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Differential equations associated with Mittag-Leffler polynomials

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Abstract

In this paper, we consider the Mittag-Leffler polynomials and derive differential equations from the generating function of these polynomials. In addition, we give some new and explicit identities for the Mittag-Leffler polynomials arising from those differential equations.

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1. Introduction

The classical Mittag-Leffler polynomials $g_n(x)$ were introduced by Mittag-Leffler in an investigation of analytic representation of the integrals and invariants of a linear homogeneous differential equation (see [13]). For the summary of basic properties of them, one may refer to the paper of Bateman (see [1, 2]). They are given by ordinary generating function as

$$\sum_{n\geq 0} g_n(x)t^n = \left(\frac{1+t}{1-t}\right)^x, \tag{1}$$

with $|t| < 1$. The first few them are $g_0(x) = 1$, $g_1(x) = 2x$, $g_2(x) = 2x^2$, $g_3(x) = \frac{4}{3}x^3 + \frac{2}{3}x$, $g_4(x) = \frac{2}{3}x^4 + \frac{4}{3}x^2$, $g_5(x) = \frac{4}{15}x^5 + \frac{4}{3}x^3 + \frac{2}{5}x$.

In fact, the Mittag-Leffler polynomials can be expressed in terms of the Gauss hypergeometric function ${}_2F_1$ as

$$g_n(x) = 2x {}_2F_1\left(\begin{array}{c} 1-n, 1-x \\ 2 \end{array} \mid 2\right),$$

for all $n \geq 1$. Following Roman, throughout this paper $M_n(x)$, also called Mittag-Leffler polynomials, are defined by the generating function as

$$\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!} \quad \text{(see [15])}. \tag{2}$$

We note that the Mittag-Lefffler polynomials form the associated Sheffer sequence for

$$f(t) = \frac{e^t - 1}{e^t + 1} \tag{3}$$

and have the following explicit expression

$$M_n(x) = \sum_{k=0}^{n} \binom{n}{k}(n-1)_{n-k}2^k(x)_k, \tag{4}$$

where $(x)_n$ is the falling factorial given by $(x)_n = x(x-1)\cdots(x-n+1)$, for $n \geq 1$, and $(x)_0 = 1$. Also, note that the Mittage-Leffler polynomials are connected with the *Pidduck polynomials* $P_n(x)$ by the expression $P_n(x) = \frac{1}{2}(e^{d/dx} + 1)M_n(x)$, where the Pidduck polynomials are defined by the generating function

$$\frac{1}{1-t}\left(\frac{1+t}{1-t}\right)^x = \sum_{n\geq 0} P_n(x)\frac{t^n}{n!}.$$

There are several interesting researches on Mittag-Leffler polynomials and their properties and generalizations (for instance, see [1, 2, 13, 18, 19]). In particular, a recent study of these polynomials together with a generalization can be found in [19] and an application of the Mittag-Leffler polynomials to an expansion for the Riemann zeta function is discussed in [17].

Recently, D. S. Kim and T. Kim derived linear and nonlinear differential equations from the generating function of many interesting polynomials and obtained many new and interesting identities involving those polynomials, for example, Changhee polynomials, actuarial polynomials, Meixner polynomials of the first kind, Poisson-Charlier polynomials, Laguerre polynomials, Hermite polynomials, and Stirling polynomials (see [3–12, 14, 16]).

The purpose of this paper is to derive differential equations from the generating function of the Mittag-Leffler polynomials and to give some new and explicit identities for these polynomials arising from those differential equations.

2. Some identities for the Mittag-Leffler polynomials

Throughout this paper, all the derivatives will be taken with respect to t. Let

$$F = F(t,x) \quad = \quad \left(\frac{1+t}{1-t}\right)^{x}. \tag{5}$$

Then

$$F^{(1)} \quad = \quad \frac{d}{dt}F = 2x(1-t)^{-2}F(t,x-1), \tag{6}$$

$$F^{(2)} \quad = \quad \left(\frac{d}{dt}\right)^{2}F = 4x(1-t)^{-3}F(t,x-1) + 4(x)_2(1-t)^{-4}F(t,x-2), \tag{7}$$

$$\begin{aligned} F^{(3)} \quad = \quad & \left(\frac{d}{dt}\right)^{3}F = 12x(1-t)^{-4}F(t,x-1) + 24(x)_2(1-t)^{-5}F(t,x-2) \\ & +8(x)_3(1-t)^{-6}F(t,x-3). \end{aligned} \tag{8}$$

From this observation, we are led to put

$$F^{(N)}(t,x) \quad = \quad \left(\frac{d}{dt}\right)^{N}F(t,x) = \sum_{i=1}^{N} a_i(N)(x)_i(1-t)^{-N-i}F(t,x-i), \tag{9}$$

for $N = 1, 2, \cdots$. By taking the derivative with respect to t of (8), we have

$$F^{(N+1)} \quad = \quad \sum_{i=1}^{N}(N+i)a_i(N)(x)_i(1-t)^{-N-1-i}F(t,x-i)$$

$$
\begin{aligned}
&+\sum_{i=1}^{N} a_i(N)(x)_i(1-t)^{-N-i}F^{(1)}(t,x-i)\\
=\ &\sum_{i=1}^{N}(N+i)a_i(N)(x)_i(1-t)^{-N-1-i}F(t,x-i)\\
&+\sum_{i=1}^{N} a_i(N)(x)_i(1-t)^{-N-i}2(x-i)(1-t)^{-2}F(t,x-i-1)\\
=\ &\sum_{i=1}^{N}(N+i)a_i(N)(x)_i(1-t)^{-N-1-i}F(t,x-i)\\
&+\sum_{i=1}^{N} 2a_i(N)(x)_{i+1}(1-t)^{-N-2-i}F(t,x-i-1)\\
=\ &\sum_{i=1}^{N}(N+i)a_i(N)(x)_i(1-t)^{-N-1-i}F(t,x-i)\\
&+\sum_{i=2}^{N+1} 2a_{i-1}(N)(x)_i(1-t)^{-N-1-i}F(t,x-i). \qquad (10)
\end{aligned}
$$

On the other hand, replacing N by $N+1$ in (8), we obtain

$$F^{(N+1)} = \sum_{i=1}^{N+1} a_i(N+1)(x)_i(1-x)^{-N-1-i}F(t,x-i). \qquad (11)$$

Comparing (9) and (10), we immediately get the following recurrence relations:

$$
\begin{aligned}
a_1(N+1) &= (N+1)a_1(N),\\
a_{N+1}(N+1) &= 2a_N(N),\\
a_i(N+1) &= 2a_{i-1}(N)+(N+i)a_i(N), \quad (2 \le i \le N). \qquad (12)
\end{aligned}
$$

Also, we observe that

$$
\begin{aligned}
2x(1-t)^{-2}F(t,x-1) &= F^{(1)}(t,x)\\
&= a_1(1)x(1-t)^{-2}F(t,x-1). \qquad (13)
\end{aligned}
$$

So, we get

$$a_1(1) = 2. \qquad (14)$$

Now, we have

$$
\begin{aligned}
a_1(N+1) &= (N+1)a_1(N) = (N+1)Na_1(N-1)\\
&= \cdots
\end{aligned}
$$

$$\begin{aligned} &= (N+1)N\cdots 2a_1(1) \\ &= 2(N+1)!, \end{aligned} \tag{15}$$

$$\begin{aligned} a_{N+1}(N+1) &= 2a_N(N) = 2^2 a_{N-1}(N-1) \\ &= \cdots \\ &= 2^N a_1(1) = 2^{N=1}. \end{aligned} \tag{16}$$

Next, we consider

$$a_i(N+1) = 2a_{i-1}(N) + (N+i)a_i(N), \quad (2 \le i \le N). \tag{17}$$

For $i = 2$,

$$\begin{aligned} a_2(N+1) &= 2a_1(N) + (N+2)a_2(N) \\ &= 2a_1(N) + (N+2)(2a_1(N-1) + (N+1)a_2(N-1)) \\ &= 2(a_1(N) + (N+2)a_1(N-1)) + (N+2)_2 a_2(N-1) \\ &= 2(a_1(N) + (N+2)a_1(N-1)) \\ &\quad +(N+2)_2(2a_1(N-2) + Na_2(N-2)) \\ &= 2(a_1(N) + (N+2)a_1(N-1) \\ &\quad +(N+2)_2 a_1(N-2)) + (N+2)_3 a_2(N-2) \\ &= \cdots \\ &= 2\sum_{k=0}^{N-2}(N+2)_k a_1(N-k) + (N+2)_{N-1}a_2(2) \\ &= 2\sum_{k=0}^{N-1}(N+2)_k a_1(N-k), \end{aligned} \tag{18}$$

Proceeding analogously to the case of (17), we can easily get the followings.

$$a_3(N+1) = 2\sum_{k=0}^{N-2}(N+3)_k a_2(N-k), \tag{19}$$

$$a_4(N+1) = 2\sum_{k=0}^{N-3}(N+4)_k a_3(N-k). \tag{20}$$

Thus we can deduce that, for $2 \le i \le N$,

$$a_i(N+1) = 2\sum_{k=0}^{N-i+1}(N+i)_k a_{i-1}(N-k). \tag{21}$$

Explicit expressions for $a_i(N+1)$ can be obtained from (2).

$$a_2(N+1) = 2\sum_{k_1=0}^{N-1}(N+2)_{k_1} a_1(N-k_1)$$

$$\begin{aligned} &= 2\sum_{k_1=0}^{N-1}(N+2)_{k_1}2(N-k_1)! \\ &= 2^2\sum_{k_1=0}^{N-1}(N+2)_{k_1}(N-k_1)!, \end{aligned} \tag{22}$$

$$\begin{aligned} a_3(N+1) &= 2\sum_{k_2=0}^{N-2}(N+3)_{k_2}a_2(N-k_2) \\ &= 2\sum_{k_2=0}^{N-2}(N+3)_{k_2}2^2\sum_{k_1=0}^{N-2-k_2}(N+1-k_2)_{k_1}(N-1-k_2-k_1)! \\ &= 2^3\sum_{k_2=0}^{N-2}\sum_{k_1=0}^{N-2-k_2}(N+3)_{k_2}(N+1-k_2)_{k_1}(N-1-k_2-k_1)!. \end{aligned} \tag{23}$$

Similarly to the cases of (21) and (22), we can obtain

$$\begin{aligned} a_4(N+1) &= 2^4\sum_{k_3=0}^{N-3}\sum_{k_2=0}^{N-3-k_3}\sum_{k_1=0}^{N-3-k_3-k_2}(N+4)_{k_3} \\ &\quad \times(N+2-k_3)_{k_2}(N-k_3-k_2)_{k_1}(N-2-k_3-k_2-k_1)! \end{aligned} \tag{24}$$

Continuing in this fashion, we can deduce that, for $2 \leq i \leq N$,

$$\begin{aligned} a_i(N+1) &= 2^i\sum_{k_{i-1}=0}^{N-i+1}\sum_{k_{i-2}=0}^{N-i+1-k_{i-1}}\cdots\sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2} \\ &\quad \times\prod_{l=2}^{i}(N-i+2l-\sum_{j=1}^{i-1}k_j)_{k_{l-1}}(N-i+2-\sum_{j=1}^{i-1}k_j)! \end{aligned} \tag{25}$$

Remark 2.1. We note here that (24) holds also for $i = N+1$.

Now, we have the following theorem.

Theorem 2.2. The following family of differential equations

$$F^{(N)} = (1-t)^{-N}\left(\sum_{i=1}^{N}a_i(N)(x)_i(1+t)^{-i}\right)F \quad (N = 1, 2, 3, \cdots) \tag{26}$$

have a solution

$$F = F(t,x) = \left(\frac{1+t}{1-t}\right)^x, \tag{27}$$

where

$$a_1(N) = 2N!, \tag{28}$$

$$\begin{aligned} a_i(N) &= 2^i \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-i-k_{i-1}} \cdots \sum_{k_1=0}^{N-i-k_{i-1}-\cdots-k_2} \\ &\quad \times \prod_{l=2}^{i} (N-i-1+2l-\sum_{j=l}^{i-1} k_j)_{k_{l-1}} (N-i+1-\sum_{j=1}^{i-1} k_j)!, \end{aligned} \tag{29}$$

$(2 \leq i \leq N)$.

3. Applications

In this section, we recall that the Mittag-Leffler polynomials $M_n(x)$ are given by the generating function

$$F = F(t,x) = \left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} M_n(x) \frac{t^n}{n!}. \tag{30}$$

From (26) and (30), we have

$$\begin{aligned} \sum_{n=0}^{\infty} M_{n+N}(x) \frac{t^n}{n!} &= F^{(N)}(t,x) \\ &= \sum_{i=1}^{N} a_i(N)(x)_i (1+t)^{-N-i} F(t, x+N) \\ &= \sum_{i=1}^{N} a_i(N)(x)_i \sum_{l=0}^{\infty} (N+i+l-1)_l (-1)^l \frac{t^l}{l!} \\ &\quad \times \sum_{m=0}^{\infty} M_m(x+N) \frac{t^m}{m!} \\ &= \sum_{i=1}^{N} a_i(N)(x)_i \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \\ &\quad \times (N+i+l-1)_l (-1)^l M_{n-l}(x+N) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (\sum_{i=1}^{N} \sum_{l=0}^{n} (-1)^l \binom{n}{l} (N+i+l-1)_l \\ &\quad \times a_i(N)(x)_i M_{n-l}(x+N)) \frac{t^n}{n!}. \end{aligned} \tag{31}$$

Comparing the left anf right sides of (30), we finally get the following theorem.

Theorem 3.1. For $n = 0, 1, 2, \ldots$ and $N = 1, 2, 3, \ldots$, we have

$$\begin{aligned} M_{n+N}(x) &= \sum_{i=1}^{N}\sum_{l=0}^{n}(-1)^l\binom{n}{l}(N+i+l-1)_l \\ &\quad \times a_i(N)(x)_i M_{n-l}(x+N), \end{aligned} \tag{32}$$

where $a_i(N)$'s are as in Theorem 2.1.

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