S$. Since $< x_i > (m_0x) = i$, thus $\nu(m_0x) \vee \eta(m_0x) = i$. Without less of generality let $\nu(m_0x) = i$. We claim that $\nu(mx) = i$ and $\eta(mx) = 0$, for every $m \in S$. If there exists $m_1 \in S$ such that $\eta(m_1x) = i$, then

$$i = \nu(m_0x) \leq \nu(m_1m_0x) \leq \nu(m_1m_0x) \vee \eta(m_1m_0x) =< x_i > (m_1m_0x) = i,$$

that is, $\nu(m_1m_0x) = i$. Also

$$\begin{aligned} i &= \eta(m_1x) \leq \eta(m_0m_1x) = \eta(m_1m_0x) \\ &\leq \nu(m_1m_0x) \vee \eta(m_1m_0x) =< x_i > (m_1m_0x) = i, \end{aligned}$$

that is $\eta(m_1m_0x) = i$. Therefore $\eta(m_1m_0x) \wedge \nu(m_1m_0x) = i \neq 0$, which is a contradiction. Since $\eta(mx) \vee \nu(mx) =< x_i > (mx) = i$, for every $m \in S$, thus $\nu(mx) = i$, for every $m \in S$. Also $\eta(mx) \wedge \nu(mx) = 0$, implies that $\eta(mx) \wedge i = 0$. Thus $\eta(mx) = 0$, for every $m \in S$. Now let $a \in A \setminus Sx$. Then

$$\eta(a) \leq \eta(a) \vee \nu(a) =< x_i > (a) = 0.$$

Therefore $\eta(a) = 0$, for every $a \in A \setminus Sx$. So $\eta = 0$, which is a contradiction. Thus cyclic fuzzy S-poset $< x_i >$ is indecomposable, as required. ■

Theorem 4.3. Let S be a pomonoid, $\left\{A^{(\mu_i)}\right\}_{i\in I}$ be a family of indecomposable fuzzy S-poset such that $\wedge_{i\in I}\mu_i \neq 0$. Then $\vee_{i\in I}\mu_i$ is indecomposable.

Proof. Since $\wedge_{i\in I}\mu_i \neq 0$, thus there exist $x_0 \in A$, such that $\mu_i(x_0) \neq 0$, for every $i \in I$. Let $\vee_{i\in I}\mu_i$ be decomposable. Then there exist $\nu_1, \nu_2 \leq \vee_{i\in I}\mu_i$ such that $\nu_1, \nu_2 \neq 0$, $\vee_{i\in I}\mu_i = \nu_1 \vee \nu_2$ and $\nu_1 \wedge \nu_2 = 0$. Since $\vee_{i\in I}\mu_i(x_0) \neq 0$, thus at most one of $\nu_1(x_0)$ or $\nu_2(x_0)$ is not zero, because if $\nu_1(x_0) \neq 0$ and $\nu_2(x_2) \neq 0$, then $\nu_1(x_0) \wedge \nu_2(x_0) \neq 0$, which is a contradiction. Without less of generality let $\nu_1(x_0) \neq 0$. Then $\mu_i = \mu_i \wedge (\vee_{i\in I}\mu_i) = (\mu_i \wedge \nu_1) \vee (\mu_i \wedge \nu_2)$, for every $i \in I$. $(\mu_i \wedge \nu_1) \wedge (\mu_i \wedge \nu_2) = \mu_i \wedge (\nu_1 \wedge \nu_2) = 0$, for every $i \in I$. Since μ_i is indecomposable, for every $i \in I$, thus $\mu_i \wedge \nu_1 = 0$ or $\mu_i \wedge \nu_2 = 0$. But $\mu_i \wedge \nu_1 \neq 0$, since $\nu_1(x_0) \neq 0$ and $\mu_i(x_0) \neq 0$, for every $i \in I$. Therefore $\mu_i \wedge \nu_2 = 0$, for every $i \in I$, that is $\nu_2 = 0$, which is a contradiction. Hence $\vee_{i\in I}\mu_i$ is indecomposable. ■

Definition 4.4. Let S be a pomonoid. A fuzzy S-poset $A^{(\mu)}$ is called *finitely generated* whenever $\mu =< \cup_{i=1}^{n}(x_i)_{\alpha_i} >$, where $\alpha_i \in [0, 1]$.

In the following theorem we show that the image of finitely generated fuzzy S-poset, is also finitely generated.

Theorem 4.5. Let S be a pomonoid, $f : A \to B$ be an S-poset homomorphism, which is surjective and $A^{(\mu)}$ be a fuzzy S-poset. If μ is finitely generated, then $B^{(f(\mu))}$ is finitely generated.

Proof. Let $\mu =< \cup_{i=1}^{n}(x_i)_{\alpha_i} >$. Then

$$(f(\mu))(b) = \vee\{\mu(a) \mid f(a) = b\} = \vee\left\{< \cup_{i=1}{}^{n}(x_i)_{\alpha_i} > (a) \mid f(a) = b\right\}.$$

Also,

$$\langle \cup_{i=1}^{n} f((x_i)_{\alpha_i})\rangle(b) = \vee\left\{\langle \cup_{i=1}^{n}(x_i)_{\alpha_i}\rangle(a) \mid f(a) = b\right\}.$$

Therefore $f(\mu) =< \cup_{i=1}^{n} f((x_i)_{\alpha_i}) >$, and so $B^{(f(\mu))}$ is finitely generated. ■

From Theorem 4.5 we have the following corollary.

Corollary 4.6. Let S be a pomonoid, $f : A \to B$ be a left S-poset homomorphism, which is surjective and $A^{(\mu)}$ be a fuzzy S-poset. Let $\alpha \in [0, 1]$, if $\mu =< x_\alpha >$, then $f(\mu) =< f(x_\alpha) >$.

References

[1] S. Bulman-Fleming, M. Mahmoudi, The category of S-posets, *Semigroup Forum*, **71** (2005), 443–461.

[2] N. Kehayopulu, M. Tsingelis, Fuzzy sets in ordered groupoids, *Semigroup Forum*, **65** (2002), 128–132.

[3] N. Kehayopulu, M. Tsingelis, The embedding of an ordered groupoid into a *poe*-groupoid in terms of fuzzy sets, *Inform. Sci.*, **152** (2003), 231–236.

[4] M. Kilp, U. Knauer, A. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York (2000).

[5] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.*, **35** (1971), 512–517.