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Stability and bifurcation synthesis in a nonlinear chemostat model

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Abstract

This study focuses on a wastewater treatment problem. It explores the equilibria and their stability providing hence conditions for the local and global stability. Our aim is to provide a qualitative study of the stability with respect to three parameters: residence time τ, rate of air/liquid oxygen transfer K_La and the dissolved oxygen saturation coefficient C_s. The study analyzes all situations that may occur and establishes a synthesis of bifurcations with diagrams showing our results. The results reveal that the no-washout equilibrium can be reached without the need to increase the residence time, by means of an adequate choice of K_La and C_s.

AMS subject classification:
Keywords: Dynamical system, Stability, Bifurcation, Wastewater treatment.

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1. Introduction

This study investigates a model for wastewater treatment, namely, the activated sludge process. The working principle can be briefly described as follows, see for instance [1, 2, 3, 4, 5, 6, 7]: the influent is fed into an aerator. At the first stage, a bacteria population degrades the pollutant in the biological oxidation of the substrates. The reaction makes an aerobic environment by consuming the oxygen. In a second phase the mixture is forwarded to a settler tank. Here, due to gravity, the solid components settle and concentrate at the bottom. Part of the sludge containing bacteria biomass is recycled into the aerator to stimulate the oxidation.

The model adopted in this study is inspired from [2], (see also [1, 3, 4, 7, 8, 9] for substrate and bacteria evolutions and [3, 4, 9] for oxygen evolution). It involves substrate, bacteria and oxygen dynamics. As mostly reported in the literature, see [1, 3, 4, 9], the specific growth kinetic is assumed to depend on both substrates and oxygen states, and follows the modified Monod law. A bacteria death rate is considered.

After exploring the equilibria and their stability, it appears that there always exists a trivial equilibrium with zero biomass of bacteria (washout equilibrium), whereas, under some conditions on parameters, there exists an equilibrium with non null components, (no-washout equilibrium). We exhibit suitable conditions for local asymptotic stability of the no-washout equilibrium and show that if those conditions are not fulfilled then the trivial equilibrium is locally asymptotically stable (l.a.s.).

An other aim of this work concerns the bifurcation behavior of the system. Although the wastewater treatment has been the subject of several mathematical studies, where different aspects are investigated: dynamic behavior [5, 8], control [2, 10, 11], the anaerobic case was often investigated and the bifurcation is analyzed for one parameter, namely, the residence time, see for instance [8]. The feature of this work is to provide a synthesis of the change of the stability, occurring if three parameters are changed, namely, the residence time (τ), the air/liquid oxygen transfer coefficient ($K_L a$) and the dissolved oxygen saturation coefficient (C_s).

Hence, we firstly provide the condition under which transcritical bifurcation occurs with respect residence time. Secondly, we establish that by fixing C_s and varying $K_L a$ with respect to τ or inversely by fixing $K_L a$ and varying C_s with respect to τ. According to the values of the residence time, a transcritical bifurcation occurs for the values of $K_L a$ and C_s that cross some critical values $K_L^{cr} a$ and C_s^{cr}. This result may be useful since we can achieve the no-washout equilibrium without being forced to choose τ too large, by means of an adequate choice of $K_L a$ and C_s with respect to the critical values, while τ which is too large leads to more energy consumption. Finally, some computer simulations to illustrate our results are introduced.

2. Mathematical model

The mathematical model is formulated as a nonlinear dynamical system. Three phenomena are considered: reaction kinetics in the aerator linked to microbial growth, substrate degradation and finally the consumption of oxygen. By introducing dimensionless variables, the mass equilibrium of the various components around the aerator leads to

$$\frac{dS}{dt} = \frac{1}{\tau}(1-S) - X\frac{S}{K_s+S}\frac{C}{K_c+C}, \tag{2.1}$$

$$\frac{dX}{dt} = \frac{1}{\tau}X(R-1) + X\frac{S}{K_s+S}\frac{C}{K_c+C} - K_d X, \tag{2.2}$$

$$\frac{dC}{dt} = -X\frac{1}{\alpha_0}\frac{S}{K_s+S}\frac{C}{K_c+C} + \frac{1}{\tau}(1-C) + K_L a(C_s - C), \tag{2.3}$$

where τ denotes the residence time, S, X and C denote respectively, the concentration of the substrate species, the microorganisms, and the oxygen.

The parameters are R, the recycle concentration rate for the settling unit; K_d, the death coefficient; K_s, the substrate saturation coefficient; K_c, the oxygen saturation coefficient; $K_L a$, the oxygen transfer coefficient; C_s, the dissolved oxygen saturation concentration; α_0, the yield factor. As used in several papers, the growth of microorganism and removal of substrate and oxygen is modeled, using the modified Monod function,

$$\mu(S, C) = \frac{S}{K_s+S}\frac{C}{K_c+C}.$$

See ([1, 4, 9]) and section "Microbial Growth on Multiple Substrates" in ([3]), we can also see the preprint of Katebi ([12]), who proposes a design of software to control the wastewater process using a model based on this modified Monod kinetic.

Assumptions

Two hypotheses have been formulated for the purpose of this study

H_1- The parameters τ, K_s, K_c, K_d, $K_L a$, C_s are positive constants and $0 \leq R < 1$.

H_2- $K_d < \inf\{\frac{C_s}{(K_s+1)(K_c+C_s)}; \frac{1}{(K_s+1)(K_c+1)}\}$.

The hypothesis H_1 means that we model two situations, the treatment with imperfect recycle ($0 < R < 1$) and without recycle ($R = 0$). The case of perfect recycle ($R = 1$) is not considered, since generally not all sludge is recycled to aerator, see ([8]) and for the treatment of this case.

The second assumption means that $\mu(1, C_s) > K_d$. Recall that the substrate concentration in the feed stream S_{in} in the dimensionless context is equal to 1. Hence, H_2 translates the fact that the growth of bacteria at the substrate concentration in the feed stream level, must be upper than the death coefficient K_d to guarantee the growth of the bacteria, otherwise the population goes to extinction.

3. Equilibria and local stability

In [6], we proved that the system is positively invariant and that all trajectories starting in the positive octant are uniformly bounded in it. Our goal in this section, is to identify the equilibria together with their local stability.

3.1. Equilibria

To simplify the analysis, we define the parameters

$$\beta_1 := \alpha_0(1+\tau K_L a), \tag{3.4}$$

$$\beta_2 := 1-\alpha_0(1+\tau K_L a C_s), \tag{3.5}$$

$$\beta_3 := K_d + \frac{(1-R)}{\tau}, \tag{3.6}$$

$$T := (1+\tau K_L a C_s)(1-\beta_3(1+K_s)) - (1+\tau K_L a)\beta_3 K_c(K_s+1). \tag{3.7}$$

Proposition 3.1. The system (2.1-2.3) admits always a trivial equilibrium point $P_0 = (1, 0, C_e)$, where

$$C_e := \frac{1-\beta_2}{\beta_1}. \tag{3.8}$$

If $T > 0$, then there exists a second equilibrium point $P_1 = (S^*, X^*, C^*)$ belonging to the interior of Ω, where

$$S^* = \beta_2 + \beta_1 C^*;$$

$$X^* = \frac{1}{\tau\beta_3}(1-S^*)$$

and

$$C^* = \frac{b+\sqrt{\Delta}}{-2a}.$$

with

$$a = (\beta_3 - 1)\beta_1;\, b = (\beta_3-1)\beta_2 + \beta_3(\beta_1 K_c + K_s)$$

and

$$\Delta = [(\beta_3-1)\beta_2 + \beta_3(K_s - \beta_1 K_c)]^2 + 4\beta_1\beta_3 K_s K_c.$$

The proof is given in Appendix A.

3.2. Stability

It emerges from the analysis below, that the interior equilibrium P_1, if it exists, is l.a.s. Otherwise, the boundary equilibrium point P_0 is l.a.s. or instable.

The proofs are given in the Appendix B. For the local stability of P_0, we have

Lemma 3.2. If

1. $\dfrac{C_e}{(1+K_s)(K_c+C_e)} < \beta_3,$
   or equivalently

2. $T < 0$

Then P_0 is l.a.s.

For the l.a. stability of P_1, we have

Lemma 3.3. The following assertions are equivalents

1. P_1 is l.a.s.,
2. $\dfrac{C_e}{(1+K_s)(K_c+C_e)} > \beta_3$,
3. $T > 0$

4. Bifurcation synthesis

Many works were interested by the bifurcations in the wastewater models, but generally, the parameter used to study the bifurcation was the residence time τ, see for instance [8, 13, 14]. In this section, we give a synthesis on the bifurcation conditions on both $K_L a$ and C_s, with respect to τ.

4.1. Bifurcation in residence time

It is useful to notice that by varying the residence time τ, transcritical bifurcation can occur.

Lemma 4.1. Under assumptions $H_1 - H_2$, there exists a critical residence time τ_c such that:
if $\tau < \tau_c$ then P_0 is l.a.s., and P_1 doesn't exist,
if $\tau = \tau_c$ then $P_0 = P_1$,
if $\tau > \tau_c$ then P_1 is l.a.s. and P_0 is unstable.
Consequently the system (2.1-2.3) present a transcritical bifurcation at $\tau = \tau_c$.

See Appendix C for the proof of this lemma.

The figure 1, (resp. 2), illustrates the bifurcation diagram with respect to τ, on the $C-$component, (resp. $S-$component). We observe that if $\tau < \tau_c$ then P_0 is l.a.s., conversely if $\tau > \tau_c$ then P_1 is l.a.s. and if $\tau = \tau_c$ then $P_0 = P_1$.

4.2. Bifurcation in C_s with respect to τ

We fix $K_L a$ and check for the impact of the variation of C_s, with respect to the values of τ, on the change of stability. Three situations appear depending on whether τ is larger, smaller or between two critical values. As seen in lemmas 3.2 and 3.3, the stability of P_0 and P_1 can be handled by evaluating the positivity or vanishing of T.

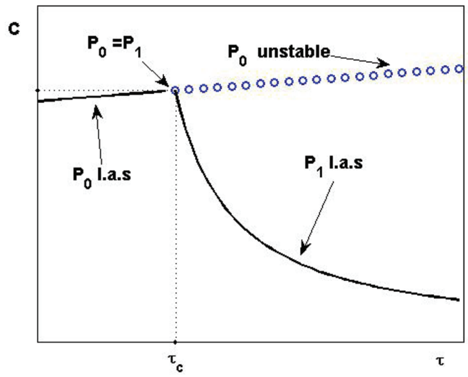

Figure 1: Bifurcation diagram with respect to τ, for C-component.

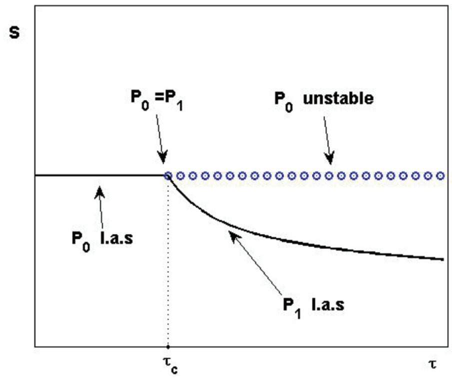

Figure 2: Bifurcation diagram with respect to τ, for S-component.

Consider

$$\tau_1 := \frac{1-R}{\frac{1}{(K_s+1)(K_c+1)} - K_d};$$
$$\tau_0 := \frac{1-R}{\dfrac{1}{K_s+1} - K_d};$$
$$K_0 := \frac{1-\beta_3(1+K_s)(1+K_c)}{\tau\beta_3 K_c(1+K_s)}$$

Remark that under assumptions $H_1 - H_2$,

$$0 < \tau_0 < \tau_1 \ .$$

Case 1: $\tau > \tau_1$
We summarize the situations in the following result

Theorem 4.2. According to value of $K_L a$, we have
If $K_L a > K_0$, there exists

$$C_s^{cr} := \frac{1}{\tau K_L a}\left[\frac{(1+\tau K_L a)\beta_3 K_c(K_s+1)}{1-\beta_3(1+K_s)} - 1\right]$$

such that a transcritical bifurcation occurs as:
if $C_s = C_s^{cr}$ then $P_0 = P_1$,
if $C_s < C_s^{cr}$ then P_0 is l.a.s., and P_1 doesn't exist,
if $C_s > C_s^{cr}$ then P_1 is l.a.s. and P_0 is unstable.
If $K_L a \leq K_0$
then, independently of C_s, P_1 is l.a.s. and P_0 is unstable.

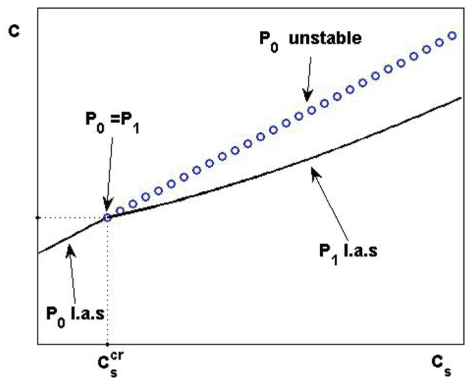

Figure 3: C-component bifurcation diagram with respect to C_s.

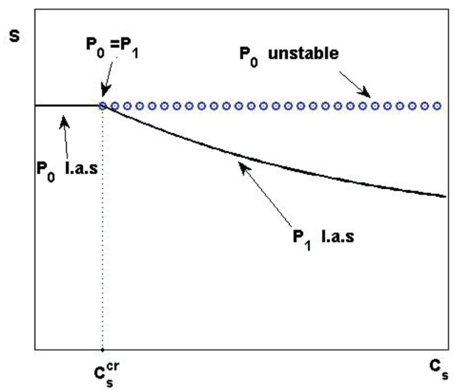

Figure 4: S-component bifurcation diagram with respect to C_s.

The figure 3, (resp. 4), illustrates the bifurcation diagram with respect to C_s, on the $C-$component, (resp. $S-$component), of the equilibria P_0 and P_1 in the case where $K_L a > K_0$ and $\tau > \tau_1$. We observe that if $C_s < C_s^{cr}$ then P_0 is l.a.s., conversely, if $C_s > C_s^{cr}$ then P_1 is l.a.s. and if $C_s = C_s^{cr}$ then $P_0 = P_1$.

Proof. First of all, note that K_0 is positive. Indeed, from $\tau > \tau_1$ we deduce that

$$\frac{1}{(K_s+1)(K_c+1)} > \frac{1-R}{\tau} + K_d,$$

and hence using the expression of β_3 we have

$$1 - \beta_3(1+K_s)(1+K_c) > 0, \tag{4.9}$$

and by the way that $K_0 > 0$.
i) Suppose now that $K_La > K_0$. Since $\beta_3 > 0$, it follows from the inequality (4.9) that

$$1 - \beta_3(1+K_s) > 0. \tag{4.10}$$

Furthermore, we have $K_La > K_0$, i.e.

$$K_La > \frac{1-\beta_3(1+K_s)(1+K_c)}{\tau\beta_3 K_c(1+K_s)}$$

which is equivalent to

$$\beta_3 K_c(1+K_s)(1+\tau K_La) > 1 - \beta_3(1+K_s).$$

Taking account of the inequality (4.10), we deduce that $K_La > K_0$ is equivalent to

$$\frac{(1+\tau K_La)\beta_3 K_c(K_s+1)}{1-\beta_3(1+K_s)} - 1 > 0, \tag{4.11}$$

Hence, C_s^{cr} is well defined and positive. On the other hand, from (3.7)

$$T(C_s,\tau) := T = (1-\beta_3(1+K_s))\left[\tau K_La C_s - \left(\frac{(1+\tau K_La)\beta_3 K_c(K_s+1)}{1-\beta_3(1+K_s)} - 1\right)\right]. \tag{4.12}$$

Therefore, according to lemmas 3.2 and 3.3,
if $C_s > C_s^{cr}$ then $T > 0$, so, P_1 is l.a.s. and P_0 is unstable.
if $C_s < C_s^{cr}$ then $T < 0$, so, P_0 is l.a.s. and P_0 doesn't exists.
if $C_s = C_s^{cr}$ then $T = 0$, so according to (C.43) and the proof given thereafter in the Appendix C, we conclude that $P_0 = P_1$.
ii) Now, if $K_La \leq K_0$ this leads, according to (4.11), that

$$\frac{(1+\tau K_La)\beta_3 K_c(K_s+1)}{1-\beta_3(1+K_s)} - 1 \leq 0,$$

so, from (4.12) and (4.10), we deduce that $T > 0$ and that P_1 is l.a.s. ■

Case 2: $\tau_0 < \tau \leq \tau_1$

Theorem 4.3. Consider
$$C_s^{cr} := \frac{1}{\tau K_L a}\left[\frac{(1+\tau K_L a)\beta_3 K_c(K_s+1)}{1-\beta_3(1+K_s)} - 1\right].$$
Then a transcritical bifurcation occurs as:
if $C_s = C_s^{cr}$ then $P_0 = P_1$,
if $C_s < C_s^{cr}$ then P_0 is l.a.s., and P_1 doesn't exist,
if $C_s > C_s^{cr}$ then P_1 is l.a.s and P_0 is unstable.

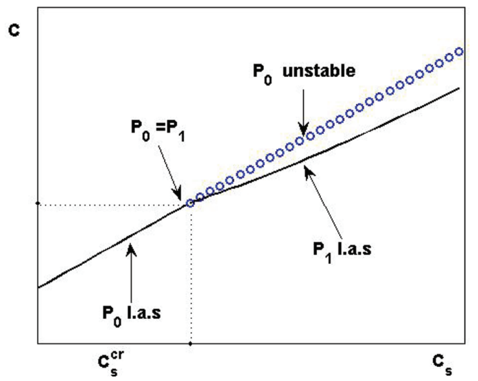

Figure 5: C-component Bifurcation diagram with respect to C_s.

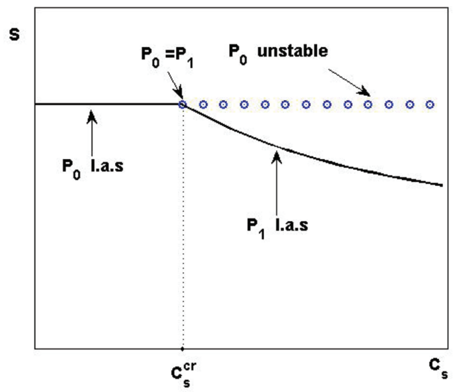

Figure 6: S-component Bifurcation diagram with respect to C_s.

The figure 5, (resp. 6), illustrates the bifurcation diagram with respect to C_s, on the $C-$component, (resp. S-component), of the equilibria P_0 and P_1 in the case where $\tau_0 < \tau \leq \tau_1$. We observe that if $C_s < C_s^{cr}$ then P_0 is l.a.s., conversely if $C_s > C_s^{cr}$ then P_1 is l.a.s. and if $C_s = C_s^{cr}$ then $P_0 = P_1$.

Proof. Recall from (4.12) that
$$T(C_s,\tau) = (1-\beta_3(1+K_s))\left[\tau K_L a C_s - \left(\frac{(1+\tau K_L a)\beta_3 K_c(K_s+1)}{1-\beta_3(1+K_s)} - 1\right)\right] \quad (4.13)$$

Remark that

$$\frac{(1+\tau K_L a)\beta_3 K_c(K_s+1)}{1-\beta_3(1+K_s)} - 1 = \frac{\tau K_L a\beta_3 K_c(K_s+1) + [\beta_3(1+K_s)(1+K_c)-1]}{1-\beta_3(1+K_s)},$$

but since $\tau \le \tau_1$ we have

$$\beta_3(1+K_s)(1+K_c) - 1 \ge 0,$$

and since $\tau > \tau_0$ we know that

$$1-\beta_3(1+K_s) > 0.$$

Hence, for all $K_L a$, we deduce that

$$C_s^{cr} := \frac{1}{\tau K_L a}\left[\frac{(1+\tau K_L a)\beta_3 K_c(K_s+1)}{1-\beta_3(1+K_s)} - 1\right] > 0.$$

By the same arguments of the proof of the previous theorem, applied to $T(C_s, \tau)$, we derive the desired results. ∎

Case 3: $\tau \le \tau_0$

Lemma 4.4. If $\tau \le \tau_0$ then, independently of C_s, P_0 is l.a.s.

Proof. We have $\tau \le \tau_0$ then

$$1-\beta_3(1+K_s) \le 0.$$

Hence (3.7) allows us to say that

$$T < 0$$

and then that P_0 is l.a.s. as required. ∎

4.3. Bifurcation in $K_L a$ with respect to τ

In this third step, we fix C_s and investigate the impact of variation of $K_L a$ with respect to the values of τ, on the change of stability of the model. Consider the functions

$$G_1(\tau) = \beta_3(1+K_s)(1+K_c) - 1 = \left(K_d + \frac{(1-R)}{\tau}\right)(1+K_s)(1+K_c) - 1, \quad (4.14)$$

$$G_2(\tau) = C_s - \beta_3(1+K_s)(C_s+K_c) = C_s - \left(K_d + \frac{(1-R)}{\tau}\right)(1+K_s)(C_s+K_c), \quad (4.15)$$

and the criterion test (3.7)

$$\begin{aligned} T(K_L a, \tau) &:= T, \\ &= K_L a\tau[C_s - \beta_3(1+K_s)(C_s+K_c)] - [\beta_3(1+K_s)(1+K_c) - 1]. \end{aligned}$$

It is clear that

$$T(K_La, \tau) = K_La\tau G_2(\tau) - G_1(\tau). \tag{4.16}$$

Note that

$$G_1(\tau) = 0 \Leftrightarrow \tau = \tau_1 = \frac{1 - R}{\frac{1}{(K_s+1)(K_c+1)} - K_d}. \tag{4.17}$$

$$G_2(\tau) = 0 \Leftrightarrow \tau = \tau_2 := \frac{1 - R}{\frac{C_s}{(K_s+1)(K_c+C_s)} - K_d}. \tag{4.18}$$

Obviously

$$\tau_0 < \inf\{\tau_1, \tau_2\}.$$

Before pursuing, we exclude the case when $\tau \leq \tau_0$.

Lemma 4.5. If $\tau \leq \tau_0$ then, independently of K_La, P_0 is l.a.s.

Proof. In this case, as shown in lemma 4.4, $(1 - \beta_3(1 + K_s)) \leq 0$ holds which leads to

$$T < 0,$$

which leads to desired results. ■

Now if $\tau > \tau_0$, three cases may occur, depending upon the value of C_s. Indeed:

Case 1: $C_s = 1$

Lemma 4.6. Independently of K_La, we have
if $\tau < \tau_1$ then P_0 is l.a.s., and P_1 doesn't exist,
if $\tau = \tau_1$ then $P_0 = P_1$,
if $\tau > \tau_1$ then P_1 is l.a.s. and P_0 is unstable.

Proof. In the case where $C_s = 1$ we have $\tau_1 = \tau_2$ and $G_1(\tau) = -G_2(\tau)$. So,

$$T(K_La, \tau) = (K_La\tau + 1)G_2(\tau).$$

Remark that $G_2(\tau)$ is an increasing function.
If $\tau < \tau_1$ then

$$G_2(\tau) < G_2(\tau_1) = 0.$$

It follows that

$$T < 0,$$

which implies that P_0 is l.a.s.
If $\tau = \tau_1$ then

$$G_2(\tau) = 0.$$

It follows that

$$T = 0,$$

and hence according to equation (C.43) in the Appendix C, we conclude that $P_0 = P_1$.
If $\tau > \tau_1$ then

$$G_2(\tau) > 0.$$

It follows that

$$T > 0,$$

and hence P_1 is l.a.s. ■

Case 2: $C_s < 1$

Theorem 4.7. The following cases hold:
If $\tau \in]\tau_1, \tau_2[$ then there exists

$$K_L^{cr} a := \frac{G_1}{\tau G_2}$$

such that a transcritical bifurcation occurs as:
if $K_L a = K_L^{cr} a$ then $P_0 = P_1$,
if $K_L a < K_L^{cr} a$ then P_1 is l.a.s and P_0 is unstable.
if $K_L a > K_L^{cr} a$ then P_0 is l.a.s.,
If $\tau \in]\tau_0, \tau_1]$ then, independently of $K_L a$, P_0 is l.a.s.,
If $\tau \in [\tau_2, +\infty[$ then, independently of $K_L a$, P_1 is l.a.s.

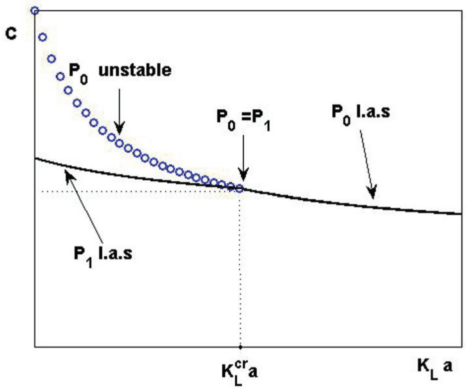

Figure 7: C-component bifurcation diagram with respect to $K_L a$.

The figure 7, (resp. 8), illustrates the bifurcation diagram with respect to $K_L a$, on the C component, (resp. S component), of the equilibria P_0 and P_1 in the case where $\tau \in]\tau_1, \tau_2[$. We observe that if $K_L a < K_L^{cr} a$ then P_1 is l.a.s., conversely if $K_L a > K_L^{cr} a$ then P_0 is l.a.s. and if $K_L a = K_L^{cr} a$ then $P_0 = P_1$.

Proof. First of all, remark that according to the condition $C_s < 1$, it follows that

$$\tau_1 < \tau_2.$$

Suppose that $\tau \in]\tau_1, \tau_2[$. In this case, $G_1(\tau) \neq 0$ and $G_2(\tau) \neq 0$, so from (4.16), we deduce that $K_L^{cr} a$ is well defined and $K_L^{cr} a \neq 0$. So by taking two inequalities $\tau_1 < \tau$ and

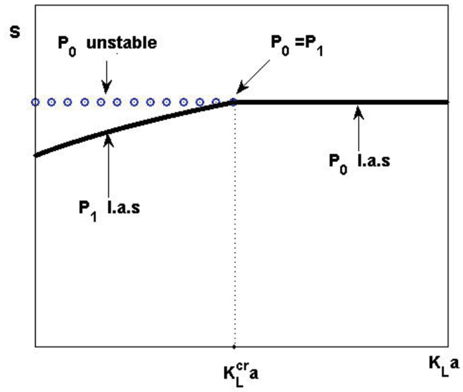

Figure 8: S-component bifurcation diagram with respect to K_La.

$\tau_2 > \tau$ and relations (4.17) and (4.18) and the fact that $G_1(\tau)$ and $G_2(\tau)$ are respectively decreasing and increasing functions, we have

$$G_1(\tau) < 0 \text{ and } G_2(\tau) < 0.$$

So,

$$K_L^{cr}a > 0.$$

If $K_La = K_L^{cr}a$ then according to (4.16), $T = 0$. If $K_La < K_L^{cr}a$ then according to (4.16), $T > 0$. If $K_La > K_L^{cr}a$ then according to (4.16), $T < 0$.
The desired results derived hence.
Now we consider the case $\tau \in]\tau_0, \tau_1]$. By the same way, taking two inequalities $\tau \leq \tau_1$ and $\tau > \tau_0$ and using the monotony of G_1 and G_2 we get that

$$G_1(\tau) \geq 0 \text{ and } G_2(\tau) < 0$$

which leads, according to (4.16), that $T < 0$, so P_0 is l.a.s.
Finally if $\tau \in [\tau_2, +\infty[$ then $G_1(\tau) < 0$ and $G_2(\tau) \geq 0$, so $T > 0$ and hence P_1 is l.a.s.
■

Case 3: $C_s > 1$

Theorem 4.8. The following cases hold:
If $\tau \in]\tau_2, \tau_1[$, there exists

$$K_L^{cr}a := \frac{G_1}{\tau G_2}$$

such that a transcritical bifurcation occurs as
if $K_La = K_L^{cr}a$ then $P_0 = P_1$,
if $K_La < K_L^{cr}a$ then P_0 is l.a.s., and P_1 doesn't exist,
if $K_La > K_L^{cr}a$ then P_1 is l.a.s. and P_0 is unstable.
If $\tau \in]\tau_0, \tau_2]$ then, independently of K_La, P_0 is l.a.s.
If $\tau \in [\tau_1, +\infty[$ then, independently of K_La, P_1 is l.a.s.

Proof. The proof is similar to the proof of Theorem 4.7. ■

5. Conclusion

In this paper, an aerobic model of activated sludge process is studied and equilibria and their local stability are identified. In a second time, the conditions under which bifurcation occurs is investigated. We firstly looked for bifurcation with respect to residence time and, secondly, we provided bifurcation conditions by fixing C_s and varying K_La with respect to τ or inversely by fixing K_La and varying C_s with respect to τ. The results reveal that the change of stability occurs for some critical values $K_L^{cr}a$ and C_s^{cr}. Our hope in the next works, is to study the global behavior of the system together with Hopf bifurcation under more general kinetic laws.

A. Appendix: Equilibria proof

In the system (2.1-2.3), if $\dot{X} = 0$ then either $X = 0$ or $\dfrac{S}{K_s + S}\dfrac{C}{K_c + C} = \beta_3$
Case 1: $X = 0$. In this case, we have $S = 1$ and

$$\begin{aligned} C_e &= \frac{1 + \tau K_L a C_s}{1 + \tau K_L a} \\ &= \frac{1 - \beta_2}{\beta_1} \end{aligned} \tag{A.19}$$

This gives the equilibrium $P_0 = (1, 0, C_e)$.

Case 2:

$$\frac{S}{K_s + S}\frac{C}{K_c + C} = \beta_3. \tag{A.20}$$

In this case we have $0 < \beta_3 < 1$. By substituting the term $\dfrac{S}{K_s + S}\dfrac{C}{K_c + C}$ by β_3 in the first and third equations of (2.1-2.3), we obtain

$$0 = \frac{1}{\tau}(1 - S) - X\beta_3, \tag{A.21}$$

$$0 = -X\frac{1}{\alpha_0}\beta_3 + \frac{1}{\tau}(1 - C) + K_L a(C_s - C). \tag{A.22}$$

Combining the equations (A.21) and (A.22), it follows

$$S = 1 - \alpha_0(1 + \tau K_L a C_s) + \alpha_0(1 + \tau K_L a)C, \tag{A.23}$$

which yields, taking account of the values of β_1 and β_2 given by (3.4-3.5)

$$S = \beta_2 + \beta_1 C. \tag{A.24}$$

In equation (A.20) we substitute S by its value in (A.24). This leads to the following second order equation

$$aC^2 + bC + e = 0$$

with

$$a = (\beta_3 - 1)\beta_1, \tag{A.25}$$

$$e = \beta_3 K_c(\beta_2 + K_s), \tag{A.26}$$

$$b = (\beta_3 - 1)\beta_2 + \beta_3(\beta_1 K_c + K_s). \tag{A.27}$$

By algebra calculus, we deduce that there exist two solutions

$$C_1 = \frac{b + \sqrt{\Delta}}{-2a}, \text{ and } C_2 = \frac{b - \sqrt{\Delta}}{-2a}.$$

where

$$\Delta = [(\beta_3 - 1)\beta_2 + \beta_3(K_s - \beta_1 K_c)]^2 + 4\beta_1\beta_3 K_s K_c > 0. \tag{A.28}$$

if $\underline{\beta_2 > -K_s}$

Since $\beta_3 < 1$ then $-2a > 0$ and hence $\Delta - b^2 = -4ae > 0$. So, $\sqrt{\Delta} > |b|$.
In this case only one solution is positive, that is

$$C^* = \frac{b + \sqrt{\Delta}}{-2a}. \tag{A.29}$$

From (A.24) and (A.29) we have

$$S^* = \frac{\beta_1}{-2a}[\beta_2(1 - \beta_3) + \beta_3(\beta_1 K_c + K_s) + \sqrt{\Delta}]$$

and from (A.21), we deduce that

$$X^* = \frac{1}{\beta_3 \tau}(1 - S^*).$$

S^* is also positive. Indeed, if we consider $\delta := \beta_2(1-\beta_3)+\beta_3(\beta_1 K_c+K_s)$, we compare δ^2 with Δ by using the following relationship

$$\delta^2 - \Delta = 4(1 - \beta_3)\beta_3 K_s(\beta_2 - \beta_1 K_c)$$

- if $-K_s < \beta_2 < \beta_1 K_c$ then $|\delta| < \sqrt{\Delta}$ and hence $S^* > 0$
- otherwise if $\beta_2 > \beta_1 K_c$ then $\delta > 0$ similarly $S^* > 0$.

It remains now to provide conditions under which $X^* > 0$. This fact is equivalent to $S^* < 1$ which is true iff

$$\sqrt{\Delta} < (1 - \beta_3)(2 - \beta_2) - \beta_3(\beta_1 K_c + K_s).$$

Equivalently we have

$$(1-\beta_3)(2-\beta_2) > \beta_3(\beta_1 K_c + K_s), \tag{A.30}$$

and

$$[(1-\beta_3)(2-\beta_2) - \beta_3(\beta_1 K_c + K_s)]^2 > \Delta \tag{A.31}$$

The condition (A.30) is equivalent to

$$\beta_2 < 2 - \frac{\beta_3}{1-\beta_3}(\beta_1 K_c + K_s), \tag{A.32}$$

and the condition (A.31) is equivalent to

$$(1-\beta_2)(1-\beta_3 K_s - \beta_3) > \beta_1\beta_3 K_c(1+K_s). \tag{A.33}$$

In order to fulfill the last condition, we require

$$\beta_3 < \frac{1}{1+K_s}.$$

From this condition (which is necessary) and the condition (A.33) we obtain

$$\beta_2 < 1 - \frac{\beta_1\beta_3 K_c(1+K_s)}{1-\beta_3 K_s - \beta_3}.$$

But

$$1 - \frac{\beta_1\beta_3 K_c(1+K_s)}{1-\beta_3 K_s - \beta_3} < 2 - \frac{\beta_3}{1-\beta_3}(\beta_1 K_c + K_s).$$

By the way (A.32) holds, but since this last inequality is equivalent to (A.30), this fact says that the condition (A.31) implies (A.30).
It follows that the existence condition of solutions is given only by (A.33).
Finally, remark that, by substituting in (A.33), β_1 and β_2 by their values, this condition becomes equivalent to

$$T > 0,$$

where T is given by equation (3.7).

Now if $\beta_2 < -K_s$

In this case $b > 0$ since $(\beta_3 - 1)\beta_2 > 0$. Hence, $\sqrt{\Delta} < b$. Consequently, two positive solutions exist

$$C_1 = \frac{b+\sqrt{\Delta}}{-2a} \text{ and } C_2 = \frac{b-\sqrt{\Delta}}{-2a}$$

which leads to corresponding solutions S_1 and S_2 but it must verify the positivity of

$$S_1 = \frac{\beta_1}{-2a}(\delta + \sqrt{\Delta}), \text{ and } S_2 = \frac{\beta_1}{-2a}(\delta - \sqrt{\Delta}).$$

Using the similar arguments to above, we have

$$\delta^2 - \Delta = 4(1-\beta_3)\beta_3 K_s(\beta_2 - \beta_1 K_c).$$

Since $\beta_2 < -K_s$ then $\beta_2 < 0$, so $\beta_2 - \beta_1 K_c < 0$ and hence $\delta^2 - \Delta < 0$ i.e. $|\delta| < \sqrt{\Delta}$. Then the solution S_1 is positive and the condition of X^* remains the same.

To resume, under the condition $T > 0$ there exists a physically meaningful (no washout) equilibrium $P_1 = (S^*, X^*, C^*)$ with

$$C^* = \frac{b + \sqrt{\Delta}}{-2a},$$

$$S^* = \beta_2 + \beta_1 C^*$$

and

$$X^* = \frac{1}{\tau\beta_3}(1 - S^*).$$

B. Appendix: Stability proof

B.1. Proof of the l.a.s. of P_0: Lemma 3.2

Remark firstly that

$$\begin{aligned} & \frac{C_e}{(1+K_s)(K_c+C_e)} < \beta_3, \\ \Leftrightarrow \quad & T < 0. \end{aligned} \tag{B.34}$$

Indeed, By substituting C_e by its value in equation (A.19), we obtain

$$\frac{1-\beta_2}{(1+K_s)(K_c\beta_1 + 1 - \beta_2)} < \beta_3.$$

But since $1 - \beta_2 > 0$ and $\beta_1 > 0$, we obtain

$$(1-\beta_2)(1-\beta_3(1+K_s)) < \beta_1\beta_3 K_c(K_s+1), \tag{B.35}$$

or equivalently, taking account of (3.7) and values of β_1 and β_2,

$$T < 0.$$

The same arguments can be used to prove that

$$\begin{aligned} & \frac{C_e}{(1+K_s)(K_c+C_e)} = \beta_3, \\ \Leftrightarrow \quad & T = 0. \end{aligned} \tag{B.36}$$

Let us now, prove the l.a. stability of P_0. This fact can be handled by computing the eigenvalues of the Jacobian matrix at P_0 of (2.1-2.3)

$$\det(\lambda I - J(P_0)).$$

The eigenvalues are

$$\lambda_1 = -\frac{1}{\tau} < 0, \tag{B.37}$$

$$\lambda_2 = \frac{C_e}{(1+K_s)(K_c+C_e)} - \beta_3, \tag{B.38}$$

$$\lambda_3 = -\frac{1}{\tau} - K_L a < 0. \tag{B.39}$$

P_0 is l.a.s. iff $\lambda_2 < 0$, or equivalently

$$\frac{C_e}{(1+K_s)(K_c+C_e)} < \beta_3.$$

So, taking account of the equivalence (B.34), we prove the l.a.s. of P_0 as shown in lemma (3.2).

B.2. Proof of the l.a.s. of P_1: Lemma 3.3

Consider the Jacobian matrix at P_1 of (2.1-2.3).

$$\mathbf{J}(P_1) = \begin{pmatrix} \frac{\partial f_1}{\partial S}(P_1) & \frac{\partial f_1}{\partial X}(P_1) & \frac{\partial f_1}{\partial C}(P_1) \\ \frac{\partial f_2}{\partial S}(P_1) & \frac{\partial f_2}{\partial X}(P_1) & \frac{\partial f_2}{\partial C}(P_1) \\ \frac{\partial f_3}{\partial S}(P_1) & \frac{\partial f_3}{\partial X}(P_1) & \frac{\partial f_3}{\partial C}(P_1) \end{pmatrix}$$

with

$$f_1 = \frac{1}{\tau}(1-S) - X\frac{S}{K_s+S}\frac{C}{K_c+C},$$

$$f_2 = \frac{1}{\tau}X(R-1) + X\frac{S}{K_s+S}\frac{C}{K_c+C} - K_d X,$$

$$f_3 = -X\frac{1}{\alpha_0}\frac{S}{K_s+S}\frac{C}{K_c+C} + \frac{1}{\tau}(1-C) + K_L a(C_s - C).$$

Then we give the derivatives

$$\frac{\partial f_1}{\partial X}(P_1) = -\beta_3,\ \frac{\partial f_2}{\partial X}(P_1) = 0 \text{ and } \frac{\partial f_3}{\partial X}(P_1) = \frac{-\beta_3}{\alpha_0}.$$

So the characteristic polynomial is

$$P(\lambda) = \det(\lambda \mathbf{I} - \mathbf{J}(P_1)) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$$

with

$$a_1 = -\frac{\partial f_1}{\partial S} - \frac{\partial f_3}{\partial C}$$

$$a_2 = \frac{\partial f_1}{\partial S}\frac{\partial f_3}{\partial C} + \frac{\beta_3}{\alpha_0}\frac{\partial f_2}{\partial C} + \beta_3\frac{\partial f_2}{\partial S} - \frac{\partial f_1}{\partial C}\frac{\partial f_3}{\partial S}$$

$$a_3 = -\frac{\partial f_1}{\partial S}\frac{\partial f_2}{\partial C}\frac{\beta_3}{\alpha_0} - \frac{\partial f_2}{\partial S}\frac{\partial f_3}{\partial C}\beta_3 + \frac{\beta_3}{\alpha_0}\frac{\partial f_1}{\partial C}\frac{\partial f_2}{\partial S} + \beta_3\frac{\partial f_2}{\partial C}\frac{\partial f_3}{\partial S}$$

We use the Routh-Hurwitz criterion to verify that $a_1 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$, for local stability of P_1. Among references discussing this criterion, we can see [15, 16].

Verification:

We have

$$a_1 = \frac{1}{\tau} + \frac{XK_s\beta_3}{S(K_s+S)} + \frac{XK_c\beta_3}{\alpha_0(K_c+C)C} + \frac{\beta_1}{\alpha_0\tau} > 0.$$

Using standard algebra calculus, we find that a_3 can be written as

$$a_3 = \frac{1}{\alpha_0\tau}\frac{X\beta_3^2K_c}{C(K_c+C)} + \frac{\beta_1}{\alpha_0\tau}\frac{X\beta_3^2K_s}{S(K_s+S)} > 0.$$

Now, we need to prove that $a_1a_2 - a_3 > 0$. We have

$$\begin{aligned} a_2 &= \frac{\partial f_1}{\partial S}\frac{\partial f_3}{\partial C} + \frac{\beta_3}{\alpha_0}\frac{\partial f_2}{\partial C} + \beta_3\frac{\partial f_2}{\partial S} - \frac{\partial f_1}{\partial C}\frac{\partial f_3}{\partial S} \\ &= M_1 + \frac{X\beta_3^2K_c}{\alpha_0 C(K_c+C)} + \frac{X\beta_3^2K_s}{S(K_s+S)} \end{aligned}$$

$$a_1 = \frac{1}{\tau} + \frac{\beta_1}{\alpha_0\tau} + M_2,$$

with

$$\begin{aligned} M_1 &= \frac{1}{\tau}\left(\frac{\beta_1}{\alpha_0\tau} + \frac{X\beta_3K_c}{\alpha_0C(K_c+C)}\right) + \frac{\beta_1}{\alpha_0\tau}\frac{X\beta_3K_s}{S(K_s+S)}, \\ M_2 &= \frac{XK_s\beta_3}{S(K_s+S)} + \frac{XK_c\beta_3}{\alpha_0(K_c+C)C}. \end{aligned}$$

Then

$$\begin{aligned} a_1a_2 - a_3 &= M_1\left(\frac{1}{\tau} + \frac{\beta_1}{\alpha_0\tau} + M_2\right) + \left(\frac{X\beta_3^2K_c}{\alpha_0C(K_c+C)}\right)\left(\frac{\beta_1}{\alpha_0\tau} + M_2\right) \\ &+ \frac{X\beta_3^2K_s}{S(K_s+S)}\left(\frac{1}{\tau} + M_2\right) > 0. \end{aligned}$$

We deduce that all eigenvalues of the Jacobian matrix $J(P_1)$ have negative real part and thereafter that P_1 is l.a.s. if it exists.

C. Appendix: Residence time bifurcation (Proof of the Lemma 4.1)

We give a proof of the transcritical bifurcation on the residence time τ.

- $\tau < \tau_c$: Taking account of the lemma 3.2, we recall that the equilibrium point P_0 is l.a.s. iff

$$\frac{C_e}{(1+K_s)(K_c+C_e)} < \beta_3. \tag{C.40}$$

To carry out τ_c, we substitute C_e by its value given in (3.8), and substituting also β_3 by its value in (3.6), the equivalence becomes: P_0 is l.a. stable iff

$$A\tau^2 + B\tau + D > 0, \tag{C.41}$$

where

$$A = K_La[K_d(K_s+1)(K_c+C_s) - C_s],$$

$$B = K_La(K_s+1)(K_c+C_s)(1-R) + K_d(K_s+1)(K_c+1) - 1$$

and

$$D = (K_s+1)(K_c+1)(1-R).$$

According to hypothesis H_1, H_2, we have $A < 0$ and $D > 0$, it follows that there exist two solutions

$$\tau_p = -\frac{B+\sqrt{\Delta}}{2A} > 0 \text{ and } \tau_n = -\frac{B-\sqrt{\Delta}}{2A} < 0.$$

where $\Delta := B^2 - 4AD$. We conclude that P_0 is l.a. stable iff

$$\tau < \tau_c := \tau_p.$$

- $\tau > \tau_c$: Obviously if $\tau > \tau_c$ by equivalence, we get

$$A\tau^2 + B\tau + D < 0,$$

and hence,

$$\frac{C_e}{(1+K_s)(K_c+C_e)} > \beta_3.$$

Then by lemmas (3.2) and (3.3) we get that P_0 is unstable and P_1 is l.a. stable.

- $\tau = \tau_c$: If $\tau = \tau_c$ then inequality (C.41) becomes

$$A\tau_c^2 + B\tau_c + D = 0.$$

Using the same arguments of the equivalence between (C.40) and (C.41), the last equality is equivalent

$$\frac{C_e(\tau_c)}{(1+K_s)(K_c + C_e(\tau_c))} = \beta_3. \tag{C.42}$$

But as shown in lemma 3.3, the above equality implies that the in the domain D, P_0 is globally stable.
It remains to show that in this case $P_0 = P_1$.

The equivalence (B.36) and the equation (C.42) says that in the case where $\tau = \tau_c$, we have

$$T = 0. \tag{C.43}$$

On the other hand, recall from the equation (A.28) in the Appendix A, that P_1 exists iff $\Delta > 0$. Our hope is to prove that, in the case where $\tau = \tau_c$, we have

$$\sqrt{\Delta} = (1-\beta_3)(2-\beta_2) - \beta_3(\beta_1 K_c + K_s).$$

For this, we calculate $\xi = \Delta - \eta^2$, with $\eta := (1-\beta_3)(2-\beta_2) - \beta_3(\beta_1 K_c + K_s)$, The algebraic calculus gives

$$\xi = 4(1-\beta_3)[(1-\beta_3)(\beta_2 - 1) + \beta_3(\beta_1 K_c + K_s)(1-\beta_2) + \beta_1\beta_3 K_c(\beta_2 + K_s)].$$

Consider now

$$\xi_0 = (1-\beta_3)(\beta_2 - 1) + \beta_3(\beta_1 K_c + K_s)(1-\beta_2) + \beta_1\beta_3 K_c(\beta_2 + K_s),$$

then we have

$$\xi_0 = (1-\beta_2)[\beta_3(K_s+1) - 1] + \beta_1\beta_3 K_c(K_s+1).$$

So, according to equality (C.43), we obtain that $\xi_0 = 0$, and by the way that

$$\sqrt{\Delta} = (1-\beta_3)(2-\beta_2) - \beta_3(\beta_1 K_c + K_s). \tag{C.44}$$

Hence the equilibrium $P_1 = (S^*, X^*, C^*)$ becomes

$$S^* = \frac{\beta_1}{-2a}[\beta_2(1-\beta_3) + \beta_3(\beta_1 K_c + K_s) + \sqrt{\Delta}] = 1, \text{ and } X^* = \frac{1}{\beta_3\tau_c}(1-S^*) = 0.$$

From (A.25), it follows that

$$C^* = \frac{(1-\beta_2)}{\beta_1}.$$

We conclude that $P_1 = P_0$ and that the bifurcation is transcritical.

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