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An Optical-Mechanical Analogy And The Problems Of The Trajectory-Wave Dynamics

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Abstract

On the basis of existing variational principles, an optical-mechanical analogy is carried out only at the level of geometrical optics. A local variational principle (LVP) is formulated. On the basis of this principle, new formulations of the direct and inverse problems are considered. An optical-mechanical analogy which has a new extension in comparison with the existing optical-mechanical analogies is presented. According to the method of V-function, the trajectory motion of an object is connected with its wave motion. A linear harmonic oscillator is considered, and its energy levels are found. These results correlate with the energy levels of the real microscopic oscillators interacting with the light. While simulating the electron motion in the Coulomb field, the considered method allows setting a rule of energy quantization of a hydrogen-like atom, which completely coincides with the classical results of Schrödinger and Bohr.

Keywords: local variational principle, trajectory motion, wave motion, wave function, optical-mechanical analogy, harmonic oscillator, hydrogen-like atom

1. Introduction

An optical-mechanical analogy is, first of all, a view on the nature of light, its corpuscular and wave properties. Hamilton [1] was the first, who turned attention to the analogy between the motion of mechanical conservative systems and the propagation of light beams in an optically inhomogeneous medium. So, from the Hamilton-Jacobi equation, written for a stationary system,

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0 \qquad (01)$$

we obtain, setting in (01) $S(q) = Et - W(q)$:

$$H\left(\frac{\partial W}{\partial q}, q\right) = E \tag{02}$$

where *W(q)* is the characteristic Hamilton function, *E* is the total energy of the conservative system, H is the Hamiltonian function.
From (02), written for a single particle:

$$\sum_{i=1}^{3}\left(\frac{\partial W}{\partial q_i}\right)^2 = 2m(E - U), \tag{03}$$

where *U(q)* is the potential energy of the particle, *E* is the total energy of the particle, *m* is the mass of the particle; and the eikonal equation, which describes the propagation of the light beam:

$$\sum_{i=1}^{3}\left(\frac{\partial \varphi}{\partial q_i}\right)^2 = \frac{\nu^2 n^2}{c^2} = \frac{1}{\lambda^2}, \tag{04}$$

where $\varphi(q)$ is the eikonal function (the light wave phase), ν is the frequency of the monochromatic light wave, *n* is the refractive index of the medium, *c* is the velocity of light, λ is the light wavelength, it follows that the equations are similar in appearance. That is, this analogy is based not on the real nature of phenomena, but on the similarity of the mathematical form of expression of the laws of these physical phenomena.
Moreover, (04) is obtained from Fresnel's scalar wave equation,

$$\frac{\partial^2 \Phi(q,t)}{\partial t^2} - \frac{c^2}{n^2}\sum_{i=1}^{3}\frac{\partial^2 \Phi(q,t)}{\partial q^2{}_i} = 0 \tag{05}$$

when a monochromatic wave $\Phi(q,t) = e^{2\pi i(\nu t - \varphi(q))}$ is considered for $\lambda \to 0$.
The same analogy can be demonstrated on the basis of Fermat's principle of the light beam propagation, written in the form:

$$\delta(\int n ds) = 0, \tag{06}$$

where *n* is the index of refraction, *ds* is the beam path element,
and Maupertuis' principle of least action:

$$\delta(\int m\vartheta ds) = 0, \tag{07}$$

where $m\vartheta$ is the particle momentum, *ds* is the particle trajectory element.
In addition, if the refractive index varies according to the law $n = m\vartheta = \sqrt{2m(E - U)}$, then the path of the light beam coincides with the trajectory of the particle.

Louis de Broglie shed new light into the optical-mechanical analogy [2, 3]. He considered a correspondence between wave and particle on the basis of equations (03) and (04), and on the basis of variational principles of Maupertuis and Fermat [4]. If, instead of the function *W*, we take *W/h* in the equation (03), where *h* is the Planck constant, and put $\varphi = W/h$, then from (03) and (04) we have

$$\frac{1}{\lambda^2} = \frac{2m(E-U)}{h^2} = \frac{p^2}{h^2}, \qquad (08)$$

From this we obtain the famous Louis de Broglie's formula for determining the wavelength of the particle:

$$\lambda = \frac{h}{p} \qquad (09)$$

The idea that a wave motion is hidden behind the motion of particles has become particularly productive for physics [1]. The studies of Louis de Broglie have served Schrödinger as a basis for the formulation of the wave equation [15] which is now one of the fundamentals of quantum mechanics.

The recent experimental advances in the study of behavior of individual microscopic systems, in turn, are reviving sustained interest in the problem of the wave-particle dualism, in the role of information in the theoretical description of the behavior of micro-particles [5, 6]. The ongoing attempts to understand the paradoxical manifestations of wave-corpuscle dualism in the motion of an electron (and other microparticles) also stimulate the creation of new theories that somehow develop Louis de Broglie’s idea of a pilot-wave [7-9]. In this paper, we propose a new approach in this direction, based on the wave-corpuscle monism, in order to explain the nature of a particle (an object). Namely, the theory developed below uses the description of physical reality, which takes into account the presence of the particle’s trajectory, which is a reflection of the existence of the particle; at the same time, it is assumed that the motion of the particle is determined by a physical wave *V(x,t)*.

2. Local variational principle (LVP)

Let us introduce $x(t) = (x_1, x_2, \ldots, x_n)^T$, a vector of the phase coordinates, $x \in R^n$, where R^n is *n*-dimensional Euclidean space and time $t \in T$, where *T* is a time interval. Consider a system of differential equations:

$$\dot{x} = f(x) \qquad (1)$$

We say that the equation (1), describing the motion of an object along a path, determines the state of the object being studied.

Now, we introduce a function *V=V(x,t)* ($x \in R^n, t \in T$), which we call a wave function or V-function, and its rate (speed) of change

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}^T f \tag{2}$$

according to the system (1).
Consider an isochronous variation of the rate of change of the wave function

$$\delta\left(\frac{dV}{dt}\right) = \delta\left(\frac{\partial V}{\partial t}\right) + \delta\left(\frac{\partial V}{\partial x}^T f\right)$$

where

$$\delta\left(\frac{\partial V}{\partial t}\right) = \frac{\partial}{\partial t}(\delta V)$$

$$\delta\left(\frac{\partial V}{\partial x}^T f\right) = \frac{\partial \delta V}{\partial x}^T f + \frac{\partial V}{\partial x}^T \delta f$$

$$\delta V = \frac{\partial V}{\partial x}^T \delta x; \quad \delta f = \frac{\partial f}{\partial x}\delta x$$

When the rate of change of the wave function is varied, the object goes from a certain state into a new state. Such a transition will be called a wave transition of the object to a new state. The quantity δV will be called a possible wave transition from the initial state into a new state. Moreover, δx determines the trajectory variations.
We now formulate a local variational principle (LVP) [10-11]:

Among all possible transitions into a new state, a transition is realized, in which, at any given time, the rate of change of the wave function *V(x,t)* assumes a stationary value

$$\delta\left(\frac{dV}{dt}\right) = 0 \tag{3}$$

Introduce the total variation of the rate of change of the wave function:

$$\Delta\left(\frac{dV}{dt}\right) = \delta\left(\frac{dV}{dt}\right) + \frac{d}{dt}\left(\frac{dV}{dt}\right)\Delta t, \quad (\Delta t = dt) \tag{4}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}^T \frac{dx}{dt}$

Let the wave function (V-function) be a finite, twice differentiable function of its arguments, satisfying the equation:

$$\frac{\partial^2 V(x,t)}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial^2 V(x,t)}{\partial x_i \partial x_j} f_i(x) f_j(x) = \sum_{i=1}^{n} \frac{\partial V(x,t)}{\partial x_i} \frac{df_i(x)}{dt}, \tag{5}$$

where $f_i(x)$ are the components of *n*-dimensional vector-function of the right parts of the equations (1) of the object motion.

Theorem I
For the transition into a new state, it is necessary and sufficient the existence of *V*-function, satisfying the condition:

$$\Delta\left(\frac{dV}{dt}\right) = 0. \tag{6}$$

Theorem II.
The motion of the object (1) occurs so that, at each time, the phase velocity vector is co-directional with the gradient of the wave function, i.e.

$$\frac{\partial V}{\partial x}^T f = |\lambda|| \dot{x}|. \tag{6a}$$

3. New formulation of the direct and inverse dynamics problem based on the method of V-function.
A direct dynamics problem can be formulated in the following form:
Let differential equations be given describing the trajectory of the object (1).
It is required to determine a wave function *V(x,t)* satisfying the equation (5)
The initial and boundary conditions (5) are determined from Theorems I, II and from the condition of connectedness of the wave function *V(x,t)* with the trajectory of the object motion (1). The conditions of connectedness provide the initial condition for the wave function:

$$V(x,t)\big|_{t=0} = V(x,0) = 0 \tag{7}$$

and the boundary condition for the wave function:

$$V(x,t)\big|_{x=0} = V(0,t) = 0. \tag{8}$$

Two other conditions follow from Theorems I and II. From the condition of Theorem I

$$\Delta\left(\frac{dV}{dt}\right) = \frac{d}{dt}\left(\frac{\partial V}{\partial x}^{\mathrm{T}} \delta x\right) + \frac{d}{dt}\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}^{\mathrm{T}} \dot{x}\right) dt = 0, \tag{9}$$

taking into account

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x}^T \delta x\right) = \frac{d}{dt}\left(\frac{\partial V}{\partial x}^T \dot{x}\varepsilon\right) = 0 \Rightarrow \frac{\partial V}{\partial x}^T \dot{x} = const. \tag{10}$$

we derive the equality:

$$\frac{d}{dt}\left(\frac{\partial V}{\partial t}\right) = 0 \tag{11}$$

Hence we obtain:

$$\frac{\partial V}{\partial t} = const \tag{12}$$

Then the second initial condition for the equation (5) will have the form:

$$\left.\frac{\partial V(x,t)}{\partial t}\right|_{t=0} = \frac{\partial V(x,0)}{\partial t} = const \tag{13}$$

From the condition of Theorem II, it follows:

$$\frac{\partial V}{\partial x} = k^{-1}\dot{x}, \tag{14}$$

Hence, the second boundary condition is obtained:

$$\left.\frac{\partial V(x,t)}{\partial x}\right|_{x=0} = \frac{\partial V(0,t)}{\partial x} = k^{-1}\dot{x}(t) = k^{-1} f(x=0). \tag{15}$$

For the case of $\dot{x} = \vartheta$ (n=1), we obtain the solution of the equation (5) in view of (7)–(8), (13), (15) in the following form:

$$V(x,t) = \pm Ae^{\pm i(\frac{\omega}{\vartheta}x - \omega t)} \tag{16}$$

The inverse problem of dynamics on the basis of the V-function method is stated as follows:

For a given wave function *V(x,t)* satisfying the equation (5) which we write in the following form:

$$\frac{\partial^2 V}{\partial t^2} - \dot{x}^T W \dot{x} = \frac{\partial V^T}{\partial x}\frac{d\dot{x}}{dt}, \quad W = \left[\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\right], \tag{17}$$

it is required to determine the differential equations (1) of the object motion.

For a given wave function, the solution of the inverse problem of dynamics immediately follows from (6a):

$$\dot{x}_i = k\frac{\partial V}{\partial x_i} \tag{18}$$

Using (10) and Theorem II, it can be shown that the following is true:

$$\frac{\partial V^T}{\partial x}\frac{d}{dt}\dot{x} = 0 \tag{19}$$

From (19), taking into consideration (18), we get the following equality for *k=1*:

$$\frac{1}{2}\dot{s}^2 = \frac{1}{2}\dot{x}^T\dot{x} = \frac{1}{2}\sum_{i=1}^{n}\dot{x}_i^{\,2} = c_1 \tag{20}$$

Solving the inverse problem, we not only have obtained the equation of motion (18), the right-hand sides of which depend on the way of defining the V-function, but also an access to the foundations of H. Hertz's mechanics, which follows from (20).
The principle of straightest path, as H. Herts [12] showed, is more general in the sense that there follow from it the integral energy principles, the principle of least curvature and the least time principle. Also, it follows from (20) and (10) that H. Hertz's principle is contained in our principle.

Besides, if the condition $\dot{x}_j = \lambda_i \dfrac{\partial \dot{x}_i}{\partial x_j}$ $(i, j = \overline{1, n})$ is fulfilled, then the equation (17), in view of (18), assumes the form:

$$\frac{\partial^2 V}{\partial t^2} - \vartheta^2 \nabla^2 V = 0, \tag{21}$$

where $\vartheta^2 = \sum_{i=1}^{n} \dot{x}_i^{\,2} = \dot{x}^T \dot{x}$ and $\nabla^2 = \sum_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$.

For the one-dimensional case (*n=1*), the equation (17), taking into account (19), assumes the form:

$$\frac{\partial^2 V(x,t)}{\partial t^2} - \frac{\partial^2 V(x,t)}{\partial x^2} \dot{x}^2 = 0 \tag{21a}$$

Let the wave function be given in the form of the equation (16) of a plane wave which disseminates in the direction of the object motion. Then (16) will satisfy (21a), provided $\dot{x} = \vartheta$.
In addition, from the equation (12) where the wave function is given in the form (16), it follows that

$$\frac{\partial V(x,t)}{\partial t} = \mp iA\omega e^{\pm i(\frac{\omega}{\vartheta}x - \omega t)} = const. \tag{22}$$

The constant in the right-hand side of (22) is a real number. Therefore, in order to satisfy the condition (22), the phase should assume the value:

$$\varphi = \left(\frac{\omega}{\vartheta}x - \omega t\right) = \omega\left(\frac{x}{\vartheta} - t\right) = \frac{\pi}{2} + \pi n, \ (n=0,1,2,3....) \tag{23}$$

Since $\dot{x} = \vartheta \Rightarrow \dfrac{x}{\vartheta} - t = C$, the equation (23) takes the form:

$$\omega C = \frac{\pi}{2} + \pi n \Rightarrow \omega = \frac{\pi}{2C}(1+2n) = \frac{\omega_0}{2}(1+2n), \tag{24}$$

i. e. in the solution (16), the natural frequencies may assume only certain discrete values. Then (22), in view of (23) and (24), will assume the form:

$$A\omega = A\frac{\omega_0}{2}(1+2n) = const \tag{25}$$

This means that, there are only discrete values in the equation (22), and the object motion will proceed along the direction of movement of the wave front.

Moreover, from the equation (14), taking into consideration (16), it follows

$$\frac{\partial V(x,t)}{\partial x} = \pm i\frac{A\omega}{\vartheta}e^{\pm i(\frac{\omega}{\vartheta}x-\omega t)} = k^{-1}\dot{x} = k^{-1}\vartheta, \tag{26}$$

Taking into account (22), we obtain

$$\frac{\partial V(x,t)}{\partial x}\vartheta = k^{-1}\vartheta^2 = const. \tag{27}$$

The equality (27) is nothing more than the realization of (10) for $n=1$.

4. Extension of the Optical-Mechanical Analogy

Consider the trajectory motion of a particle satisfying the equation (14) $\dot{x} = k\frac{\partial V}{\partial x}$. The trajectory motion of the particle, as it follows from (21), corresponds to the wave motion that satisfies the wave equation:

$$\frac{\partial^2 V(x,t)}{\partial t^2} - \left(k\frac{\partial V}{\partial x}\right)^2\frac{\partial^2 V(x,t)}{\partial x^2} = 0. \tag{28}$$

The function (16) $V(x,t) = Ae^{i(\frac{\omega}{\vartheta}x-\omega t)}$ will satisfy the equation (28), provided the equality (24) is fulfilled. In this case, we get $|A| = \frac{k^{-1}\vartheta^2}{\omega}$. Suppose that $k^{-1} = m$ is the mass of the particle. Then the amplitude $|A|$ has the dimension of action. If we take $A = \frac{h}{2\pi} = \hbar$, where h is the Planck constant, then there follows from (25) the rule of energy quantization, which is the same as the one considered by Schrödinger in the case of Planck's oscillator. Moreover, by considering (23) we obtain from (26)

$$\frac{\hbar\omega}{\vartheta} = m\vartheta. \tag{29}$$

Using these results, it is possible to indicate the following relations between wave and particle:

$$\vartheta = \vartheta,\ \ \omega = \frac{m\vartheta^2}{\hbar} = \frac{2E}{\hbar},$$
$$\lambda = \frac{h}{m\vartheta},\ \ A = \hbar \tag{30}$$

Moreover, the wave and trajectory measurements can be described by one and the same wave function:

$$V(x,t) = Ae^{\pm i(\frac{\omega}{\vartheta}x - \omega t)} = \hbar e^{\pm i\frac{1}{\hbar}(\frac{\hbar\omega}{\vartheta}x - \hbar\omega t)} = \hbar e^{\pm i\frac{1}{\hbar}(m\vartheta x - Et)} \tag{31}$$

In the relations (30), the basic fact is the equality between the phase velocity of wave and the particle velocity, while in quantum mechanics the particle velocity is equal to the group velocity of Louis de Broglie's waves. The condition (25) of energy quantization is produced naturally as a result of solving the inverse problem.

Consider now in what way the V-function method is related to the optical-mechanical analogy of N.H. Chetaev [13, 14]. To this end, we first show how the local variational principle is connected with the Hamilton principle.

If the Hamilton principle $\delta S = \delta\left(\int\limits_{t0}^{t} Ldt\right) = 0$ determines the state of a mechanical system on a fixed time interval, and the equations of motion derived from this principle determine the state at each time point, then the local principle determines also the wave process within the state.

Suppose that $V(q,t) = e^{i\frac{S(q,t)}{h}}$, where S is the main Hamiltonian function. Then

$$\delta\left(\frac{dV}{dt}\right) = \delta\left(\frac{i}{h}\frac{dS}{dt}e^{\frac{iS}{h}}\right) = \frac{i}{h}\delta\left(\frac{dS}{dt}\right)e^{\frac{iS}{h}} - \frac{1}{h^2}\frac{dS}{dt}\delta S e^{\frac{iS}{h}} = 0\,. \tag{32}$$

Hence, the following equalities should hold separately:

$$\delta\left(\frac{dS}{dt}\right) = 0, \tag{33}$$

$$\delta S = 0\ . \tag{34}$$

As seen from (34), for the representation $V = e^{\frac{iS}{h}}$ the local principle contains the Hamiltonian principle. At the same time, the Hamiltonian action $S = \int\limits_{t0}^{t} Ldt$ should satisfy the additional condition (33).

In his work [13], N.H. Chetaev demonstrated how a stable motion of a holonomic conservative system relates to the wave equation, a mathematical theory of light propagation due to Cauchy. He proceeded from the stability of the equations in variations for the reduced system:

$$\delta\dot{q}_k = \sum_{i,j=1}^{n} \frac{\partial}{\partial q_i}\left(a_{kj} \frac{\partial W}{\partial q_j} \right) \delta q_i . \tag{35}$$

Here the condition of stability is taken in the form:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial q_i}\left(a_{ij} \frac{\partial W}{\partial q_j} \right) = 0 . \tag{36}$$

Using this condition for the function *Φ(Et+W)*, he obtained the wave equation:

$$\frac{2(E+U)}{E^2}\frac{\partial^2 \Phi}{\partial t^2} = \sum_{i,j=1}^{n} \frac{\partial}{\partial q_i}\left(a_{ij} \frac{\partial \Phi}{\partial q_j} \right) \tag{37}$$

where
E is the constant of kinetic energy,
U is the force function of the system.
From the equation (33), we have:

$$\delta\left(\frac{dS}{dt}\right) = \frac{d}{dt}\sum_i \frac{\partial S}{\partial q_i}\delta q_i = \sum_i \frac{d}{dt}\left(\frac{\partial S}{\partial q_i}\right)\delta q_i + \sum_i \frac{\partial S}{\partial q_i}\delta\dot{q}_i =$$
$$= \sum_i \frac{\partial}{\partial t}\left(\frac{\partial S}{\partial q_i}\right)\delta q_i + \sum_{i,j} \frac{\partial}{\partial q_j}\left(\frac{\partial S}{\partial q_i}\right)\dot{q}_j)\delta q_i + \sum_i \frac{\partial S}{\partial q_i}\delta\dot{q}_i = 0. \tag{38}$$

Taking into account (35), and that *S=Et+W*, as well as

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\frac{1}{2}\sum_{ij} a_{ij} p_i p_j - U(q)\right) = \sum_{ij} a_{ij} p_j = \sum_{ij} a_{ij}\frac{\partial W}{\partial q_j}$$

we have from (38), passing to the matrix representation $(A = A^T)$:

$$0 + \left(\frac{\partial}{\partial q}\left(\frac{\partial W}{\partial q}\right)^T A \frac{\partial W}{\partial q}\right)^T + \frac{\partial W}{\partial q}^T A \frac{\partial W}{\partial q}\frac{\partial}{\partial q}^T =$$
$$= 2\frac{\partial W}{\partial q}^T A \frac{\partial W}{\partial q}\frac{\partial}{\partial q}^T = 0 \tag{39}$$

The equation (39) holds provided that:

$$Tr\left(\frac{\partial}{\partial q}\left(A\frac{\partial W}{\partial q}\right)^T \right) = \frac{\partial}{\partial q}^T \left(A\frac{\partial W}{\partial q}\right) = 0 . \tag{40}$$

which coincides with the condition (36) for a stable motion of the system (35).

5. A Harmonic Oscillator

Consider a linear harmonic oscillator. The equation of the trajectory motion of an object (particle)

$$m\ddot{x} = -kx \tag{41}$$

allows the first integral $\frac{m\dot{x}^2}{2} + \frac{kx^2}{2} = E$.

The square of the particle velocity assumes the form

$$\dot{x}^2 = \frac{2E - kx^2}{m}. \tag{42}$$

Substituting (42) into the equation (21) yields:

$$\frac{\partial^2 V}{\partial t^2} - \left(\frac{2E - kx^2}{m}\right)\frac{\partial^2 V}{\partial x^2} = 0 \tag{43}$$

We look for a wave function *V*(*x*,*t*) in the form of $V(x,t) = \psi(x)\varphi(t)$. From the equation (43), the following stationary equation is obtained:

$$\psi'' + \frac{m\omega^2}{2E - kx^2}\psi = 0 \tag{44}$$

The initial conditions (8), (15) for the function $\psi(x)$ have the form:

$$\psi(x)\big|_{x=0} = \psi(0) = 0,\ \ \psi'(x)\big|_{x=0} = \psi'(0) = C_1; \tag{45}$$

As it is seen from the equation (44), $\psi\left(x = \sqrt{\frac{2E}{k}}\right) = 0$. The fulfillment of this condition is only possible for the specific values of the natural frequencies of the equation (44). Let us reduce the equation (44) to the form

$$\psi'' + \frac{\eta^2}{1-\xi^2}\psi = 0, \tag{46}$$

where $\xi = \frac{x}{\sqrt{\frac{2E}{k}}}$ is a dimensionless quantity, $\eta^2 = \frac{m\omega^2}{k} = \frac{\omega^2}{\omega_0^2}$.

Let us determine these frequencies numerically by solving the equation (46) with the initial conditions (45). To this end, we introduce the auxiliary functions

$\psi_1 = \psi$;

$\psi_2 = \psi'$.

and solve the system

$$\begin{cases} \psi_1{}' = \psi_2; \\ \psi_2{}' = \dfrac{\eta^2}{\xi^2 - 1}\psi_1. \end{cases}$$

by the fourth-order Runge-Kutta method with the initial values $\psi_1(0) = 0,\ \psi_2(0) = C_1$. We obtain

$$\eta_1^2 = \frac{\hbar^2\omega_1^2}{\hbar^2\omega_0^2} = 6,\ \eta_2^2 = \frac{\hbar^2\omega_2^2}{\hbar^2\omega_0^2} = 20,\ \eta_3^2 = \frac{\hbar^2\omega_3^2}{\hbar^2\omega_0^2} = 42,\ \eta_4^2 = \frac{\hbar^2\omega_4^2}{\hbar^2\omega_0^2} = 72, \ldots$$

Taking into account the results of the optical-mechanical analogy $2E = \hbar\omega,$ we arrive at the quantization rule of the harmonic oscillator energy in the following form

$$E_{n+2}^2 - 2E_{n+1}^2 + E_n^2 = \Delta\Delta E_n^2 = 2\hbar^2\omega_0^2 \tag{47}$$

Thus, in the case when the trajectory motion of the object is directly connected with the wave motion, the harmonic oscillator energy can assume only certain discrete values: $E_1^2 = 6\hbar^2\omega_0^2,\ E_2^2 = 20\hbar^2\omega_0^2,\ E_3^2 = 42\hbar^2\omega_0^2,\quad E_4^2 = 72\hbar^2\omega_0^2 \ldots$

These values can be also obtained analytically using the Maple software complex

$$\psi(x) = \frac{1}{E}C_1 x(E - \frac{kx^2}{2})\text{hypergeom}\left(\left[\frac{5}{4} + \frac{1}{4}\sqrt{\frac{4m\omega^2}{k} + 1}\right], \left[\frac{5}{4} - \frac{1}{4}\sqrt{\frac{4m\omega^2}{k} + 1}\right], \left[\frac{3}{2}\right], \frac{kx^2}{E}\right)$$

and as a series

$$\psi(x) = \frac{1}{E}C_1 x(E - \frac{kx^2}{2})(1 + \frac{1}{12}\frac{6k - m\omega^2}{E}x^2 + \frac{1}{480}\frac{(6k - m\omega^2)(20k - m\omega^2)}{E^2}x^4 +$$

$$\frac{1}{40320}\frac{(6k - m\omega^2)(20k - m\omega^2)(42k - m\omega^2)}{E^3}x^6 +$$

$$\frac{1}{5806080}\frac{(6k - m\omega^2)(20k - m\omega^2)(42k - m\omega^2)(72k - m\omega^2)}{E^4}x^8 + \ldots)$$

As is known, Schrödinger obtained a rule of energy quantization for the harmonic oscillator in the form $E_n = \left(n + \dfrac{1}{2}\right)\hbar\omega_0$. If these results are substituted into the equation (47), we have $\left(\left(n + 2 + \dfrac{1}{2}\right)^2 - 2\left(n + 1 + \dfrac{1}{2}\right)^2 + \left(n + \dfrac{1}{2}\right)^2\right)\hbar^2\omega_0^2 = 2\hbar^2\omega_0^2$, i.e. we have an identity.

6. Motion of an electron in a hydrogen-like atom

Consider the motion of an object (particle) in a 3-dimensional potential field of forces in the Cartesian coordinate system. Let the trajectory equations of the object (particle) (5) allow the first integral of motion in the form of the energy conservation law of the particle, i.e.

$$\frac{m(\dot{x}^2+\dot{y}^2+\dot{z}^2)}{2}+U(x,y,z)=E, \tag{48}$$

where m is the particle mass, E is the total energy of particle, U is the potential energy of the particle. Then the motion of the object (particle) is completely described by the following system of equations (48) and (21):

$$\begin{cases} \dfrac{m\vartheta^2}{2}+U=E, \\ \dfrac{\partial^2 V}{\partial t^2}-\vartheta^2\nabla^2 V=0. \end{cases} \tag{49}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the Laplace operator, $\vartheta^2=\dot{x}^2+\dot{y}^2+\dot{z}^2$ is the square of the particle velocity. Hence, the second equation, taking into account the first, assumes the form:

$$\frac{\partial^2 V}{\partial t^2}-\frac{2(E-U)}{m}\nabla^2 V=0, \tag{50}$$

We apply the method of separation of variables in the equation (50) $\left(V=X(x,y,z)T(t)\right)$,

$$\frac{\frac{d^2T(t)}{dt^2}}{T(t)}=\frac{2(E-U)\nabla^2 X(x,y,z)}{mX(x,y,z)}=-\omega^2. \tag{51}$$

As a result, we obtain the following stationary equation

$$\frac{2(E-U)}{m}\nabla^2 X+\omega^2 X=0 \tag{52}$$

It is known that the potential energy of a hydrogen-like atom is equal to

$$U(r)=-Ze^2/r \tag{53}$$

In this case, the equation (52) takes the form $\frac{2(E+Ze^2/r)}{m}\nabla^2 X+\omega^2 X=0,$ or

$$\left(-\beta_0^2+\frac{\alpha}{r}\right)\nabla^2 X+\omega^2 X=0, \tag{54}$$

where $\beta_0^2=-\frac{2E}{m},\ \alpha=\frac{2Ze^2}{m}$.

In the equation (54), we pass to the spherical coordinate system:
$$\left(-\beta_0^2+\frac{\alpha}{r}\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]X+\omega^2X=0. \quad (55)$$
where $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}=\nabla^2$ is the Laplace operator in the spherical coordinate system.

Applying again the method of separation of variables $(X=R\Phi\Theta)$, we arrive at the following equation:
$$\frac{d^2\Phi}{d\varphi^2}+k_1^2\Phi=0,$$
$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{k_1^2}{\sin^2\theta}=const=-l(l+1),$$
$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{r^2\omega^2}{-\beta_0^2+\frac{\alpha}{r}}R-l(l+1)R=0. \quad (56)$$
The solutions of the first two equations in (56) are known. It is the third equation that is of interest. If we make a change $R=u/r$, then we get
$$\frac{d^2u}{dr^2}+\left(\frac{k_0^2\alpha}{\alpha-\beta_0^2r}-k_0^2-\frac{l(l+1)}{r^2}\right)u=0, \quad (57)$$
where $k_0^2=\frac{\omega^2}{\beta_0^2}=-\frac{\omega^2m}{2E}$.

The resulting solution of the direct problem of dynamics for the equation (57) must satisfy the natural condition $u(r=r_0)=0$ (where $r_0=\alpha/\beta_0^2=-Ze^2/E=Ze^2/|E|$), which corresponds to the fulfillment of the boundary condition (7), when the wave amplitude becomes zero for $r=r_0$, where, respectively, as a solution of the inverse problem, there arises a trajectory of the particle (electron). Taking into account asymptotic solution of the equation (57) ($r\to\infty$), let us write its general solution in the form $u=c_1u_-(r)+c_2u_+(r)=e^{-k_0r}f_-(r)+e^{k_0r}f_+(r)$. Substituting it into (57), we obtain the following equation:
$$f_\pm''(r)\pm2k_0f_\pm'(r)+\left(\frac{\beta_1}{r_0-r}-\frac{l(l+1)}{r^2}\right)f_\pm(r)=0, \quad (58)$$
where $\beta_1=\mathrm{k}_0^2\alpha/\beta_0^2=\frac{1}{2}\mathrm{Ze}^2\omega^2\mathrm{m_e}/\mathrm{E}^2$.

The solution of the equation (58) for $l=0$ will be looked for as the following power series $f_\pm(r)=\sum_{m=0}^{\infty}a_m^{(\pm)}(r_0-r)^m$. The equation (58) after this substitution takes the form

$$\sum_{m=0}^{\infty} m(m-1)a_m^{(\pm)}(r_0-r)^{m-2} \mp 2k_0\sum_{m=0}^{\infty} m a_m^{(\pm)}(r_0-r)^{m-1} + \beta_1\sum_{m=0}^{\infty} a_m^{(\pm)}(r_0-r)^{m-1} = 0, \;\Rightarrow$$

$$\sum_{n=0}^{\infty}[(n+1)na_{n+1}^{(\pm)} \mp 2k_0 n a_n^{(\pm)} + \beta_1 a_n^{(\pm)}](r_0-r)^{n-1} = 0\,, \tag{59}$$

The equation (59) is identically satisfied only when $r = r_0$ or when all the coefficients of the obtained series are equal to zero, i.e. $(n+1)na_{n+1}^{(\pm)} \mp 2k_0 n a_n^{(\pm)} + \beta_1 a_n^{(\pm)} = 0$. It follows that $a_0 = 0$, and the coefficients $a_{n+1}^{(\pm)}$ satisfy the recurrence relation

$$a_{n+1}^{(\pm)} = \frac{\pm 2k_0 n - \beta_1}{(n+1)n} a_n^{(\pm)}\,, \tag{60}$$

because on the basis of the inverse problem of dynamics, we seek the trajectory of a particle (an electron). Besides, we have to satisfy the relations (15) and (7), which hold under the condition

$$\beta_1 = 2k_0 n \;\; (\beta_1 = \tfrac{1}{2}Ze^2\omega^2 m_e / E^2,\; k_0^2 = -\frac{\omega^2 m}{2E},\; \beta_1 > 0, k_0 > 0). \tag{61}$$

This condition is satisfied only when the series $f_+(r) = \sum_{m=1}^{\infty} a_m^{(+)}(r_0-r)^m$ terminates, i.e. $a_m^{(+)} = 0$ for $m \geq n+1$. This results in the following solution

$$u_{+,n}(r) = C\exp\{k_{0,n}r\}\sum_{m=1}^{n} a_m^{(+)}(r_{0,n}-r)^m\,, \tag{62}$$

where C is a constant,

$$r_{0,n} = 2\hbar^2 n^2/(Ze^2 m_e)\,, \tag{63}$$

is the radius of the *n*-th state of the particle (electron), which is obtained from (61) on the basis of the connection between frequency and energy $2E = \hbar\omega,$ which follows from the optical-mechanical analogy. Also, from the equation (61) which takes the form $E^3/\omega^2 = -\frac{1}{8}Z^2e^4 m_e/n^2$, in view of $\omega^2 = (2\mathrm{E}/\hbar)^2$, we found the value of the energy of the *n*-th state of the particle (electron)

$$E_n = -\frac{Z^2e^4 m_e}{2\hbar^2}\frac{1}{n^2}. \tag{64}$$

Note that the energy of the n-th state is exactly the same as the solution obtained in Bohr's model [15] or on the basis of the stationary Schrödinger equation [16].

7. Discussion and conclusion

The described research indicates that Louis de Broglie's desire to overcome the wave-particle dualism by means of the concept of the pilot-wave finds a justification here through extension of the optical-mechanical analogy which is solved at the level of wave optics. Moreover, the wave function *V(x,t)* is not only somehow connected with the motion of the particle, but directly expresses the motion itself, which always has the wave nature whether it be light or any other object. If the wave function (*V*-function) has the dimension of action ([kg][m/s][m]), then the energy quantization of an object (particle) for the case of uniform rectilinear motion occurs according to the same rule as Schrödinger's rule for the harmonic oscillator. At the same time, it becomes obvious that the resolution of the classical quantum physics is insufficient for the detection of energy quantization in the rectilinear motion with constant velocity.

The *V*-function method for the harmonic oscillator allows establishing the appropriate picture of the energy quantization (64). As it is known, for the actual microscopic oscillators interacting with the light [17], the transitions can take place only between neighboring levels which is completely consistent with our results.

In simulating the electron motion in the Coulomb field, the method of *V*-function allows establishing a rule of energy quantization of a hydrogen-like atom which fully agrees with the classical results of Schrödinger and Bohr.

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