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Third-order Composite Runge Kutta Method for Solving Fuzzy Differential Equations

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Abstract.

In this paper a third-order composite Runge Kutta method is applied for solving fuzzy differential equations based on generalized Hukuhara differentiability. This study intends to explore the explicit methods which can be improved and modified to solve fuzzy differential equations. Some definitions and theorem are reviewed as a basis in solving fuzzy differential equations. Some numerical examples are given to illustrate the accuracy of the method. The comparisons with the existing method are also discussed. Based on the results, the proposed method gives the better result. Hence, the method can be used to solve fuzzy differential equations.

Keywords: Runge-Kutta, Fuzzy differential equations, Hukuhara differentiability.

1. INTRODUCTION

Differential equations play an important role in various fields such as physics, economy, science and engineering. It is widely used by experts in this field to model problems in their study. However, in some cases, information regarding the physical phenomena involved are often represented not by crisp values, but instead the values are represented by uncertainty. Klir et al. [3] defines the concept of fuzzy numbers which exist from many phenomena that cannot be measured with precise values.

Nowadays, fuzzy differential equations (FDEs) have started to grow rapidly. The concept of FDEs was first introduced by Chang and Zadeh [13]. Later, Dubois and Prade [1] expanded the principle approach in solving FDEs. Kaleva [8,9] and Seikkala [12] managed to solve FDEs with fuzzy initial value problem.

Ma et al. [6] first introduced the classic Euler method as a numerical method to solve the problem of fuzzy differential equations. Abbasbandy and Allahviranloo [11] studied Taylor method to solve the problem of fuzzy differential equations. Subsequently, the field is growing rapidly with a variety of numerical methods to solve the problem of fuzzy differential equations. The numerical methods such as Adam Bashford [14], Runge Kutta of order five [15], block methods [16], and Runge-Kutta Method with Harmonic Mean of Three Quantities [2] were investigated to find solutions for fuzzy differential equations problem.
In this study, third-order composite Runge-Kutta method [10] is applied for solving fuzzy differential equations. In second section, some necessary preliminaries is given. Transformation of FDEs into the fuzzy parametric form is shown in the third section. Next, the derivation of third-order composite Runge-Kutta method is presented and the stability of the proposed method is given in section five. Section six discussed the formulation of the third-order composite Runge-Kutta method for solving FDEs. The proposed method is compared with Runge-Kutta Method with Harmonic Mean of Three Quantities and a numerical result is illustrated in section seven. The discussion is provided in section eight and the conclusion is given in the last section.

2. PRELIMINARIES

In this section, some basic definitions of fuzzy numbers are reviewed:

Definition 1 : A fuzzy number is a fuzzy subset of the real line with normal, convex and upper semi continuous membership function of bounded support. A fuzzy number y is determined by any pair $y = (\underline{y}(r), \overline{y}(r)),\ \ 0 \le r \le 1,$ which satisfy the three conditions:

1. $\underline{y}(r)$ is a bounded left continuous increasing function $\forall r \in [0,1]$.
2. $\overline{y}(r)$ is a bounded left continuous decreasing function $\forall r \in [0,1]$.
3. $\underline{y}(r) \le \overline{y}(r),\ \ 0 \le r \le 1.$

Definition 2: Let I be a real interval. A mapping $F : I \to E$ is called a fuzzy process. We denote its *r*-level set of y for $I \in [t_0, T]$

$[F(t)]^r = [\underline{y}^r(t), \overline{y}^r(t)],\ \ r \in [0,1].$

The Seikkala derivatives $y'(t)$ of a fuzzy process y is defined by

$[F'(t)]^r = [(\underline{y}^r)'(t), (\overline{y}^r)'(t)],\ \ r \in [0,1].$

Bede and Stefanini [5] introduced the definition and theorem of the generalized Hukuhara as follows:

Definition 3: Let $F : T \to E(\mathbb{R})$ and $t_0 \in (a,b)$. F is differentiable at t_0 if

Form I. For all $h > 0$ sufficiently close to 0, the Hukuhara differences $F(t_0 + h) - F(t_0)$ and $F(t_0) - F(t_0 - h)$ and the limits (D-metric)

$$\lim_{h\to 0^+}\frac{F(t_0+h)-\mathrm{F}(\mathrm{t}_0)}{h}=\lim_{h\to 0^+}\frac{\mathrm{F}(\mathrm{t}_0)-F(t_0-h)}{h}=F'(t_0).$$

Form II. For all $h<0$ sufficiently close to 0, the Hukuhara differences $F(t_0+h)-F(t_0)$ and $F(t_0)-F(t_0-h)$ and the limits (D-metric)

$$\lim_{h\to 0^-}\frac{F(t_0+h)-\mathrm{F}(\mathrm{t}_0)}{h}=\lim_{h\to 0^-}\frac{\mathrm{F}(\mathrm{t}_0)-F(t_0-h)}{h}=F'(t_0).$$

Theorem 1: Let $F:T\to E(\mathbb{R})$ where $t_0\in(a,b)$ and E is a fuzzy function and denote $[F(t)]^r=[\underline{y}^r(t),\overline{y}^r(t)]$ for each $r\in[0,1]$. Then

Case 1. If F is differentiable in the first form I, then $\underline{y}^r(t)$ and $\overline{y}^r(t)$ are differentiable functions and

$[F'(t)]^r=[(\underline{y}^r)'(t),(\overline{y}^r)'(t)]$.

Case 2. If F is differentiable in the first form II, then $\underline{y}^r(t)$ and $\overline{y}^r(t)$ are differentiable functions and

$[F'(t)]^r=[(\overline{y}^r)'(t),(\underline{y}^r)'(t)]$.

3. FUZZY DIFFERENTIAL EQUATIONS

Mathematically, FDEs can be defined as follows [7]:

$$y'(t)=f(t,y(t)) \qquad (1)$$

where y is a fuzzy function of t, $f(t,y(t))$ is fuzzy function of the crisp variable y and y' is a fuzzy derivative of y.

If an initial value $y(t_0)=y_0$ is given, a fuzzy Cauchy problem of first order is obtained:

$$y'(t)=f(t,y(t)),\quad y(t_0)=y_0 \qquad (2)$$

It is possible to replace Eq.(2) by the following equivalent system

$$\begin{aligned}&\underline{y}'(t)=f(t,\underline{y}(t))=F(t,\underline{y},\overline{y}),\quad \underline{y}(t_0)=\underline{y}_0,\\&\overline{y}'(t)=f(t,\overline{y}(t))=G(t,\underline{y},\overline{y}),\quad \overline{y}(t_0)=\overline{y}_0,\end{aligned} \qquad (3)$$

where

$$\begin{aligned}&F(t,\underline{y},\overline{y})=\min\ f(t,\underline{y},\overline{y}),\\&G(t,\underline{y},\overline{y})=\max f(t,\underline{y},\overline{y}).\end{aligned} \qquad (4)$$

Based on Theorem 1, the parametric form of two different cases are as follows:

Case 1:

$$\left.\begin{aligned}&\underline{y}'(t,r)=F\left(t,\underline{y}(t,r),\overline{y}(t,r)\right),\qquad \underline{y}(t_0,r)=\underline{y}_0(r)\\&\overline{y}'(t,r)=G\left(t,\underline{y}(t,r),\overline{y}(t,r)\right),\qquad \overline{y}(t_0,r)=\overline{y}_0(r)\end{aligned}\right\}, \qquad (5)$$

Case 2:

$$\left.\begin{array}{ll} \underline{y}'(t,r) = G\ \ t, \underline{y}(t,r), \overline{y}(t,r)\ \ , & \underline{y}(t_0,r) = \underline{y}_0(r) \\ \overline{y}'(t,r) = F\ \ t, \underline{y}(t,r), \overline{y}(t,r)\ \ , & \overline{y}(t_0,r) = \overline{y}_0(r) \end{array}\right\}, \tag{6}$$

where $t \in [t_0, T]$ and $r \in [0,1]$.

4. THIRD-ORDER COMPOSITE RUNGE KUTTA METHOD

Ahmad and Yaacob [10] introduced third-order composite Runge-Kutta method for solving ordinary differential equations. This method is the combination of the harmonic and arithmetic means of the Runge-Kutta formulations. The formulation of the third-order composite of the arithmetic and harmonic means is shown as follows:
By using the concept of the arithmetic mean, we have

$$\begin{aligned} k_1 &= f(t_n, y_n) = f, \\ k_2 &= f(t_n + a_2 h, y_n + a_2 h k_1), \\ k_3 &= f\left(t_n + a_3 h, y_n + a_3 h\left(\frac{k_1 + k_2}{2})\right)\right), \\ y_{n+1} &= y_n + \frac{h}{2}\left(\frac{2k_1k_2}{k_1 + k_2} + \frac{2k_2k_3}{k_2 + k_3}\right). \end{aligned} \tag{7}$$

Since Eq. (7) involved the division of two series,

$$\frac{2k_i k_{i+1}}{k_i + k_{i+1}}, \quad i = 1,2,3 \quad , \tag{8}$$

then it cannot be solved by a direct substitutions. This problem is alleviated by cross-multiplying the terms by the common denominator $(k_1 + k_2)(k_2 + k_3)$ which can be written as

$$y_{n+1} = y_n + \frac{TOP}{BOTTOM}, \tag{9}$$

where

$$Top = h[k_2k_3(k_1 + k_2) + k_1k_2(k_2 + k_3)],$$

$$BOTTOM = (k_1 + k_2)(k_2 + k_3).$$

Taylor series expansion of $y(t_{n+1})$ may be written as

$$Taylor = y_n + hf + \frac{1}{2}h^2 ff_y + \frac{1}{6}h^3\ \ ff_y^{\ 2} + f^2 f_{yy}\ \ + \frac{1}{24}h^{\ 4}\ \ f^3 f_{yyy} + 4f^2 f_y f_{yy} + ff_y^{\ 3}\ \ + \ldots$$

Since the error of the method can be determined using the expression

$$Error = y(t_{n+1}) - y_{n+1},$$

we obtain

$$Error = Taylor + \frac{TOP}{BOTTOM},$$

which can be written as

$$Error \times BOTTOM = Taylor \times BOTTOM - TOP. \tag{10}$$

The coefficients of the same terms in (10) up to the terms h^3 are compared and two sets of parameter are obtained

$a_2 = 0, \quad a_3 = 2$ and $a_2 = \dfrac{3}{5}, \quad a_3 = \dfrac{4}{5}.$

The composite arithmetic–harmonic mean Runge–Kutta formula can be represented as follows.

SET 1:

$$\begin{aligned}
k_1 &= f(t_n, y_n),\\
k_2 &= f(t_n, y_n) = k_1,\\
k_3 &= f\left(t_n + 2h, y_n + 2h\left(\frac{k_1 + k_2}{2}\right)\right) = f(t_n + 2h, y_n + 2hk_1),\\
y_{n+1} &= y_n + \frac{h}{2}\left(\frac{2k_1k_2}{k_1 + k_2} + \frac{2k_2k_3}{k_2 + k_3}\right) = y_n + \frac{h}{2}\left(k_1 + \frac{2k_1k_3}{k_1 + k_3}\right).
\end{aligned} \tag{11}$$

Since $k_1 = k_2$.

SET 2:

$$\begin{aligned}
k_1 &= f(t_n, y_n),\\
k_2 &= f\left(t_n + \frac{3}{5}h, y_n + \frac{3}{5}hk_1\right),\\
k_3 &= f\left(t_n + \frac{4}{5}h, y_n + \frac{4}{5}h\left(\frac{k_1 + k_2}{2}\right)\right),\\
y_{n+1} &= y_n + \frac{h}{2}\left(\frac{2k_1k_2}{k_1 + k_2} + \frac{2k_2k_3}{k_2 + k_3}\right).
\end{aligned} \tag{12}$$

According to [10], SET 2 gives the ‘best’ result. Hence in this study, SET 2 is used to solve fuzzy differential equations.

5. STABILITY ANALYSIS

The stability of the third-order composite Runge-Kutta method can be calculated as follows:

Consider the test problem

$$y' = \lambda y. \tag{13}$$

By substituting eq. (12) into the test problem, we have

$$R(z) = 1 + \frac{z(5 + 3z)}{(10 + 3z)}\left[\frac{50 + 30z + 6z^2}{5(5 + 2z)}\right], \quad z \neq -2.5, \ -10/3. \tag{14}$$

The stability regions for the eq.(14) is illustrated in Figure 1 and Figure 2.

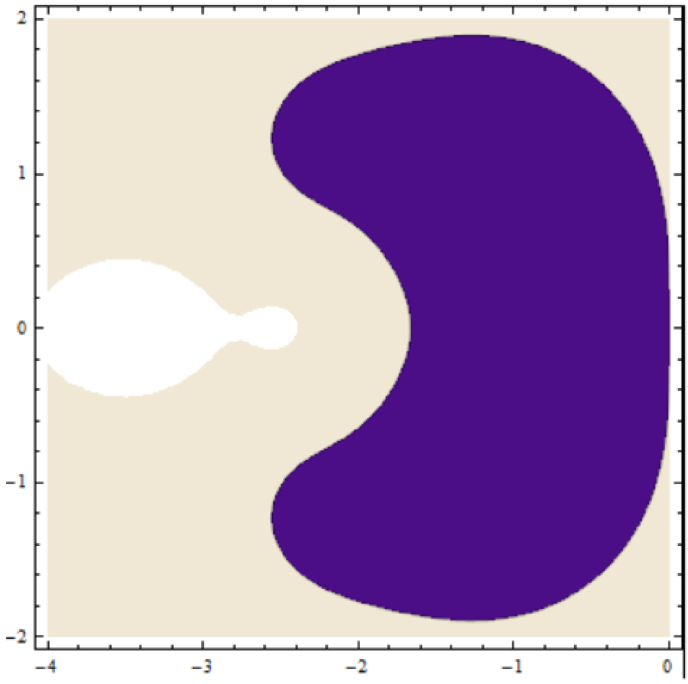

Figure 1: The 2D stability region plotted for the third-order composite Runge-Kutta method.

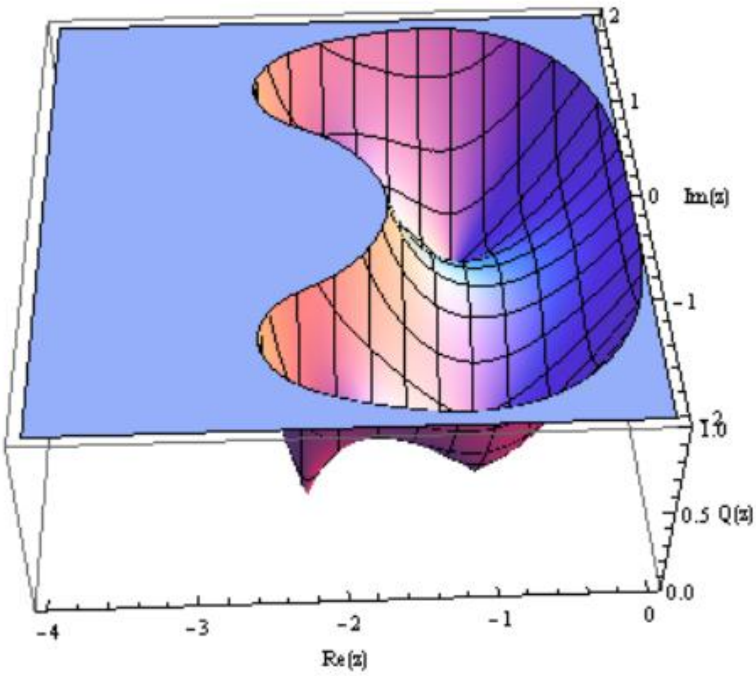

Figure 2: The 3D stability region plotted for the third-order composite Runge-Kutta method.

6. FUZZY CONFIGURATION FOR THIRD-ORDER COMPOSITE RUNGE KUTTA METHOD

Let the exact solution $[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)]$ is approximated by some $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$.

The solution is calculated by grid points at

$$h = \frac{T - t_0}{N}, t_i = t_0 + ih, \quad 0 \le i \le N \tag{15}$$

From (7), we define

$$\begin{aligned} \underline{y}(t_{n+1};r) &= \underline{y}(t_n;r) - \sum\nolimits_{i=1}^{3} w_i \underline{k}_i(t_n, \underline{y}(t_n;r)) \\ \overline{y}(t_{n+1};r) &= \overline{y}(t_n;r) - \sum\nolimits_{i=1}^{3} w_i \overline{k}_i(t_n, \overline{y}(t_n;r)) \end{aligned} \tag{16}$$

where w_i's are weighted value which are constants

$$k_i(t, y(t;r)) = \left[\underline{k}(t, y(t;r)), \overline{k}(t, y(t;r))\right], \qquad i = 1,2,3 \tag{17}$$

whereby

$$\begin{aligned} \underline{k}_1(t, y(t;r)) &= \min \{ f(t,u) \,|\, u \in [\underline{y}(t;r), \overline{y}(t;r)] \}, \\ \overline{k}_1(t, y(t;r)) &= \max \{ f(t,u) \,|\, u \in [\underline{y}(t;r), \overline{y}(t;r)] \}, \\ \underline{k}_2(t, y(t;r)) &= \min \left\{ f\left(t + h\frac{3}{5}, u \right) | \, u \in [\underline{z}_1(t;r), \overline{z}_1(t;r)] \right\}, \\ \overline{k}_2(t, y(t;r)) &= \max \left\{ f\left(t + h\frac{3}{5}, u \right) | \, u \in [\underline{z}_1(t;r), \overline{z}_1(t;r)] \right\}, \\ \underline{k}_3(t, y(t;r)) &= \min \left\{ f\left(t + h\frac{4}{5}, u \right) | \, u \in [\underline{z}_2(t;r), \overline{z}_2(t;r)] \right\}, \\ \overline{k}_3(t, y(t;r)) &= \max \left\{ f\left(t + h\frac{4}{5}, u \right) | \, u \in [\underline{z}_2(t;r), \overline{z}_2(t;r)] \right\}, \end{aligned} \tag{18}$$

where in the third-order composite Runge-Kutta method for solving FDEs is given as follows

$$\begin{aligned} \underline{z}_1(t, y(t;r)) &= \underline{y}(t;r) + \frac{3}{5}\underline{k}_1(t, y(t;r)), \\ \overline{z}_1(t, y(t;r)) &= \overline{y}(t;r) + \frac{3}{5}\overline{k}_1(t, y(t;r)), \\ \underline{z}_2(t, y(t;r)) &= \underline{y}(t;r) + \frac{4}{5}h\left(\frac{\underline{k}_1(t, y(t;r)) + \underline{k}_2(t, y(t;r))}{2} \right) \\ \overline{z}_2(t, y(t;r)) &= \overline{y}(t;r) + \frac{4}{5}h\left(\frac{\overline{k}_1(t, y(t;r)) + \overline{k}_2(t, y(t;r))}{2} \right) \end{aligned} \tag{19}$$

with the following equations

$$\underline{y}(t_{n+1};r)=\underline{y}(t_{n+1};r)+\frac{h}{2}\left(\left(\frac{2\underline{k}_1(t,y(t;r))\cdot\underline{k}_2(t,y(t;r))}{\underline{k}_1(t,y(t;r))+\underline{k}_2(t,y(t;r))}\right)+\left(\frac{2\underline{k}_2(t,y(t;r))\cdot\underline{k}_3(t,y(t;r))}{\underline{k}_2(t,y(t;r))+\underline{k}_3(t,y(t;r))}\right)\right)$$
$$\overline{y}(t_{n+1};r)=\overline{y}(t_{n+1};r)+\frac{h}{2}\left(\left(\frac{2\overline{k}_1(t,y(t;r))\cdot\overline{k}_2(t,y(t;r))}{\overline{k}_1(t,y(t;r))+\overline{k}_2(t,y(t;r))}\right)+\left(\frac{2\overline{k}_2(t,y(t;r))\cdot\overline{k}_3(t,y(t;r))}{\overline{k}_2(t,y(t;r))+\overline{k}_3(t,y(t;r))}\right)\right) \quad (20)$$

7. NUMERICAL EXAMPLES

In this section, the composite arithmetic–harmonic mean Runge–Kutta is tested on the following fuzzy initial value problems.

EXAMPLE 1:

Consider the fuzzy initial value problem [6].

$y'(t)=y(t) \qquad t\in[0,1],$

$y(0)=(0.8+0.125r,\ 1.1-0.1r).$

The exact solution at $t=1$ is

$Y(1,r)=\ (0.8+0.125r)e,\quad 1.1-0.1r\ \ e\ .$

In this paper, by considering Case 1 (for $h>0$) in Theorem 1, then the system of ODEs is given as follows:

$\underline{y}'(t)=\underline{y}(t), \qquad \underline{y}(t_0)=0.8+0.125r,$

$\overline{y}'(t)=\overline{y}(t), \qquad \overline{y}(t_0)=1.1-0.1r.$

Error $=\left|\ \underline{y}-\underline{Y}\ +\ \overline{y}-\overline{Y}\ \right|.$

Table 1. Numerical results for Example 1 using third-order composite RK and RK harmonic mean 3 quantities with $h=0.1$.

r	EXACT	THIRD-ORDER COMPOSITE RK	ERROR	RK HARMONIC MEAN 3 QUANTITIES	ERROR
0.0	2.174625462 2.990110011	2.171470132 2.985771432	7.49×10^{-3}	2.161248706 2.971716971	3.18×10^{-2}
0.2	2.242582508 2.935744374	2.239328574 2.931484678	7.51×10^{-3}	2.228787727 2.917685752	3.19×10^{-2}
0.4	2.310539554 2.881378738	2.307187015 2.877197925	7.53×10^{-3}	2.29632675 2.863654536	3.19×10^{-2}
0.6	2.378496600 2.827013101	2.375045454 2.822911171	7.55×10^{-3}	2.363865773 2.809623317	3.20×10^{-2}
0.8	2.446453645 2.772647465	2.442903898 2.768624419	7.57×10^{-3}	2.431404795 2.755592101	3.21×10^{-2}
1.0	2.514410691 2.718281828	2.510762339 2.714337666	7.59×10^{-3}	2.498943816 2.701560883	3.22×10^{-2}

Table 2. Numerical results for Example 1 using third-order composite RK and RK harmonic mean 3 quantities with $h = 0.01$.

r	EXACT	THIRD-ORDER COMPOSITE RK	ERROR	RK HARMONIC MEAN 3 QUANTITIES	ERROR
0.0	2.174625462 2.990110011	2.174591654 2.990063521	8.03×10^{-5}	2.173404265 2.988430855	2.90×10^{-3}
0.2	2.242582508 2.935744374	2.242547643 2.935698728	8.05×10^{-5}	2.241323148 2.934095748	2.91×10^{-3}
0.4	2.310539554 2.881378738	2.310503631 2.881333935	8.07×10^{-5}	2.309242032 2.879760641	2.92×10^{-3}
0.6	2.378496600 2.827013101	2.378459620 2.826969147	8.09×10^{-5}	2.377160914 2.825425534	2.92×10^{-3}
0.8	2.446453645 2.772647465	2.446415610 2.772604355	8.11×10^{-5}	2.445079799 2.771090427	2.93×10^{-3}
1.0	2.514410691 2.718281828	2.514371600 2.718239564	8.14×10^{-5}	2.512998681 2.716755321	2.94×10^{-3}

Table 1 and 2 shows the error in the solution by the third-order composite RK of two different step size ($h = 0.1$ and $h = 0.01$) along with the numerical result obtained by the RK harmonic mean 3 quantities.

EXAMPLE 2:

Consider the fuzzy initial value problem [6].

$$y'(t) = tf(t) \qquad t \in [0,1],$$

$$y(0) = (1.01 + 0.1r\sqrt{e}, 1.5 - 0.1r\sqrt{e}).$$

The exact solution at $t = 1$ is

$$Y(1,r) = \left(1.01 + 0.1r\sqrt{e} \;\; e^{\frac{1}{2}},\; 1.5 - 0.1r\sqrt{e} \;\; e^{\frac{1}{2}} \right)$$

In this paper, by considering Case 1 (for $h > 0$) in Theorem 1, then the system of ODEs is given as follows:

$$\underline{y}'(t) = t\underline{f}(t.y(t)), \qquad \underline{y}(t_0) = 1.01 + 0.1r\sqrt{e},$$

$$\overline{y}'(t) = t\overline{f}(t.y(t)), \qquad \overline{y}(t_0) = 1.5 - 0.1r\sqrt{e},$$

Error $= \left| \; \underline{y} - \underline{Y} \; + \; \overline{y} - \overline{Y} \; \right|$.

Table 3. Numerical results for Example 2 using third-order composite RK and RK harmonic mean 3 quantities with $h = 0.1$.

r	EXACT	THIRD-ORDER COMPOSITE RK	ERROR	RK HARMONIC MEAN 3 QUANTITIES	ERROR
0.0	1.665208484 2.473081906	1.658517822 2.463145280	1.66×10^{-2}	1.640208439 2.435953128	6.21×10^{-2}
0.2	1.719574120 2.418716271	1.712665022 2.408998080	1.66×10^{-2}	1.693757875 2.382403692	6.21×10^{-2}
0.4	1.773939757 2.364350633	1.766812222 2.354850880	1.66×10^{-2}	1.747307311 2.328854257	6.21×10^{-2}
0.6	1.828305393 2.309984997	1.820959423 2.300703679	1.66×10^{-2}	1.800856748 2.275304818	6.21×10^{-2}
0.8	1.882671031 2.25561936	1.875106623 2.246556479	1.66×10^{-2}	1.854406184 2.221755382	6.21×10^{-2}
1.0	1.937036666 2.201253724	1.929253823 2.192409279	1.66×10^{-2}	1.907955621 2.168205946	6.21×10^{-2}

Table 4. Numerical results for Example 2 using third-order composite RK and RK harmonic mean 3 quantities with $h = 0.01$.

r	EXACT	THIRD-ORDER COMPOSITE RK	ERROR	RK HARMONIC MEAN 3 QUANTITIES	ERROR
0.0	1.665208484 2.473081906	1.665119959 2.472950436	2.20×10^{-4}	1.663807345 2.471001009	3.48×10^{-3}
0.2	1.719574120 2.418716271	1.719482705 2.418587689	2.20×10^{-4}	1.718127238 2.416681117	3.48×10^{-3}
0.4	1.773939757 2.364350633	1.773845451 2.364224943	2.20×10^{-4}	1.772447130 2.362361225	3.48×10^{-3}
0.6	1.828305393 2.309984997	1.828208199 2.309862196	2.20×10^{-4}	1.826767021 2.308041332	3.48×10^{-3}
0.8	1.882671031 2.25561936	1.882570944 2.255499450	2.20×10^{-4}	1.881086915 2.25372144	3.48×10^{-3}
1.0	1.937036666 2.201253724	1.936933692 2.201136705	2.20×10^{-4}	1.935406806 2.199401548	3.48×10^{-3}

Table 3 and 4 shows the approximated numerical result and the comparison error with two different step size ($h = 0.1$ and $h = 0.01$).

8. DISCUSSION

Based on Table 1, the numerical result obtained through third-order composite Runge-Kutta method produces numerical values approximately close to the exact solution in comparison to the values obtained through Runge-Kutta Method with Harmonic Mean of Three Quantities. The difference of error between these two methods is by order one. In Table 2, as the step size decrease, third-order composite Runge-Kutta method shows outstanding result in comparison to Runge-Kutta Method with Harmonic Mean of Three Quantities. The error is obtained by order two.

In Table 3, third-order composite Runge-Kutta method yields the numerical results slightly better than Runge-Kutta Method with Harmonic Mean of Three Quantities which mean the error between this two methods is almost the same. In Table 4, with h=0.01, third-order composite Runge-Kutta method gives a better numerical solutions than Runge-Kutta Method with Harmonic Mean of Three Quantities. The error is obtained by order one.

9. CONCLUSIONS

In this paper, we have applied the third-order composite Runge-Kutta method for finding the numerical solution of fuzzy differential equations. Comparison of the solutions of Example 1 and 2, it proved that third-order composite Runge-Kutta method gives better solution than Runge-Kutta Method with Harmonic Mean of Three Quantities when compared with the exact solutions.

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REFERENCES

[1] D. Dubois, H. Prade, 1980, *Fuzzy Sets and System : Theory and Application*, New York: Academic Press.

[2] D. Paul Dhayabaran & J. Christy Kingston, 2015, *A Method for Solving Fuzzy Differential Equations Using Runge-Kutta Method with Harmonic Mean of Three Quantities*. International Journal of Engineering Science and Innovative Technology, 4(3), 90-96.

[3] G.J. Klir, U.H. St Clair, B. Yuan, 1997, *Fuzzy Set Theory: Foundations and Applications*, NJ: Prentice Hall.

[4] J.H. Mathew and K.D. Fink, 1999, *Numerical Methods using Matlab*. 4th Edition.

[5] L. Stefanini, B. Bede, 2009, *Generalized Hukuhara differentiability of interval-valued functions and interval differential equations*, Nonlinear Analysis, 71, 1311-1328.

[6] M. Ma, M. Friedman, A. Kandel, 1999, *Numerical solutions of fuzzy differential equation*, Fuzzy Sets and System, 105, 133-138.

[7] M. Puri, D. Ralescu, 1983, *Differential of Fuzzy Functions*, Journal of Mathematical Analysis and Applications. 91, 552-558.

[8] O. Kaleva, 1987, *Fuzzy Differential Equations*, Fuzzy Sets and System. 24, 301-317.

[9] O. Kaleva, 1990, *The Cauchy problem for fuzzy differential equations*, Fuzzy Sets and System. 35, 389-396.

[10] R.R. Ahmad and N. Yaacob, 2005, *Third-order composite Runge-Kutta method for stiff problems*. International Journal of Computer Mathematics. 82(10) :1221-1226.

[11] S. Abbasbandy, T, Allahviranloo, 2002, *Numerical Solutions of Fuzzy Differential Equations by Taylor Method*, Journal of Computational Methods in Applied Mathematics, 2, 113-124.

[12] S. Seikkala, 1987, *On the fuzzy initial value problem*, Fuzzy Sets and System, 24, 319-330.

[13] S. L. Chang, L.A. Zadeh, 1972, *On Fuzzy Mapping and Control*, IEEE Trans. System Man Cybernet, 2, 30-34.

[14] T. Allahviranloo, 2004, *Numerical Solution of Fuzzy Differential Equation by Adam-Bashforth two Step Method*, Journal of Applied Mathematics, 36-47.

[15] T. Jayakumar, D. Maheskumar, K. Kanarajan, 2012, *Numerical solution of fuzzy differential equations by Runge Kutta method of order five*, Applied Mathematical Sciences, 6, 2989-3002.

[16] Z.A. Majid, S. Mehkarnoon, M. Suleiman, 2009, *Block Method for Numerical Solution of Fuzzy Differential Equation*, International Mathematical Forum. 4, 2269-2280.

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