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Riesz basis property and stability of a beam equation with conjugate variables assigned at the same boundary point

Touré K. Augustin
Institut National Polytechnique
Houphouët-Boigny de Yamoussoukro,
BP 1093 Yamoussoukro, Côte d'Ivoire

Koua B. Jean-Claude
Université Felix Houphouët-Boigny and
UFR Mathématiques Appliquées et Informatique,
Côte d'Ivoire.
E-mail: k_brou@hotmail.com

Diop Fatou N.
Université Felix Houphouët-Boigny and
UFR Mathématiques Appliquées et Informatique,
Côte d'Ivoire.

Abstract

We study a Euler-Bernoulli beam using a special boundary feedback at the free and. The closed-loop system is shown to be non dissipative. This gives rise to difficulties in analyzing the well-posedness and the stability of the considered system using the traditional dissipativity based-method. The major difficulty in answering this question comes from its special boundary conditions: the physical variables and their conjugate variables are assigned simultaneously at the same boundary point. Despite the lack of the dissipativity we obtain the Riesz basis property. As consequences, both the spectrum-determined growth conditions and exponential stability are concluded.

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1. Introduction

Let us consider the following Euler-Bernoulli beam subject to the boundary bending moment feedback:

$$\begin{cases} y_{tt} + y_{xxxx} = 0,\ 0 < x < 1,\ \ t \geq 0, \\ y(0,t) = y_x(0,t) \ = 0, \qquad t \geq 0, \\ y_{xxx}(1,t) \ = 0, \qquad t \geq 0, \\ -\, y_{xx}(1,t) \quad = \alpha y_{xt}(1,t) + \beta y_x(1,t),\ t \geq 0. \end{cases} \qquad (S)$$

It has been proved when $\beta = 0$ in [13] and when $\beta > 0$ in [17] that the above system is Riesz spectral, and is exponentially stable in the energy state-pace. Formally, let $u(x,t) = y_x(x,t)$. Then u satisfies the following set of equations

$$u_{tt} + u_{xxxx} = 0,\ 0 < x < 1,\ t \geq 0, \tag{1}$$

$$u(0,t) = u_{xxx}(0,t) = 0,\ t \geq 0, \tag{2}$$

$$u_{xx}(1,t) = 0,\ t \geq 0, \tag{3}$$

$$-u_x(1,t) = \alpha u_t(1,t) + \beta u(1,t),\ t \geq 0. \tag{4}$$

A question was proposed in [6]. What can we say about the well posedness of system (1)–(4)

Let us recall the physical meanings of the variables (the reader can be referred to [6] and the references therein):

$y(x,t) =$ displacement $\qquad y_t(x,t) =$ velocity;

$y_x(x,t) =$ rotation $\qquad y_{xt}(x,t) =$ angular velocity;

$-y_{xx}(x,t) =$ bending moment $\qquad y_{xxx}(x,t) =$ shear force.

at point x and time t; and the mutual conjugacy of the physical variables is indicated as follows:

$$y(\text{or } y_t) \longleftrightarrow y_{xxx}(\text{or } y_{xxxt})$$

$$y_x(\text{or } y_{xt}) \longleftrightarrow y_{xx}(\text{or } y_{xxt}).$$

The general applied mathematics principle for assigning boundary conditions is that the physical variable and its conjugate variable cannot be imposed simultaneously at the same boundary point. The major difficulty in answering the above question comes from its special boundary condition: the physical variables and their conjugate variables are assigned simultaneously at the same boundaries.The pair of conjugate variables u and u_{xxx} are assigned simultaneously at $x = 0$.

From the system theoritic point of view, the system (1)–(4) is not passive, hence its associated generator in the state space with energy norm is not dissipative. This gives rise to difficulties in analyzing the well-posedness and the stability of the system (1)–(4) using the traditional dissipativity-based method. In this paper, we give a positive answer to the question proposed above, using the Riesz basis approach. Actually, we go beyond the question of well-posedness:

- we show that the system (1)–(4) verifies the Riesz basis property which means that there is a sequence of generalized eigenfunctions of generator of system (1)–(4) which forms a Riesz basis for the state space with the energy norm (which is a Hilbert space);
- the spectrum-determined growth condition holds;
- we obtain exponential stability.

These results show that there is more freedom in the design of boundary control for the suppression of vibrations of flexible structures. Therefore the applied mathematics principle mentioned earlier can be relaxed.

The rest of this paper is organized as follows: we rewrite the two systems in their operators forms, then we study the spectrum and prove the Riesz basis property for the system. The exponential stability of (1)–(4) is also concluded.

2. Energy space and energy norm

In this section we rewrite the two systems in their evolutive forms. Let us introduce the following spaces:

$$L = \left\{u \in H^2(0,1); u(0) = u_x(0) = 0\right\}$$

$$K = \left\{(u,v)^T; u \in L, v \in L^2(0,1)\right\} = L \times L^2(0,1)\},$$

$$V = \left\{u \in H^2(0,1); u(0) = u_{xxx}(0) = 0\right\}, \tag{5}$$

$$H = \left\{(u,v)^T; u \in V, v \in L^2(0,1)\right\} = V \times L^2(0,1)\}, \tag{6}$$

$D\,(0,1) =$ the space of smooth functions with compact support,

$D'\,(0,1) =$ the space of continuous linear functions $f : D\,(0,1) \to \mathbb{C}$.

The superscript T stands for the transpose and the spaces $L^2\,(0,1)$ and $H^k\,(0,1)$ are defined as

$$L^2\,(0,1) = \left\{y : [0,1] \to \mathbb{R} \middle| \int_0^1 |y|^2\, dx < \infty\right\}. \tag{7}$$

$$H^k\,(0,1) = \left\{y : [0,1] \to \mathbb{R} \middle| y, y^{(1)}, \ldots, y^{(k)} \in L^2\,(0,1)\right\}. \tag{8}$$

We consider the system (S) and let $z = y_t,\ R = (y,z)^T$. The space K is called energy space of the system (S). We define the inner product on the space K as follows:

$$\forall (y_1,z_1) \in K, \forall (y_2,z_2) \in K, \langle (y_1,z_1)\,, (y_2,z_2)\rangle = \int_0^1 (y_1'' y_2'' + z_1 z_2) dx + \beta y_1' y_2'.$$

The energy norm induced by the inner product is hence defined by

$$\|(y,z)\|_K^2 = \int_0^1 \left[(y'')^2 + z^2\right] dx + \beta \left(y'\right)^2$$

Letting

$$B = \begin{pmatrix} 0 & 1 \\ \dfrac{d^4}{dx^4} & 0 \end{pmatrix}$$

The system (S) can be written as

$$\begin{cases} \dfrac{dR\,(t)}{dt} = BR\,(t) \\ R\,(0) = R_0 \in K, \end{cases}$$

where $D(B)$ the domain of the operator is defined as follows:

$$\begin{aligned} D\,(B) &= \Big\{ (y,z)^T \in (H^4\,(0,1) \cap L) \times L \\ y_{xxx}\,(1) &= 0,\ y_{xx}\,(1) = -\alpha z_x(1) - \beta z_x(1) \Big\}. \end{aligned}$$

We consider the system (1)–(4) and let $v = u_t,\ W = (u, v)^T$. The space H is called energy space of the system. We define the inner product on the space H as follows:

$$\forall (u,v) \in H, \forall (u_1, v_1) \in H, \langle (u_1, v_1), (u_2, v_2) \rangle = \int_0^1 (u_1'' u_2'' + v_1 v_2) dx + \beta u_1 u_2.$$

The energy norm induced by the inner product is hence defined by

$$\|(u,v)\|_H^2 = \int_0^1 \left[(u'')^2 + v^2 \right] dx + \beta u^2.$$

Finally letting

$$A = \begin{pmatrix} 0 & 1 \\ \dfrac{d^4}{dx^4} & 0 \end{pmatrix}. \tag{9}$$

The system (1)−(4) can be written as

$$\begin{cases} \dfrac{dW\,(t)}{dt} = AW\,(t) \\ W\,(0) = W_0 \in H, \end{cases} \tag{10}$$

where $D(A)$ the domain of the operator is defined as follows:

$$D\,(A) = \left\{ (u,v)^T \in (H^4\,(0,1) \cap V) \times V \middle| u_{xx}\,(1) = 0, u_x\,(1) = -\alpha v(1) - \beta u(1) \right\}.$$

3. Spectral Analysis and the Riesz basis property

In this section, we show that there is a sequence of generalized eigenvectors of the operator A which forms a Riesz basis for the energy space H. The study of the spectral problem associated with the evolutive system reveals that the spectral parameter appears in boundary conditions. For this kind of problem, the classical theorem of Bari seems very difficult to apply [5]. Let us recall the basic idea of Bari's theorem: if $\{\Phi_n\}_1^\infty$ is a Riesz basis for a Hilbert space H and if another $\omega-$linearly independent basis sequence$\{\Psi_n\}_1^\infty$ from H satisfies

$$\sum_{n=1}^{\infty} \|\Psi_n - \Phi_n\|^2 < \infty,$$

then $\{\Psi_n\}_1^\infty$ is also a Riesz basis for H.

In this paper, we use a method due to Shkalikov [14]. The basis idea of the method is to build using the operator A a new operator called the *Shkalikov's linearized operator* which will give the Riesz basis property and then deduce the same property for the operator A. Here, we must work in the complexified Hilbert spaces of spaces V, $L^2(0, 1)$ and H. For convenience, we do not change the notation for these spaces.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and let $W = (u, v)^T \in D(A)$ be a corresponding eigenvector. The equation $AW = \lambda W$ leads to the following set of equations

$$\begin{cases} \lambda u - v = 0, \\ \lambda v + u_{xxxx} = 0, \\ u_{xxx}(0) = u_{xx}(1) = u(0) = 0, \\ u_x(1) = -\alpha v(1) - \beta u(1). \end{cases}$$

Eliminating v from the above equations, we get

$$\begin{cases} u_{xxxx} + \lambda^2 u = 0, \\ u_{xxx}(0) = 0, \\ u_{xx}(1) = 0, \\ u_x(1) = -(\alpha\lambda + \beta)u(1), \\ u(0) = 0. \end{cases}$$

Letting $\lambda = \tau^2$, we get

$$u_{xxxx} + \tau^4 u = 0 \tag{11}$$

$$u_{xxx}(0) = 0 \tag{12}$$

$$u_{xx}(1) = 0 \tag{13}$$

$$u_x(1) = -(\alpha\tau^2 + \beta)u(1) \tag{14}$$

$$u(0) = 0 \tag{15}$$

The Shkalikov's characteristic polynomial ([13, p. 1314]) associated with equation (11) is:

$$\omega^4 + 1 = \left(\omega^2 - \sqrt{2}\omega + 1\right)\left(\omega^2 + \sqrt{2}\omega + 1\right) = 0. \tag{16}$$

The zeros of the above polynomial are the complex numbers

$$\omega_1 = \frac{1+i}{\sqrt{2}}, \quad \omega_2 = \frac{-1+i}{\sqrt{2}}, \quad \omega_3 = \frac{-1-i}{\sqrt{2}}, \quad \omega_4 = \frac{1-i}{\sqrt{2}}.$$

The solutions of (11) can be then found in the following form:

$$u(x) = C_1 e^{\tau\omega_1 x} + C_2 e^{\tau\omega_2 x} + C_3 e^{\tau\omega_3 x} + C_4 e^{\tau\omega_4 x}. \tag{17}$$

Let

$$\begin{aligned} F_i &= \tau^2\omega_i^2 e^{\tau\omega_i}, \quad i = 1, \ldots, 4 \\ G_i &= \left(\alpha\tau^2 + \tau\omega_i + \beta\right) e^{\tau\omega_i} \quad i = 1, \ldots, 4. \end{aligned}$$

We get the following matrix equation:

$$\begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ G_1 & G_2 & G_3 & G_4 \\ \tau^3\omega_1^3 & \tau^3\omega_2^3 & \tau^3\omega_3^3 & \tau^3\omega_4^3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

A necessary and sufficient condition for this matrix equation to have nontrivial solutions for C_1, C_2, C_3, and C_4 is that the following characteristic determinant

$$\Delta(\tau) = \begin{vmatrix} F_1 & F_2 & F_3 & F_4 \\ G_1 & G_2 & G_3 & G_4 \\ \tau^3\omega_1^3 & \tau^3\omega_2^3 & \tau^3\omega_3^3 & \tau^3\omega_4^3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

vanishes. By developing $\Delta(\tau)$ with respect to the last row, we get

$$\begin{aligned} \Delta(\tau) = \tau\Big\{ & (\omega_3 - \omega_4)(F_1G_2 - F_2G_1) + (\omega_2 - \omega_3)(F_1G_4 - F_4G_1) \\ & + (\omega_3 - \omega_1)(F_2G_4 - F_4G_2) + (\omega_4 - \omega_1)(F_3G_2 - F_2G_3) \\ & + (\omega_2 - \omega_4)(F_3G_1 - F_1G_3) + (\omega_1 - \omega_2)(F_3G_4 - F_4G_3) \Big\}. \end{aligned}$$

Next, we set

$$\begin{aligned} W_{ij} &= \left(\omega_i - \omega_j\right) \quad 1 \le i,\ j \le 4 \\ T_{ij} &= F_iG_j - F_jG_i \quad 1 \le i,\ j \le 4 \end{aligned}$$

so that

$$\Delta(\tau) = \tau\left(W_{34}T_{12} + W_{23}T_{14} + W_{31}T_{24} + W_{41}T_{32} + W_{24}T_{31} + W_{12}T_{34}\right)$$

where

$$\begin{aligned}
W_{34} &= \sqrt{2},\ W_{23} = i\sqrt{2},\ W_{31} = (1-i)\sqrt{2} \\
W_{41} &= -i\sqrt{2},\ W_{24} = (1+i)\sqrt{2},\ W_{12} = -\sqrt{2} \\
T_{12} &= \tau^2\left[-\sqrt{2}\tau + 2i\left(\alpha\tau^2+\beta\right)\right]e^{i\tau\sqrt{2}} \\
T_{14} &= \tau^2\left[i\sqrt{2}\tau + 2i\left(\alpha\tau^2+\beta\right)\right]e^{\tau\sqrt{2}} \\
T_{24} &= \sqrt{2}\,(-1-i)\,\tau^3 \\
T_{32} &= \tau^2\left[-i\sqrt{2}\tau + 2i\left(\alpha\tau^2+\beta\right)\right]e^{-\tau\sqrt{2}} \\
T_{31} &= \sqrt{2}\,(-1+i)\,\tau^3 \\
T_{34} &= \tau^2\left[\sqrt{2}\tau + 2i\left(\alpha\tau^2+\beta\right)\right]e^{-i\tau\sqrt{2}}
\end{aligned}$$

After simplification, we get

$$\begin{aligned}
\Delta(\tau) = \tau^7\Big\{ & \left[-2\tau^{-1} + 2\sqrt{2}i\alpha + 2\sqrt{2}i\beta\tau^{-2}\right]e^{i\tau\sqrt{2}} \\
& -\left[2\tau^{-1} + 2\sqrt{2}i\alpha + 2\sqrt{2}i\beta\tau^{-2}\right]e^{-i\tau\sqrt{2}} \\
& -\left[2\tau^{-1} + 2\sqrt{2}\alpha + 2\sqrt{2}\beta\tau^{-2}\right]e^{\tau\sqrt{2}} \\
& +\left[-2\tau^{-1} + 2\sqrt{2}\alpha + 2\sqrt{2}\beta\tau^{-2}\right]e^{-\tau\sqrt{2}} - 8\tau^{-1}\Big\}.
\end{aligned}$$

We observe that for $|\tau|$ sufficiently large, the dominant term of each expression in bracket is nonzero. In the view of Shkalikov's theory, the boundary conditions are said to be *regular.* We also mention that the previous characteristic determinant is the same as that found for system (S) in [17] . Hence we obtain the following results:

Lemma 3.1. A has compact resolvent and $0 \in \rho(A)$. Therefore the eigenvalues of A are countable and isolated.

Proof. Clearly, we only need to prove that $0 \in \rho(A)$ and A^{-1} is compact on H. For any $G = (u, v) \in H$, we need to find a unique $F = (f, g) \in D(A)$ such that $AF = G$. In other words such that the following set of equations is satisfied

$$g = u \tag{18}$$

$$-f_{xxxx} = v \tag{19}$$

$$f_{xxx}(0) = f_{xx}(1) = f(0) = 0 \tag{20}$$

$$f_x\,(1) = -\alpha g\,(1) - \beta f\,(1) \tag{21}$$

By integrating (3.9) we obtain for all s in [0, 1]:

$$-\int_0^s f_{xxxx}\,(x)\,dx = \int_0^s v\,(x)\,dx$$

$$-\left[f_{xxx}\right]_0^s = \int_0^s v\,(x)\,dx.$$

Using the boundary condition (3.10) we have:

$$-f_{xxx}\,(s) = \int_0^s v\,(x)\,dx.$$

By integrating again we get for all $z \in [0, 1]$:

$$-\int_z^1 f_{xxx}\,(s)\,ds = \int_z^1 \int_0^s v\,(x)\,dxds$$

$$-\left[f_{xx}\,(s)\right]_z^1 = \int_z^1 \int_0^s v\,(x)\,dxds.$$

Using the boundary condition (3.10) we get for all $z \in [0, 1]$:

$$f_{xx}\,(z) = \int_z^1 \int_0^s v\,(x)\,dxds.$$

By integrating again we obtain for all r in [0, 1]:

$$\int_1^r f_{xx}\,(z)\,dz = \int_1^r \int_z^1 \int_0^s v\,(x)\,dxdsdz$$

$$\left[f_x\right]_1^r = \int_1^r \int_z^1 \int_0^s v\,(x)\,dxdsdz$$

$$f_x\,(r) - f_x\,(1) = \int_1^r \int_z^1 \int_0^s v\,(x)\,dxdsdz.$$

Using the boundary condition (3.11) we have for all r in [0, 1]:

$$f_x\,(r) + \alpha g(1) + \beta f(1) = \int_1^r \int_z^1 \int_0^s v\,(x)\,dxdsdz$$

Then set

$$k\,(r) = \int_1^r \int_z^1 \int_0^s v\,(x)\,dxdsdz$$

we get for all r in [0, 1]:

$$f_x(r) + \beta f(1) = -\alpha u(1) + k(r).$$

By integrating again we obtain for all m in [0, 1]:

$$\int_0^m f_x(r)\,dr + \beta \int_0^m f(1)dr = -\alpha \int_0^m u(1)\,dr + \int_0^m k(r)\,dr$$

$$f(m) + \beta m f(1) = -\alpha m u(1) + \int_0^m k(r)\,dr.$$

Next we determine $f(1)$. We get:

$$f(1) + \beta f(1) = -\alpha u(1) + \int_0^1 k(r)\,dr;$$

$$f(1) = \frac{-\alpha u(1) + \int_0^1 k(r)\,dr}{1+\beta}.$$

Then we get for all m in [0, 1]:

$$f(m) = -\beta m \left(\frac{-\alpha u(1) + \int_0^1 k(r)\,dr}{1+\beta} \right) - \alpha m u(1) + \int_0^m k(r)\,dr.$$

Obviously we have $(u, v) \in D(A)$. Therefore we get:

$$F = (f, g) = A^{-1}G = (N(m), u).$$

Where N is given by:

$$N(m) = -\beta m \left(\frac{-\alpha m(1) + \int_0^1 k(r)\,dr}{1+\beta} \right) - \alpha m u(1) + \int_0^m k(r)\,dr.$$

Finally we obtain that $0 \in \rho(A)$ and Sobolev's embedding theorem implies that A^{-1} is a compact operator on H. ■

Theorem 3.2. For $\alpha > 0$ and $\beta > 0$, we get:

1. The operators A and B have the same spectrum and for each $\lambda \in \sigma(A) = \sigma(B)$, $\mathrm{Re}(\lambda) < 0$;

2. The eigenvalues of the unbounded operator A which governs the system (S) take asymptotically the form

$$\lambda_n = -\frac{1}{\alpha} + O\left(\frac{1}{n}\right) + i\left[\left(n - \frac{1}{4}\right)^2 \pi^2 + O\left(\frac{1}{n^2}\right)\right], \quad n \in \mathbb{N};$$

and $\lim\limits_{n\to+\infty} \mathrm{Re}(\lambda_n) = -\dfrac{1}{\alpha} < 0,$

3. the eigenvalues of the operator A are asymptotically simple and isolated.

(For the proof see [17] .)

Remark 3.3. We recall that the property (3) of the above theorem is essential for applying theorem 3.1 of Shkalikov's theory [14] .

Theorem 3.4. Consider the system given by (2.6) where $\alpha > 0$ and $\beta > 0$, then there exists a fundamental system of generalized eigenvectors of the operator A which forms a Riesz basis in $H = V \times L^2(0, 1)$.

Proof. Following the notation of Shkalikov in [14] , for integer $r \geq 0$, we set

$$W_2^r = W_2^{n-1+r}(0, 1) \oplus W_2^{n-2+r}(0, 1) \oplus \cdots \oplus W_2^r(0, 1), \tag{22}$$

where $W_2^k(0, 1)$ is the Sobolev space of smooth functions on the segment [0, 1] having $k - 1$ absolutely continuous and derivatives and k-th derivative from $L^2(0, 1)$ with the norm $\|f\|_{W^k} = \left\|f^{(k)}\right\|_{L^2(0,1)} + \|f\|_{L^2(0,1)}$.

In our case $n = 4$ and we have

$$W_2^r = W_2^{3+r}(0, 1) \oplus W_2^{2+r}(0, 1) \oplus W_2^{r+1}(0, 1) \oplus W_2^r(0, 1) .$$

We rewrite equation (11) in the following from

$$l(u, \tau) = l_0(u) + \tau l_1(u) + \tau^2 l_2(u) + \tau^3 l_3(u) + \tau^4 l_4(u) = 0,$$

where $l_0(u) = u_{xxxx}$, $l_1(u) = l_2(u) = l_3(u) = 0$, $l_4(u) = u$.

Now, we consider the operator H, defined as follows:

$$\widetilde{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \in W_2^r \longmapsto H\widetilde{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ -\sum_{i=0}^{3} l_i(v_i) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ -v_0^{(4)} \end{bmatrix},$$

where $v_0 = u$, $v_1 = \tau u$, $v_2 = \tau^2 u$, $v_3 = \tau^3 u$.

We also define $H^i(\widetilde{v}) \in W_2^{r-i}$, where H^i is the i th power of H. Now, we normalize the boundary conditions (3.2) $-$ (3.5) according to Shkalikov's method [14] . First, we rewrite them as follows:

$$\begin{cases} U_1(u, \tau) = u'''(0) = 0 \\ U_2(u, \tau) = u''(1) = 0 \\ U_3(u, \tau) = u'(1) + \left(\alpha\tau^2 + \beta\right) u(1) = 0 \\ U_4(u, \tau) = u(0) = 0 \end{cases} \tag{23}$$

and in (3.9) , we make substitutions according to the rule

$$\begin{aligned}
\tau^i u^{(k)}(x) &= \left(H^i\widetilde{v}\right)_0^{(k)}(x) \text{ if } i+k<n+r, \\
\tau^i u^{(k)}(x) &= r^{i+k-n-r+1}\left(H^{n+r-k-1}\widetilde{v}\right)_0^{(k)}(x) \text{ if } i+k\geq n+r,
\end{aligned}$$

where $x = 0$ or $x = 1$, n being the number of boundary conditions and the subscript index means that we take the first component of the associated vector. As a result of these substitutions, we represent the boundary conditions in a following form

$$\widetilde{U}_i(\widetilde{v},\tau) = \sum_{k=0}^{v_i(r)} \tau^k U_i^k(\widetilde{v}), \quad 1\leq i\leq n,$$

where now the linear forms U_i^k do not depend on λ. We set

$$N_r = v_1(r) + v_2(r) + \cdots + v_q(r)$$

where the numbers $v_i(r)$ are those which appear above. If they are all zeros, then $N_r = 0$. With the previous transformations in mind, we rewrite the second above boundary conditions as follows: the term $\tau^2 u(1)$ is replaced by $v_2(1)$, the other boundary conditions remain unchanged. Hence, we can represent (3.9) as follows:

$$\begin{aligned}
\widetilde{U_1}(\widetilde{v},\tau) &= \widetilde{U_1}(\widetilde{v}) = v_0'''(0) = 0 \\
\widetilde{U_2}(\widetilde{v},\tau) &= \widetilde{U_2}(\widetilde{v}) = v_0''(1) = 0 \\
\widetilde{U_3}(\widetilde{v},\tau) &= \widetilde{U_3}(\widetilde{v}) = v_0'(1) + \left(\alpha\tau^2+\beta\right)v_0(1) = 0 \\
\widetilde{U_4}(\widetilde{v},\tau) &= \widetilde{U_4}(\widetilde{v}) = v_0(0) = 0.
\end{aligned}$$

We denote by $W_{2,U}^r$ the Shkalikov space defined as follows:

$$W_{2,U}^r = \left\{ \begin{array}{c} \widetilde{v}\in W_2^r, U_j\left(H^k(\widetilde{v})\right) = 0,\ 1\leq j\leq n \text{ for } 0\leq k\leq n+r-2 \\ \text{and all boundary conditions of order } \leq n+r-k-2 \end{array} \right\} \tag{24}$$

Following the theory of Shkalikov $W_{2,U}^r$ is a closed subspace of finite codimension in W_2^r. In our case, since $n = 4$, for $r = 0$, the Shkalikov's space $W_{2,U}^0$ ([14]) is defined as follows:

$$W_{2,U}^0 = \left\{ \begin{array}{c} \widetilde{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \in W_2^3(0,1)\oplus W_2^2(0,1)\oplus W_1^1(0,1)\oplus L^2(0,1), \\ U_j\left(H^k\widetilde{v}\right) = 0, \\ \text{for } 0\leq k\leq n+r-2 = 2 \text{ and all boundary conditions of order} \leq 2-k \end{array} \right\}.$$

We get

$$W_{2,U}^{0}=\left\{\begin{array}{c}\widetilde{v}\in W_2^3(0,1)\oplus W_2^2(0,1)\oplus W_2^1(0,1)\oplus L^2(0,1)\,,\\ v_0''(1)=0,\,v_0'(1)+\alpha v_2(1)+\beta v_0(1)=0,\,v_1'(1)\\ +\alpha v_3(1)+\beta v_1(1)=0,\\ v_0(0)=0,\,v_1(0)=0,\,v_2(0)=0\end{array}\right\}$$

We define the Shkalikov's operator as follows:

$$H_0\begin{bmatrix}v_0\\ v_1\\ v_2\\ v_3\end{bmatrix}=H\begin{bmatrix}v_0\\ v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}v_1\\ v_2\\ v_3\\ -v_0^{(4)}\end{bmatrix},\quad D(H_0)=W_{2,U}^{1}.$$

Using corollary 3.1 of Shkalikov's Theorem 3.1 [14], we deduce that there is a set of generalized eigenvectors of the operator H_0 which forms a Riesz basis of the Hilbert space $W_{2,U}^{0}$. When $r=1$, the space $W_{2,U}^{1}$ is defined as follows:

$$\begin{aligned}W_{2,U}^{1} &= \Big\{\widetilde{v}\in W_2^4(0,1)\oplus W_2^3(0,1)\oplus W_2^2(0,1)\oplus W_2^1(0,1)\,,\\ v_0'''(0) &= 0,\ v_0''(1)=0,\,v_1''(1)=0,\,v_0'(1)+\alpha v_2(1)\\ &\quad +\beta v_0(1)=0,\\ v_1'(1)+\alpha v_3(1)+\beta v_1(1) &= 0,\,v_2'(1)+\alpha v_0^{(4)}(1)+\beta v_2(1)=0,\ v_0(0)=0,\\ v_1(0) &= 0,\ v_2(0)=0,\ \ v_3(0)=0\Big\}\end{aligned}$$

Since $\lambda=\tau^2$, we define Shkalikov's operator H_0^2 as follows:

$$\begin{aligned}H_0^2\begin{bmatrix}v_0\\ v_1\\ v_2\\ v_3\end{bmatrix} &= \begin{bmatrix}v_2\\ v_3\\ -v_0''''\\ -v_1''''\end{bmatrix}\text{ and } D\left(H_0^2\right)=W_{2,U}^{2}\subset W_{2,U}^{0},\\ D\left(H_0^2\right) &= \{\widetilde{v}\in D(H_0)\,/H_0\widetilde{v}\in D(H_0)\}\\ &= \Big\{\widetilde{v}\in W_2^5(0,1)\oplus W_2^4(0,1)\oplus W_2^3(0,1)\oplus W_2^2(0,1)\,,\\ v_0^{(4)}(0) &= 0,\ v_0'''(0)=0,\ v_1'''(0)=0,\,v_0''(1)=0,\\ v_1''(1) &= 0,\,v_2''(1)=0,\\ v_0'(1)+\alpha v_2(1)+\beta v_0(1) &= 0,\ v_1'(1)+\alpha v_3(1)+\beta v_1(1)=0,\\ v_2'(1)+\alpha v_0^{(4)}(1)+\beta v_2(1) &= 0,\,v_3'(1)+\alpha v_1^{(4)}(1)+\beta v_3(1)=0,\\ v_0(0) &= 0,\ v_1(0)=0,\ v_2(0)=0,\ v_3(0)=0,\ \Big\}.\end{aligned}$$

Using Corollary 3.1 of Shkalikov's Theorem 3.1 [14], we deduce that there is a set of generalized eigenvectors of the operator H_0^2 which forms a Riesz basis of the Hilbert space $W_{2,U}^1$.

Now, we build a Riesz basis for the operator A. First, we remark that we can write operator H_0^2 as follows:

$$H_0^2 = H^1 \oplus H^2,$$

where H^2 operates on $v_1,\ v_3$ and H^1 operates on $v_0,\ v_2$. Operator H^2 is defined as follows:

$$H^2 \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ -w'''' \end{bmatrix},$$

where

$$\begin{aligned} D\left(H^2\right) &= \left\{ \begin{array}{c} \begin{pmatrix} w \\ v \end{pmatrix} \in W_2^4(0,1) \oplus W_2^2(0,1) / w'''(0) = 0, \\ w''(1) = 0,\ w'(1) + \alpha v(1) + \beta w(1) = 0, \\ v'(1) + \alpha w^{(4)}(1) + \beta v(1) = 0 \\ w(0) = 0,\ v(0) = 0 \end{array} \right\} \\ &= \left\{ \begin{array}{c} \begin{pmatrix} w \\ v \end{pmatrix},\ w \in W_2^4(0,1) \cap V,\ v \in V,\ w_{xx}(1) = 0, \\ w_x(1) + \alpha v(1) + \beta w(1) = 0,\ v_x(1) + \alpha w_{xxxx}(1) + \beta v(1) = 0 \end{array} \right\} \end{aligned}$$

Next, we prove that the spectral problem associated with operator H_0^2 is equivalent to the one defined by A. Let λ be an eigenvalue of H_0^2. We have:

$$H_0^2 U = \lambda U$$

where

$$U = (v_0, v_1, v_2, v_3)^T \in D\left(H_0^2\right)$$

is an eigenvector of H_0^2 associated with λ. We then obtain :

$$\left\{ \begin{array}{l} v_2 = \lambda v_0, \\ v_3 = \lambda v_1, \\ -v_0'''' = \lambda v_2, \\ -v_1'''' = \lambda v_3, \\ U \in D\left(H_0^2\right). \end{array} \right.$$

Now, by a substitution, we get the following set of systems:

$$\left\{ \begin{array}{c} v_1'''' + \lambda^2 v_1 = 0, \\ v_1'(1) + \alpha\lambda v_1(1) + \beta v_1(1) = 0, \\ \lambda v_1'(1) + \alpha v_1^{(4)}(1) + \lambda\beta v_1(1) = 0, \\ v_1'''(0) = v_1''(1) = v_1(0) = 0, \end{array} \right. \tag{25}$$

$$\begin{cases} v_0'''' + \lambda^2 v_0 = 0, \\ v_0'(1) + \alpha\lambda v_0'(1) + \beta v_0(1) = 0, \\ \lambda v_0'(1) + \alpha v_0^{(4)}(1) + \lambda\beta v_0(1) = 0 \\ v_0''''(0) = v_0'''(0) = v_0(0) = 0, \\ v_0''(1) = 0. \end{cases} \tag{26}$$

To obtain the last condition of (26), it suffices to remark that:

$$v_0'''' + \lambda^2 v_0 = 0 \text{ and } v_0 \in W_2^5(0,1).$$

From (25), λ is an eigenvalue of H^1 associated with the eigenvector $(v_1, v_3)^T$. From (26), we deduce that λ is an eigenvalue of H^2 associated with the eigenvector $(v_0, v_2)^T$. Next, let λ be an eigenvalue of A associated with the spectral problem (1)–(4). Then we easily deduce that λ is an eigenvalue of H_0^2. Since we know from the previous study of H_0^2 that there is a set of generalized eigenvectors of operator H_0^2 which form a Riesz basis of the Hilbert space $W_{2,U}^1$, we deduce that there is also a set of generalized eigenvectors of the operator A or H^2 which forms a Riesz basis of Hilbert space $H = V \oplus L^2(0,1)$.

Following the idea of ([5, Theorem 2.4]), all the properties of operator A found above, allows us to claim that for the semigroup e^{At} generated by A, the spectrum-determined growth condition holds:

$$\omega(A) = S(A),$$

where

$$\omega(A) = \lim_{t\to\infty} \frac{1}{t} \left\| e^{At} \right\|$$

is the growth order of e^{At} and

$$S(A) = \sup\{\mathcal{R}e\lambda / \lambda \in \sigma(A)\}$$

is the spectral bound of A. ■

Theorem 2 is the fundamental property of the evolutive system (1)–(4). Many other important properties of this system can be concluded from Theorem 2. The exponential stability stated below is one of such important property.

Theorem 3.5. System (1)–(4) is exponentially stable for any $\beta > 0$ and $\alpha > 0$. That is, there are nonnegative constants M, ω such that the energy $E(t)$ of system (1)–(4) satisfies

$$E(t) \le M E(0) e^{-\omega t}, \forall t \ge 0, \tag{27}$$

for any initial condition $(u(x,0), u_t(x,0)) \in H$.

Proof. Following the theorem 1, we get: for each $\lambda \in \sigma(A) = \sigma(B)$, $\mathrm{Re}(\lambda) < 0$, the eigenvalues of the unbounded operator A which governs the system (S) take asymptotically the form

$$\lambda_n = -\frac{1}{\alpha} + O\left(\frac{1}{n}\right) + i\left[\left(n - \frac{1}{4}\right)^2 \pi^2 + O\left(\frac{1}{n^2}\right)\right], \; n \in \mathbb{N}$$

and

$$\lim_{n \to +\infty} \mathrm{Re}\,(\lambda_n) = -\frac{1}{\alpha} < 0.$$

Moreover the spectrum-determined conditions holds hence the system (1)–(4) is exponential stable. ■

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