Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 4 (2016), pp. 2999–3007
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http://www.ripublication.com/gjpam.htm

Some properties of two dimensional q-tangent numbers and polynomials

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Abstract

In this paper we introduce two dimensional q-tangent numbers and polynomials. We also give some properties, explicit formulas, several identities, a connection with two dimensional q-tangent numbers and polynomials, and some integral formulas.

AMS subject classification: 11B68, 11S40, 11S80.
Keywords: Tangent numbers and polynomials, q-tangent numbers and polynomials, two dimensional q-tangent numbers and polynomials.

1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers (see [2, 3, 4, 6, 7, 8, 9]). In this paper, we study some properties of a new type of two dimensional q-tangent numbers and polynomials.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. For a real number (or complex number) x, q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q} \text{ if } q \neq 1, \quad [x]_q = x \text{ if } q = 1.$$

The q-binomial coefficients are defined for positive integer n, k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!},$$

where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q, n = 1, 2, 3, \ldots$ and $[0]_q! = 1$, which is known as q-factorial(see [1]). Note that

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

The q-analogue of the function $(x+y)^n$ is defined by

$$(x+y)_q^n = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{\binom{l}{2}} x^{n-l} y^l, n \in \mathbb{Z}_+.$$

For any $z \in \mathbb{C}$ with $|z| < 1$, the two form of q-exponential functions are given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \text{ and } E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}, \text{ (see [2, 5]).}$$

From this form we easily see that $e_q(z)E_q(-z) = 1$. The q-derivative operator of a any function f is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, x \neq 0, \tag{1.1}$$

and $D_q f(0) = f'(0)$, provided that f is differentiable at 0. It happens clearly that $D_q x^n = [n]_q x^{n-1}$.

The definite q-integral is defined as

$$\int_0^b f(x) d_q x = (1-q)b \sum_{j=0}^{\infty} q^j f(q^j b). \tag{1.2}$$

Clearly, if the function $f(x)$ is differentiable on the point x, the q-derivative in (1.1) tends to the ordinary derivative in the classical analysis when q tends to 1. Identically, if the function $f(x)$ is Riemann integrable on the concerned intervals, the q-integral in (1.2) tends to the Riemann integrals of $f(x)$ on the corresponding intervals when q tends to 1(see [2, 5]). In the following section, we introduce the two dimensional q-tangent numbers and polynomials. After that we will investigate some their properties. Finally, we give some relationships both between these polynomials and q-derivative operator and between these polynomials and q-integral.

2. Two dimensional q-tangent polynomials

In this section, we introduce the two dimensional q-tangent numbers and polynomials and provide some of their relevant properties.

The q-tangent polynomials $\mathbf{T}_n(x)$ are defined by the generating function:

$$\left(\frac{2}{e_q(2t)+1}\right) e_q(xt) = \sum_{n=0}^{\infty} \mathbf{T}_{n,q}(x) \frac{t^n}{[n]_q!} \quad (|2t| < \pi). \tag{2.1}$$

When $x = 0$, $\mathbf{T}_{n,q}(0) = \mathbf{T}_{n,q}$ are called the q-tangent numbers. Upon setting $p = 1$ in (2.1), we have

$$\left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} \quad (|2t| < \pi), \tag{2.2}$$

where $T_n(x)$ are called familiar Tangent polynomials. Numerous properties of tangent numbers and polynomials are known. More studies and results in this subject we may see references [4], [5], [6], [7]. About extensions for the tangent numbers can be found in [5, 7, 8].

The two dimensional q-tangent polynomials $\mathbf{T}_n(x, y)$ in x, y are defined by means of the generating function:

$$\left(\frac{2}{e_q(2t)+1}\right) e_q(xt)e_q(yt) = \sum_{n=0}^{\infty} \mathbf{T}_{n,q}(x, y) \frac{t^n}{[n]_q!} \quad (|2t| < \pi). \tag{2.3}$$

It is obvious that $\lim_{q\to 1} \mathbf{T}_{n,q}(x, y) = T_n(x + y)$ and $\mathbf{T}_{n,q}(x, 0) = \mathbf{T}_{n,q}(x)$.

By (2.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbf{T}_{n,q}(x) \frac{t^n}{[n]_q!} &= \left(\frac{2}{e_q(2t)+1}\right) e_q(xt) \\ &= \sum_{n=0}^{\infty} \mathbf{T}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q} y^l \right) \frac{t^n}{[n]_q!}. \end{aligned} \tag{2.4}$$

By comparing the coefficients on both sides of (2.4), we have the following theorem.

Theorem 2.1. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,q}(x) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q} x^l.$$

By using Definition of q-derivative operator and Theorem 2.1, we have the following theorem.

Theorem 2.2. For $n \in \mathbb{Z}_+$, we have

$$D_q \mathbf{T}_{n,q}(x) = [n]_q \mathbf{T}_{n-1,q}(x)$$

By Theorem 2.2 and Definition of the definite q-integral, we have

$$[n]_q \int_0^1 D_q \mathbf{T}_{n-1,q}(x) d_q x = \mathbf{T}_{n,q}(1) - \mathbf{T}_{n,q}(0). \tag{2.5}$$

Since $\mathbf{T}_{n,q}(0) = \mathbf{T}_{n,q}$, by (2.5), we have the following theorem.

Theorem 2.3. For $n \in \mathbb{Z}_+$, we have

$$\int_0^1 D_q \mathbf{T}_{n-1,q}(x) d_q x = \frac{\mathbf{T}_{n,q}(1) - \mathbf{T}_{n,q}}{[n]_q}.$$

Using the following identity:

$$\frac{2}{e_q(2t)+1} e_q(xt) e_q(2t) + \frac{2}{e_q(2t)+1} e_q(xt) = 2e_q(xt),$$

we have the following theorem.

Theorem 2.4. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,q}(x,2) + \mathbf{T}_{n,q}(x) = 2x^n.$$

Substituting $x = 0$ in Theorem 2.4, we have the following corollary.

Corollary 2.5. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,q} = -\mathbf{T}_{n,q}(2).$$

By (2.3) and the rule of Cauchy product, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbf{T}_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \left(\frac{2}{e_q(2t)+1} \right) e_q(xt) e_q(yt) \\ &= \sum_{n=0}^{\infty} \mathbf{T}_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} y^n \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q}(x) x^l \right) \frac{t^n}{[n]_q!}. \end{aligned} \tag{2.6}$$

By comparing the coefficients on both sides of (2.6), we have the following theorem.

Theorem 2.6. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,q}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q}(x) y^l.$$

Using the following identity:

$$\frac{2}{e_q(2t)+1}e_q(2t)+\frac{2}{e_q(2t)+1}=2,$$

we obtain the following theorem.

Theorem 2.7. For $n \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q 2^{n-l}\mathbf{T}_{l,q}+\mathbf{T}_{n,q}=\begin{cases} 2, & \text{if } n=0, \\ 0, & \text{if } n\neq 0. \end{cases}$$

By using Definition of q-derivative operator, we have the following theorem.

Theorem 2.8. For $n \in \mathbb{Z}_+$, we have

$$D_{q,y}\mathbf{T}_{n,q}(x,y)=[n]_q\mathbf{T}_{n-1,q}(x,y)$$

3. Some identities involving q-tangent numbers and polynomials

In this section, we give some relationships both between these polynomials and q-derivative operator and between these polynomials and q-integral. By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathbf{T}_{n,q}(x)\frac{t^n}{[n]_q!} &= \left(\frac{2}{e_q(2t)+1}\right)e_q(xt) \\
&= \left(\frac{2}{e_q(2t)+e_q(2t)e_{q^{-1}}(-2t)}\right)e_q(xt) \\
&= \left(\frac{2e_{q^{-1}}(-2t)}{e_{q^{-1}}(-2t)+1}\right)e_q(xt) \qquad (3.1) \\
&= \sum_{n=0}^{\infty}\mathbf{T}_{n,q^{-1}}(2)(-1)^n\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q(-1)^l\mathbf{T}_{l,q^{-1}}(2)x^{n-l}\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

By comparing the coefficients on both sides of (3.1), we have the following theorem.

Theorem 3.1. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,q}(x)=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q(-1)^l\mathbf{T}_{l,q^{-1}}(2)x^{n-l}.$$

By Definition of the definite q-integral and Theorem 2.1, we get

$$\begin{aligned}\int_0^1 \mathbf{T}_{n,q}(x)d_qx &= \int_0^1 \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q}x^l d_qx \\ &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q}\frac{1}{[l+1]_q}.\end{aligned} \tag{3.2}$$

We also get

$$\begin{aligned}\int_0^1 \mathbf{T}_{n,q}(x)d_qx &= \int_0^1 \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^l\mathbf{T}_{l,q^{-1}}(2)x^{n-l}d_qx \\ &= \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^l\mathbf{T}_{l,q^{-1}}(2)\frac{1}{[n-l+1]_q}.\end{aligned} \tag{3.3}$$

By (3.2) and (3.3), we have the following theorem.

Theorem 3.2. For $n \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{\mathbf{T}_{n-l,q}}{[l+1]_q} = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^l \frac{\mathbf{T}_{l,q^{-1}}(2)}{[n-l+1]_q}$$

Using the following identity:

$$\frac{2}{e_q(2t)+1}e_q(xt)e_q(yt) = \frac{2}{e_q(2t)+1}e_q(xt)\frac{e_q(\frac{t}{m})-1}{\frac{t}{m}}\frac{\frac{t}{m}}{e_q(\frac{t}{m})-1}e_q\left(\frac{t}{m}my\right),$$

we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathbf{T}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\ &= \frac{m}{t}\left(\sum_{n=0}^{\infty}\mathbf{T}_{n,q}(x)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}m^{-n}\frac{t^n}{[n]_q!}-1\right)\left(\frac{\frac{t}{m}}{e_q(\frac{t}{m})-1}e_q\left(\frac{t}{m}my\right)\right) \\ &= m\sum_{n=0}^{\infty}\left(\mathbf{T}_{n+1,q}(x,m^{-1})-\mathbf{T}_{n+1,q}(x)\right)\frac{t^n}{[n+1]_q!}\sum_{n=0}^{\infty}\mathbf{B}_{n,q}(my)m^{-n}\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\frac{\left(\mathbf{T}_{k+1,q}(x,m^{-1})-\mathbf{T}_{n+1,q}(x)\right)m^{k-n+1}}{[k+1]_q}\mathbf{B}_{n-k,q}(my)\right)\frac{t^n}{[n]_q!}.\end{aligned}$$

Matching the coefficient of $\frac{t^n}{[n]_q!}$ of both sides gives the following theorem.

Theorem 3.3. For $n \in \mathbb{Z}_+$, we have

$$\begin{aligned}\mathbf{T}_{n,q}(x,y) &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{\left(\mathbf{T}_{k+1,q}(x,m^{-1}) - \mathbf{T}_{n+1,q}(x)\right) m^{k-n+1}}{[k+1]_q} \mathbf{B}_{n-k,q}(my) \\ &= 2^n \mathbf{E}_{n,q}\left(\frac{x}{2}, \frac{y}{2}\right).\end{aligned}$$

Here $\mathbf{B}_{n,q}(x,y)$ and $\mathbf{E}_{n,q}(x,y)$ denote the q-Bernoulli and q-Euler polynomials in x, y which are defined by

$$\mathbf{B}_{n,q}(x,y) = \frac{t}{e_q(t)-1} e_q(xt)e_q(yt) \text{ and } \mathbf{E}_{n,q}(x,y) = \frac{2}{e_q(t)+1} e_q(xt)e_q(yt).$$

By Definition (2.1) and by using the following identity:

$$\frac{t}{e_q(t)-1} e_q(xt)e_q(yt) = \frac{2}{e_q\left(2\frac{t}{m}\right)+1} e_q\left(\frac{t}{m} my\right) \frac{e_q\left(2\frac{t}{m}\right)+1}{2} \frac{t}{e_q(t)-1} e_q(xt),$$

we get

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathbf{B}_{n,q}(x,y) \frac{t^n}{[n]_q!} \\ &= \frac{1}{2} \left(\sum_{n=0}^{\infty} \mathbf{T}_{n,q}(my) m^{-n} \frac{t^n}{[n]_q!}\right) \left(\sum_{n=0}^{\infty} 2^n m^{-n} \frac{t^n}{[n]_q!} + 1\right) \left(\sum_{n=0}^{\infty} \mathbf{B}_{n,q}(x,y) \frac{t^n}{[n]_q!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathbf{B}_{k,q}(x) \sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix}_q \mathbf{T}_{l,q}(my) 2^{n-k-l-1} m^{k-n}\right) \frac{t^n}{[n]_q!} \\ &\qquad + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{-1} m^{k-n} \mathbf{B}_{k,q}(x) \mathbf{T}_{n-k,q}(my)\right) \frac{t^n}{[n]_q!}.\end{aligned}$$

By comparing coefficients of $\frac{t^n}{[n]_q!}$ in the above equation, we arrive at the following theorem.

Theorem 3.4. For $n \in \mathbb{Z}_+$, we have

$$\begin{aligned}&\mathbf{B}_{n,q}(x,y) \\ &= \frac{1}{2m^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathbf{B}_{k,q}(x) \left[\sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix}_q 2^{n-k-l} m^k \mathbf{T}_{l,q}(my) + m^k \mathbf{T}_{n-k,q}(my)\right].\end{aligned}$$

By Definition of the definite q-integral and Theorem 2.6, get

$$\begin{aligned}\int_0^1 y^n \mathbf{T}_{n,q}(x,y)d_q y &= \int_0^1 \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q}(x) y^{l+n} d_q y \\ &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q}(x) \frac{1}{[n+l+1]_q}.\end{aligned} \tag{3.4}$$

By (1.2), we see that

$$\begin{aligned}&\int_0^1 y^n \mathbf{T}_{n,q}(x,y)d_q y \\ &= y^{n+1}\frac{\mathbf{T}_{n,q}(x,y)}{[n+1]_q}\bigg|_0^1 - \int_0^1 [n]_q q^{n+1} y^{n+1} \frac{\mathbf{T}_{n-1,q}(x,y)}{[n+1]_q} d_q y \\ &= \frac{\mathbf{T}_{n,q}(x,1)}{[n+1]_q} - \frac{q^{n+1}[n]_q}{[n+1]_q}\int_0^1 y^{n+1}\mathbf{T}_{n-1,q}(x,y)d_q y \\ &= \frac{\mathbf{T}_{n,q}(x,1)}{[n+1]_q} - \frac{q^{n+1}[n]_q \mathbf{T}_{n-1,q}(x,1)}{[n+1]_q[n+2]_q} \\ &\qquad + (-1)^2 \frac{q^{n+1}q^{n+2}[n]_q}{[n+1]_q}\frac{[n-1]_q}{[n+2]_q}\int_0^1 y^{n+2}\mathbf{T}_{n-2,q}(x,y)d_q y \\ &= \frac{\mathbf{T}_{n,q}(x,1)}{[n+1]_q} + (-1)\frac{q^{n+1}[n]_q \mathbf{T}_{n-1,q}(x,1)}{[n+1]_q[n+2]_q} \\ &\quad + (-1)^2 \frac{q^{n+1}q^{n+2}[n]_q}{[n+1]_q}\frac{[n-1]_q}{[n+2]_q}\frac{\mathbf{T}_{n-2,q}(x,1)}{[n+3]_q} \\ &\quad + (-1)^3 \frac{q^{n+1}q^{n+2}q^{n+3}[n]_q}{[n+1]_q}\frac{[n-1]_q}{[n+2]_q}\frac{[n-2]_q}{[n+3]_q}\int_0^1 y^{n+3}\mathbf{T}_{n-3,q}(x,y)d_q y.\end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned}&\int_0^1 y^n \mathbf{T}_{n,q}(x,y)d_q y = \frac{\mathbf{T}_{n,q}(x,1)}{[n+1]_q} \\ &+ \sum_{m=1}^{n-1} \frac{q^{n+1}\cdots q^{n+m}[n]_q[n-1]_q\cdots[n-m+1]_q(-1)^m}{[n+1]_q[n+2]_q\cdots[n+m+1]_q}\mathbf{T}_{n-m,q}(x,1) \\ &+ (-1)^n \frac{q^{n+1}\cdots q^{2m}[n]_q!}{[n+1]_q[n+2]_q\cdots[2n]_q}\int_0^1 y^{2n}\mathbf{T}_{0,q}(x,y)d_q y\end{aligned} \tag{3.5}$$

Hence, by (3.4) and (3.5), we have the following theorem.

Theorem 3.5. For $n \in \mathbb{N}$, we have

$$\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q}(x) \frac{1}{[n+l+1]_q} = \frac{\mathbf{T}_{n,q}(x,1)}{[n+1]_q}$$
$$+ \sum_{m=1}^{n-1} \frac{q^{n+1} \cdots q^{n+m}[n]_q[n-1]_q \cdots [n-m+1]_q(-1)^m}{[n+1]_q[n+2]_q \cdots [n+m+1]_q} \mathbf{T}_{n-m,q}(x,1)$$
$$+ (-1)^n \frac{q^{n+1} \cdots q^{2m}[n]_q!}{[n+1]_q[n+2]_q \cdots [2n]_q[2n+1]_q}.$$

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