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Differential equations associated with Peters polynomials

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Abstract

In this paper, we derive a family of differential equations associated with Peters polynomials from which some new and interesting identities are obtained for those polynomials.

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1. Introduction

The *Peters polynomials* are defined by the generating function

$$\left(1+(1+t)^{\lambda}\right)^{-\mu}(1+t)^{x}=\sum_{n=0}^{\infty}P_n(x;\mu,\lambda)\frac{t^n}{n!}\ \text{(see [2, 3, 4, 9])}.$$

They form the Sheffer sequence for the pair $(g(t)=(1+e^{\lambda t})^{\mu},\ f(t)=e^{t}-1)$. When $\mu=1$, they are called Boole polynomials.

In [5, 6], the authors developed a new method for obtaining some new and interesting identities related to Bernoulli numbers of the second kind and Frobenius-Euler numbers of higher order arising nonlinear differential equations. This method of using differential equations turned out to be very useful tools for studying combinatorics of special polynomials and mathematical physics (see [1, 5, 6, 7, 8]).

In this paper, we derive a family of differential equations having the generating function for Peters polynomials as solutions. Then from the differential equations we obtain some new and interesting identities for those polynomials .

2. Some identities of Peters polynomials arising from linear differential equations

Throughout this paper, all derivatives will be taken with respect to t. Let

$$F=F(t;x,\mu,\lambda)=(1+(1+t)^{\lambda})^{-\mu}(1+t)^{x}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned}
F^{(1)}(t;x,\mu,\lambda)&=\frac{d}{dt}F(t;x,\mu,\lambda)\\
&=-\mu(1+(1+t)^{\lambda})^{-\mu-1}\lambda(1+t)^{\lambda-1}(1+t)^{x}+\left(1+(1+t)^{\lambda}\right)^{-\mu}x(1+t)^{x-1}\\
&=-\lambda\mu(1+(1+t)^{\lambda})^{-\mu-1}(1+t)^{x+\lambda-1}+x\left(1+(1+t)^{\lambda}\right)^{-\mu}(1+t)^{x-1}\\
&=-\lambda\mu F(t;x+\lambda-1,\mu+1,\lambda)+xF(t;x-1,\mu,\lambda),
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
F^{(2)}(t;x,\mu,\lambda)&=\frac{d}{dt}F^{(1)}(t;x,\mu,\lambda)\\
&=-\lambda\mu F^{(1)}(t;x+\lambda-1,\mu+1,\lambda)+xF^{(1)}(t;x-1,\mu,\lambda)\\
&=-\lambda\mu\left(-\lambda(\mu+1)F(t;x+2\lambda-2,\mu+2,\lambda)+(x+\lambda-1)F(t;x+\lambda-2,\mu+1,\lambda)\right)\\
&=(-\lambda)^2<\mu>_2 F(t;x+2\lambda-2,\mu+2,\lambda)-\lambda\mu(2x+\lambda-1)F(t;x+\lambda-2,\mu+1,\lambda)\\
&\quad+(x)_2F(t;x-2,\mu,\lambda),
\end{aligned} \tag{2.3}$$

where $< x >_0 = 1$, $(x)_0 = 1$, $< x >_n = x(x+1)\cdots(x+n-1)$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

Similarly to the above, from (2.3) we obtain

$$\begin{aligned} &F^{(3)}(t; x, \mu, \lambda) \\ &= (-\lambda)^3 < \mu >_3 F(t; x+3\lambda-3, \mu+3, \lambda) \\ &\ + (-\lambda)^2 < \mu >_2 (3x+3\lambda-3)F(t; x+2\lambda-3, \mu+2, \lambda) \\ &\ + (-\lambda)\mu\big(3x^2+3(\lambda-2)x+(\lambda-1)(\lambda-2)\big)F(t; x+\lambda-3, \mu+1, \lambda) \\ &\ + (x)_3 F(t; x-3, \mu, \lambda). \end{aligned} \tag{2.4}$$

From these observations, we are led to put

$$\begin{aligned} F^{(N)} = F^{(N)}(t; x, \mu, \lambda) &= \left(\frac{d}{dt}\right)^N F(t; x, \mu, \lambda) \\ &= \sum_{i=0}^{N} (-\lambda)^i < \mu >_i a_i(N; x, \lambda) F(t; x+i\lambda-N, \mu+i, \lambda), \end{aligned} \tag{2.5}$$

where $N = 0, 1, 2, \ldots$.

Taking the derivative of (2.5) with respect to t, we have

$$\begin{aligned} &F^{(N+1)}(t; x, \mu, \lambda) = \sum_{i=0}^{N} (-\lambda)^i < \mu >_i a_i(N; x, \lambda) F^{(1)}(t; x+i\lambda-N, \mu+i, \lambda) \\ &= \sum_{i=0}^{N} (-\lambda)^i < \mu >_i a_i(N; x, \lambda) \, (-\lambda(\mu+i)F(t; x+(i+1)\lambda-N-1, \mu+i+1, \lambda) \\ &\quad +(x+i\lambda-N)F(t; x+i\lambda-N-1, \mu+i, \lambda)) \\ &= \sum_{i=0}^{N} (-\lambda)^{i+1} < \mu >_{i+1} a_i(N; x, \lambda) F(t; x+(i+1)\lambda-N-1, \mu+i+1, \lambda) \\ &\quad + \sum_{i=0}^{N} (-\lambda)^i < \mu >_i (x+i\lambda-N) a_i(N; x, \lambda) F(t; x+i\lambda-N-1, \mu+i, \lambda) \\ &= \sum_{i=1}^{N+1} (-\lambda)^i < \mu >_i a_{i-1}(N; x, \lambda) F(t; x+i\lambda-N-1, \mu+i, \lambda) \\ &\quad + \sum_{i=0}^{N} (-\lambda)^i < \mu >_i (x+i\lambda-N) a_i(N; x, \lambda) F(t; x+i\lambda-N-1, \mu+i, \lambda). \end{aligned} \tag{2.6}$$

On the other hand, by replacing N by $N+1$ in (2.5), we get

$$\begin{aligned} & F^{(N+1)}(t;x,\mu,\lambda) \\ =& \sum_{i=0}^{N+1} (-\lambda)^i < \mu >_i a_i(N+1;x,\lambda) F(t;x+i\lambda-N-1,\mu+i,\lambda). \end{aligned} \tag{2.7}$$

Comparing the coefficients on both sides of (2.6) and (2.7), we have

$$a_0(N+1;x,\lambda) = (x-N)a_0(N;x,\lambda), \tag{2.8}$$

$$a_{N+1}(N+1;x,\lambda) = a_N(N;x,\lambda), \tag{2.9}$$

and

$$a_i(N+1;x,\lambda) = a_{i-1}(N;x,\lambda) + (x+i\lambda-N)a_i(N;x,\lambda), \tag{2.10}$$

where $1 \leq i \leq N$.

Also, from $F = F^{(0)}(t;x,\mu,\lambda) = a_0(0;x,\lambda)F(t;x,\mu,\lambda)$, we have

$$a_0(0;x,\lambda) = 1. \tag{2.11}$$

Now, by (2.8), we get

$$\begin{aligned} a_0(N+1,x,\lambda) =& (x-N)a_0(N;x,\lambda) = (x-N)(x-N+1)a_0(N-1;x,\lambda) \\ =& \cdots \\ =& (x-N)(x-N+1)\cdots x a_0(0;x,\lambda) \\ =& < x-N >_{N+1} . \end{aligned} \tag{2.12}$$

From (2.9), we note that

$$a_{N+1}(N+1;x,\lambda) = a_N(N;x,\lambda) = \cdots = a_1(1;x,\lambda) = a_0(0;x,\lambda) = 1. \tag{2.13}$$

For $i = 1$ in (2.10), we have

$$\begin{aligned}
& a_1(N+1;x,\lambda) = a_0(N;x,\lambda) + (x+\lambda-N)a_1(N;x,\lambda) \\
=& a_0(N;x,\lambda) + (x+\lambda-N)\left\{a_0(N-1;x,\lambda) + (x+\lambda-N+1)a_1(N-1;x,\lambda)\right\} \\
=& (a_0(N;x,\lambda) + (x+\lambda-N)a_0(N-1;x,\lambda)) + <x+\lambda-N>_2 a_1(N-1;x,\lambda) \\
=& (a_0(N;x,\lambda) + (x+\lambda-N)a_0(N-1;x,\lambda)) \\
& + <x+\lambda-N>_2 (a_0(N-2;x,\lambda) + (x+\lambda-N+2)a_1(N-2,x,\lambda)) \\
=& (a_0(N;x,\lambda) + (x+\lambda-N)a_0(N-1;x,\lambda) + <x+\lambda-N>_2 a_0(N-2;x,\lambda)) \\
& + <x+\lambda-N>_3 a_1(N-2,x,\lambda) \\
=& \cdots \\
=& \sum_{k=0}^{N-1} <x+\lambda-N>_k a_0(N-k;x,\lambda) + <x+\lambda-N>_N a_1(1;x,\lambda) \\
=& \sum_{k=0}^{N} <x+\lambda-N>_k a_0(N-k;x,\lambda).
\end{aligned} \tag{2.14}$$

Analogously to the case of $i = 1$, for $i = 2$ and $i = 3$, we obatin

$$a_2(N+1;x,\lambda) = \sum_{k=0}^{N-1} <x+2\lambda-N>_k a_1(N-k;x,\lambda), \tag{2.15}$$

$$a_3(N+1;x,\lambda) = \sum_{k=0}^{N-2} <x+3\lambda-N>_k a_2(N-k;x,\lambda). \tag{2.16}$$

So, we deduce that, for $1 \leq i \leq N$,

$$a_i(N+1;x,\lambda) = \sum_{k=0}^{N-i+1} <x+i\lambda-N>_k a_{i-1}(N-k;x,\lambda). \tag{2.17}$$

3. Explicit expressions

From (2.14), we have

$$\begin{aligned}
a_1(N+1;x,\lambda) &= \sum_{k_1=0}^{N} <x+\lambda-N>_{k_1} a_0(N-k_1;x,\lambda) \\
&= \sum_{k_1=0}^{N} <x+\lambda-N>_{k_1} <x-N+k_1+1>_{N-k_1},
\end{aligned} \tag{3.18}$$

In the same fashion, we can show that

$$a_2(N+1;x,\lambda)=\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-1-k_2} <x+2\lambda-N>_{k_2}<x+\lambda-N+k_2+1>_{k_1} \times <x-N+k_2+k_1+2>_{N-k_1-k_2-1}. \tag{3.19}$$

and

$$\begin{aligned}&a_3(N+1;x,\lambda)\\&=\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2} <x+3\lambda-N>_{k_3}<x+2\lambda-N+k_3+1>_{k_2}\\&\times <x+\lambda-N+k_3+k_2+2>_{k_1}<x-N+k_3+k_2+k_1+3>_{N-k_3-k_2-k_1-2}.\end{aligned} \tag{3.20}$$

Thus we have, for $1\le i\le N$, that

$$\begin{aligned}a_i(N+1;x,\lambda)=&\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-i+1-k_i}\cdots\sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2}\\&\times\prod_{l=1}^{i}\langle x+l\lambda-N+\sum_{j=l+1}^{i}k_j+i-l\rangle_{k_l}\langle x-N+\sum_{j=1}^{i}k_j+i\rangle_{N-i+1-\sum_{j=1}^{i}k_j}.\end{aligned} \tag{3.21}$$

Remark. (3.21) holds also for $i=N+1$.

Altogether, we obtain the following theorem.

Theorem 3.1. The following family of differential equations

$$\begin{aligned}F^{(N)}=&(1+t)^{-N}\sum_{i=0}^{N}(-\lambda)^i <\mu>_i a_i(N;x,\lambda)(1+(1+t)^{\lambda})^{-i}(1+t)^{i\lambda}F\\=&(1+t)^{-N}\sum_{i=0}^{N}(-\lambda)^i <\mu>_i a_i(N;x,\lambda)(1+(1+t)^{-\lambda})^{-i}F.\end{aligned}$$

have a solution

$$F=F(t;x,\mu,\lambda)=\left(1+(1+t)^{\lambda}\right)^{-\mu}(1+t)^x,$$

where $a_0(N;x,\lambda)=<x-N+1>_N$,

$$\begin{aligned}a_i(N;x,\lambda)=&\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-i-k_i}\cdots\sum_{k_1=0}^{N-i-k_i-\cdots-k_2}\prod_{l=1}^{i}\langle x+l\lambda-N+\sum_{j=l+1}^{i}k_j+i-l+1\rangle_{k_l}\\&\times\langle x-N+\sum_{j=1}^{i}k_j+i+1\rangle_{N-i-\sum_{j=1}^{i}k_j}.\end{aligned}$$

4. Applications

Recall here that the Peters polynomials $P_n(x)$ are given by the generating function

$$\begin{aligned} F = F(t; x, \mu, \lambda) &= \left(1 + (1+t)^{\lambda}\right)^{-\mu} (1+t)^x \\ &= \sum_{n=0}^{\infty} P_n(x; \mu, \lambda) \frac{t^n}{n!}. \end{aligned} \tag{4.22}$$

Thus, by (4.22), we get

$$\begin{aligned} F^{(N)}(t; x, \mu, \lambda) &= \left(\frac{d}{dx}\right)^N F(t; x, \mu, \lambda) \\ &= \sum_{k=0}^{\infty} P_{n+N}(x; \mu, \lambda) \frac{t^n}{n!}. \end{aligned} \tag{4.23}$$

From Theorem 3.1, we note that

$$\begin{aligned} &F^{(N)}(t; x, \mu, \lambda) \\ &= \sum_{i=0}^{N} (-\lambda)^i < \mu >_i a_i(N; x, \lambda) F(t; x + i\lambda - N, \mu + i, \lambda) \\ &= \sum_{i=0}^{N} (-\lambda)^i < \mu >_i a_i(N; x, \lambda) \sum_{n=0}^{\infty} P_n(x + i\lambda - N; \mu + i, \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^{N} (-\lambda)^i < \mu >_i a_i(N; x, \lambda) P_n(x + i\lambda - N; \mu + i, \lambda) \right\} \frac{t^n}{n!}. \end{aligned} \tag{4.24}$$

Therefore, by comparing the coefficients on both sides of (4.23) and (4.24), we obtain the following theorem.

Theorem 4.1. For $n, N = 0, 1, 2, \ldots,$ we have the following identity

$$P_{n+N}(x; \mu, \lambda) = \sum_{i=0}^{N} (-\lambda)^i < \mu >_i a_i(N; x, \lambda) P_n(x + i\lambda - N; \mu + i, \lambda),$$

where $a_i(N; x, \lambda)$'s are as in Theorem 3.1.

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