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The form of solution of ODEs with variable coefficients by means of the integral and Laplace transform

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Abstract

We have checked the form of solution of ODEs with variable coefficients by means of the integral and Laplace transform, and this kind of article gives some considerations to establish theories in general integral transform.

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1. Introduction

The integral transform method gives a somewhat reasonable tool on solving the nonhomogeneous ODEs and corresponding initial value problems[1, 7]. However, these integral transforms have many difficulties to deal with differential equations with variable coefficients, which makes me do this kind of research. The main difficulty is on the handling of t^2y^2 and there is no suitable solution so far. Consequently, the integral transform method is well applied to solve the forms of $£(ty)$, $£(ty')$ and $£(ty'')$, but is not in the form of $£(t^{(n)}y^{(n)})(n \geq 2)$ at all. This means that integral transform method is not appropriate for solving differential equation with variable coefficients, and this article is tried with the way of overcoming this difficulty. We would like to use the tools of integral transform and integration, and the similarity of existing integral transforms makes us confined the tool to Laplace transform only. The ultimate goal of this kind of researches is the establishment of elaborate theory with respect to general integral transform, and this kind of article gives some considerations to it [2–6, 8–11].

Let me expatiate on the term of t^2y^2. Using $£(ty) = -F'(s)$ and $Y'' = s^2Y - sy(0) - y'(0)$ for $£(y) = Y = F(s)$, we have

$$£(t^2y^2) = -\frac{d}{ds}£(ty'') = -\frac{d}{ds}(-2sY - s^2\frac{dY}{ds} + y(0))$$

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$$= 2Y + 4s\frac{dY}{ds} + s^2\frac{d^2Y}{ds^2}.$$

In here, since the dealing of d^2Y/ds^2 has some difficulties, we would like to use the following equation

$$\int s^2Y''ds = \frac{1}{3}s^3Y'', \quad \int 2sY'ds = s^2Y'.$$

This means that we can handle $Y^{(n)}$ as a constant in $\displaystyle\int f(s)Y^{(n)}\ ds$ if we endure the difference as much as the constant coefficient. The details and limitation in the application of this equation are dealt with in the next section.

2. The form of solution of ODEs with variable coefficients by means of the integral and Laplace transform

If we handle $y^{(n)}$ as a constant, the following equalities are hold.

$$\begin{aligned} &1) \quad \int s^2Y''ds = \frac{1}{3}s^3Y'' \\ &2) \quad \int 2sY'ds = s^2Y' \end{aligned}$$

In the above equations, of course $y^{(n)}$ is a function of s. However, we can handle $y^{(n)}$ as a constant if we endure the difference as much as the constant coefficient. Let us see an example. Let $Y = s^2 + 1$. Then

$$\int s^2Y''ds = \int 2s^2ds = \frac{2}{3}s^3$$

and

$$\frac{1}{3}s^3Y'' = \frac{2}{3}s^3.$$

Thus the above equation 1) holds. Next, let us check the above 2).

$$\int 2sY'ds = \int 4s^2ds = \frac{4}{3}s^3$$

and

$$s^2Y' = 2s^3.$$

In here, there is the difference as much as the constant coefficient. This means that the left-hand side has a solution $c_1 + c_2X$ when the right-hand side has a solution X, for c_1 and c_2 are constants. In here, we specify that these formulas are restrictively applied to in the form of s^n only.

Let us check the above statement some more. We suppose that the solution Y is $c_1 + c_2 X$. Then

$$\int s^2 Y'' ds = \int s^2 c_2 X'' = \frac{1}{3} c_2 X'' s^3 = \frac{1}{3} s^3 c_2 X'' = \frac{1}{3} s^3 Y''.$$

Similarly,

$$\int 2sY' ds = \int 2sc_2 X' ds = c_2 X' s^2 = s^2 c_2 X' = s^2 Y'.$$

This is the same result when the right-hand side has a solution X.

Theorem 2.1. Let us denote $£(y) = Y = F(s)$, $£(y') = Y'$ and $£(y'') = Y''$. Then Euler-Cauchy equation $t^2y'' + aty' + by = 0$, Legendre's equation $y'' - t^2y'' - 2ty' + n(n+1)y = 0$ and Bessel equation $t^2y'' + ty' + (t^2 - \nu^2)y = 0$ can be represented by

$$\begin{aligned}
&(\frac{1}{3}s^4 + \frac{4-a}{2}s^2 + b - a + 2)\ Y = \frac{1}{3}y(0)s^3 + \frac{1}{3}y'(0)s^2 + \frac{4-a}{2}y(0)s,\\
&(\frac{1}{3}s^4 + \frac{2}{3}s^2 - n(n+1))\ Y = \frac{1}{3}y(0)s^3 + \frac{1}{3}y'(0)s^2 + \frac{1}{2}y(0)s - y'(0),\\
\textit{and}\quad &(\frac{1}{3}s^4 + \frac{5}{2}s^2 + 1 - \nu^2)\ Y = \frac{1}{3}y(0)s^3 + \frac{1}{3}y'(0)s^2 + \frac{5}{2}y(0)s + y'(0),
\end{aligned}$$

respectively. Putting

$$Y_1 = \frac{d}{ds}Y,$$

we have the solution y as $y = £^{-1}(Y_1)$.

Proof. Taking Laplace transform on the above equations, we have

$$\begin{aligned}
&s^2\frac{d^2Y}{ds^2} + (4-a)s\frac{dY}{ds} + (b - a + 2)Y = 0,\\
&s^2\frac{d^2Y}{ds^2} + 2s\frac{dY}{ds} - (s^2 + n(n+1))Y + sy(0) + y'(0) = 0,\\
\textit{and}\quad &(s^2+1)\frac{d^2Y}{ds^2} + 3s\frac{dY}{ds} + (1-\nu^2)Y = 0,
\end{aligned}$$

respectively. Integrating Euler-Cauchy equation with respect to s, we have

$$\frac{1}{3}s^3Y'' + \frac{4-a}{2}s^2Y' + (b-a+2)Ys = 0.$$

Since $Y' = sY - y(0)$ and $Y'' = s^2Y - sy(0) - y'(0)$, we have

$$\frac{1}{3}s^3(s^2Y - sy(0) - y'(0)) + \frac{4-a}{2}s^2(sY - y(0)) + (b-a+2)Ys = 0.$$

Organizing this equality, we have

$$Y = \frac{\frac{1}{3}y(0)s^3 + \frac{1}{3}y'(0)s^2 + \frac{4-a}{2}y(0)s}{\frac{1}{3}s^4 + \frac{4-a}{2}s^2 + b - a + 2}.$$

Putting

$$Y_1 = \frac{d}{ds}Y,$$

we have the solution y as $y = \pounds^{-1}(Y_1)$. For a given number ν, we get the above results with the similar way. ■

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