Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 3 (2016), pp. 2745-2752

http://www.ripublication.com/gjpam.htm

Half-Sweep Newton-Gauss-Seidelforimplicit finite difference solution of 1D nonlinear Porous Medium Equations

J. V. L. Chew and J. Sulaiman

Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia.

Abstract

This paper proposes a new numerical technique called Half-Sweep Newton-Gauss-Seidel(HSNGS) iterative method in solvingone-dimensional nonlinear porous medium equations. The general form of porous medium equation (PME) is discretized by using implicit finite difference scheme which leads to a nonlinear finite difference approximation equation. The developed system of nonlinear equations is transformed by the application of Newton methodinto the corresponding system of linear equations. The numerical solutions are obtained by HSNGS iteration. Four illustrative examples are chosen in order to show the effectiveness of the proposed technique. The numerical results are compared with the Full-Sweep Newton-Gauss-Seidel (FSNGS) to demonstrate the applicability of the proposed iterative method. The HSNGS iterative method shows superiority in term of iteration number and computational time.

Keywords: Porous medium equation, finite difference scheme, Newton method, Half-Sweep iteration.

Introduction

Porous medium equation (PME) is one of the important partial differential equation as it is encountered in various realistic phenomena and scientific problems such as the spatial diffusion of biological populations [1] as well as the spreading of thin liquid film under gravity, the study of thin saturated regions in porous media and the percolation of a gas through a porous medium [2]. In addition, PME is classified under nonlinear parabolic partial differential equation. PME problems have been investigated by many researchers and the solutions are commonly obtained by usingthe variational iterative method [3-4], Adomian decomposition method [5-6] as well ashomotopy perturbation method [7].

In this paper, finite difference scheme will be considered in order to construct a reliable approximation to PME. Due to the computational stability, implicit finite difference scheme is used. By using the nonlinear finite difference approximation equation, a system of nonlinear equations is generated at each time level. In this case, Newton method is applied to linearize and transform it into a sparse systemof linear equations so that iterative method can solve it efficiently.

Therefore, this paper proposes a numerical technique called Half-Sweep-Newton-Gauss-Seidel (HSNGS) iterative method which is actually a combination of Newton method and the Half-Sweep Gauss-Seidel iteration, in solving the one-dimensional PME problems. The general PME that is being considered throughout this paper is given by

$$\frac{du}{dt} = K\frac{d}{dx}\left(u^m \frac{du}{dx}\right), \quad a \le x \le b, \qquad t > 0 \tag{1}$$

where K and m are real numbers, subject to the boundary conditions and the initial condition as

$$\begin{aligned} &u(a,t) = g_1(t), \; u(b,t) = g_2(t), \; t > 0, \\ &u(x,0) = g_3(x), \; a \le x \le b. \end{aligned} \tag{2}$$

where $g_1(t)$, $g_2(t)$ and $g_3(x)$ are the prescribed functions.

To construct the finite difference solutions, the solutions domain ofEq. (1) can be uniformly partitioned into d subintervals with distance Δx defined as follows.

$$\Delta x = \frac{(b-a)}{d}, \quad d = n+1 \tag{3}$$

Half-Sweep Finite Difference Approximation Equation

To formulate the Half-Sweep finite difference approximation equation, let consider the finite grid network that is built to guide the development of the algorithm for the proposed iterative method, shown in Fig. 1.

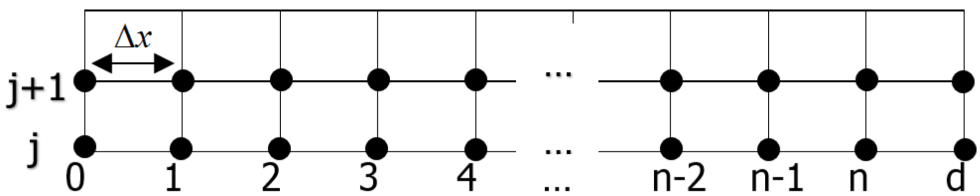

Fig. 1. Finite grid network

Noticed that Eq. (1) can be implicitly differentiated and rewritten as

$$\frac{du}{dt} = K\left[u^m \frac{d^2u}{dx^2} + mu^{m-1}\left(\frac{du}{dx}\right)^2\right] \tag{4}$$

Then, Eq. (4) can be discretized by using the implicit finite difference scheme to form the Full-and Half-Sweep finite difference approximation equations which are represented in general form as

$$\begin{aligned} &u_{i,j+1} - \alpha u_{i,j+1}^{m} u_{i+p,j+1} + 2\alpha u_{i,j+1}^{m+1} - \alpha u_{i,j+1}^{m} u_{i-p,j+1} \\ &- \beta m u_{i,j+1}^{m-1} u_{i+p,j+1}^{2} + 2\beta m u_{i,j+1}^{m-1} u_{i+p,j+1} u_{i-p,j+1} - \beta m u_{i,j+1}^{m-1} u_{i-p,j+1}^{2} = u_{i,j} \end{aligned} \quad (5)$$

where

$$\alpha = \frac{K\Delta t}{(p\Delta x)^2}, \quad \beta = \frac{K\Delta t}{4(p\Delta x)^2}$$

for $i = p, 2p, \ldots, d-p$ and $j = 0,1,2,\ldots,s$. The Full-Sweep case is represented by $p=1$ while the Half-Sweep case is by $p=2$. Actually, the Half-Sweep finite difference approximation equation can also be derived by using the equal grid spacing of $2\Delta x$. FromEq. (5), asystem of nonlinear equationsis generated at each time level j. Newton method is then used to linearize it and eventually constructs the corresponding system of linear equations. Firstly, define the nonlinear function F ofEq. (5) at each interior grid point (x_i, t_{j+1}) in Fig. 1 as

$$\begin{aligned} F_i(\bar{u}_{j+1}) = &\ u_{i,j+1} - \alpha u_{i,j+1}^{m} u_{i+p,j+1} + 2\alpha u_{i,j+1}^{m+1} - \alpha u_{i,j+1}^{m} u_{i-p,j+1} \\ &- \beta m u_{i,j+1}^{m-1} u_{i+p,j+1}^{2} + 2\beta m u_{i,j+1}^{m-1} u_{i+p,j+1} u_{i-p,j+1} - \beta m u_{i,j+1}^{m-1} u_{i-p,j+1}^{2} - u_{i,j} \end{aligned} \quad (6)$$

where $\bar{u}_{j+1} = (u_{p,j+1}, u_{2p,j+1}, \ldots, u_{d-p,j+1})^T$. Then, Eq. (6) leads to a system of nonlinear equationsas

$$F_i(\bar{u}_{j+1}) = 0, \quad i = 1,2,3,\ldots,n \quad (7)$$

At each time level j, Newton method is used to transform the system of nonlinear equations(7) into the corresponding system of linear equations

$$J(\bar{u}_{j+1}^{(k)})\Delta\bar{h}_{j+1} = -F_i(\bar{u}_{j+1}^{(k)}) \quad (8)$$

where

$$J(\bar{u}_{j+1}^{(k)}) = \begin{bmatrix} \frac{\partial f_{p,j+1}}{\partial u_p} & \frac{\partial f_{p,j+1}}{\partial u_{2p}} & \cdots & \frac{\partial f_{p,j+1}}{\partial u_{d-p}} \\ \frac{\partial f_{2p,j+1}}{\partial u_p} & \frac{\partial f_{2p,j+1}}{\partial u_{2p}} & \cdots & \frac{\partial f_{2p,j+1}}{\partial u_{d-p}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_{d-p,j+1}}{\partial u_p} & \frac{\partial f_{d-p,j+1}}{\partial u_{2p}} & \cdots & \frac{\partial f_{d-p,j+1}}{\partial u_{d-p}} \end{bmatrix}^{(k)}, \Delta\bar{h}_{j+1} = \begin{bmatrix} \Delta h_p \\ \Delta h_{2p} \\ \vdots \\ \Delta h_{d-p} \end{bmatrix}_{j+1}.$$

By solving system of linear equations (8), the unknown vector $\Delta\bar{h}$ is obtained. Lastly, the numerical solutions ofthe general PME (1) that is denoted by $\bar{u}_{j+1}^{(k+1)}$ can becomputed using the expression of

$$\bar{u}_{j+1}^{(k+1)} = \bar{u}_{j+1}^{(k)} + \Delta\bar{h}_{j+1} \quad (9)$$

with k-th iterations. Next section will discuss on the derivation of the proposed iterative methods.

Formulation And Implementation Of Gauss-Seidel Iteration Family

According to the second section, system of linear equations (8) can be considered generally as

$$A\Delta\bar{h} = b \tag{10}$$

Actually, there are various iterative methods that can be used to solve the system of linear equations (10), for instance Jacobi, Gauss-Seidel, SOR, etc. For more details see in [8-10] and the references therein. Apart from these methods, the Half-Sweep concept has been introduced by Abdullah [11] in solving two-dimensional Poisson equations. The main capability of this concept is to reduce the computational complexity of the generated system of linear equations. Advantages showed by [11] have been extensively investigated and used to solve different kind of mathematical problems such as linear Fredholm integral equation [12-13] and linear Fredholmintegro-differential equation [14], two-dimensional Helmholtz equations [15], second order two-point boundary value problems [16], and application problem such as path planning for indoor mobile robot [17].

To derive the formulation of family of Gauss-Seidel iterative methods, let the coefficient matrix A in Eq. (9) be decomposed into

$$A = D + L + U \tag{11}$$

where D is the diagonal of matrix A, L is the lower triangular part and U is the upper triangular part. It is always assumed that the entries for D are all nonzero. As mentioned in [8-10], the Gauss-Seidel iterative method can be stated as

$$\Delta\bar{h}^{(k+1)} = (D + L)^{-1}(-U\Delta\bar{h}^{(k)} + b) \tag{12}$$

The general algorithm for the derivedHSNGS iterative method is described in Algorithm 1.

Algorithm 1: HSNGS iterative method

i. Initialize $\bar{u}_{j+1}^{(0)} \leftarrow 1.0000, \varepsilon_{Newton} \leftarrow 10^{-10}, \varepsilon \leftarrow 10^{-10}$

ii. For $j = 0,1,\ldots,s$, implement

a. Set $\Delta\bar{h}_{j+1}^{(0)} = 0$,

b. Calculate $F(\bar{u}_{j+1}^{(k)})$, A and b,

c. For $i = 2,4,\ldots,d-2$, compute (12),

d. For $i = 1,3,\ldots,d-1$, compute directly.

iii. Check the convergence $\left|\Delta\bar{h}_{j+1}^{(k+1)} - \Delta\bar{h}_{j+1}^{(k)}\right| \le \varepsilon$. If yes, go to (iv). Otherwise back to (ii).

iv. Calculate $\bar{u}_{j+1}^{(k+1)} = \bar{u}_{j+1}^{(k)} + \Delta h_{j+1}^{(k)}$

v. Convergence test $\left|F_i(\bar{u}_{j+1}^{(k+1)})\right| \le \varepsilon_{Newton}$. If converges, go to next j. Otherwise go to(i).

vi. Display approximate solutions.

Numerical Experiments

In order to verify the effectiveness of the HSNGS iterative method as compared with the FSNGSiterative method, four PME problems are tested. In the numerical experiments, three criteria will be considered such as number of iterations (Iter), computational time measured in second (Time) and maximum absolute error (MAE). The implementation is based on the tolerance error of $\varepsilon = 10^{-10}$. The following are four examples of PME problems.

Example 1

$$\frac{du}{dt} = \frac{d}{dx}\left(u\frac{du}{dx}\right) \tag{13}$$

subject to initial condition $u(x,0) = C_1x + C_2$ and the exact solution is $u(x,t) = C_1x + C_1^2t + C_2$ where C_1 and C_2 are arbitrary constants [15]. Eq. (13) appears in isothermal percolation of a perfect gas through a micro-porous medium [3-5]. For numerical experiments, this study uses $C_1 = 1$ and $C_2 = 0$.

Example 2

$$\frac{du}{dt} = 0.5\frac{d}{dx}\left(u^{-1}\frac{du}{dx}\right) \tag{14}$$

subject to initial condition $u(x,0) = (C_1x + C_2)^{-1}$ and $u(x,t) = (C_1x - 0.5C_1^2t + C_2)^{-1}$ is the exact solution, where C_1 and C_2 are arbitrary constants [15]. Eq. (14) presents in the thermal limit approximation of Carleman's model of the Boltzman equation and the expansion into a vacuum of a thermalized electron cloud described by the isothermal Maxwellian distribution [3-5]. This study uses $C_1 = 0.6$ and $C_2 = 1.3$ for the implementation.

Example 3

$$\frac{du}{dt} = \frac{d}{dx}\left(u^2\frac{du}{dx}\right) \tag{15}$$

subject to initial condition $u(x,0) = (x+1)(C)^{-1}$ and exact solution is $u(x,t) = (x+1)\left(\sqrt{C^2 - t}\right)^{-1}$, with condition $t < C^2$ and C is an arbitrary constant [15]. Eq. (15) is used to model a process of melting and evaporation of metals [3-5]. Similar purpose to Examples 1 and 2, this study choses the arbitrary constant $C = 2$.

Example 4

$$\frac{du}{dt} = 0.5\frac{d}{dx}\left(u^{-2}\frac{du}{dx}\right) \tag{16}$$

subject to initial condition $u(x,0) = (2C_1x + C_2)^{-\frac{1}{2}}$ and $u(x,t) = (2C_1x - c_1^2t + C_2)^{-\frac{1}{2}}$ is the exact solution given by [15]. Eq. (16) is commonly considered as diffusion model in high-polymeric systems [3-5]. This studyset the arbitrary constants $C_1 = 0.35$ and $C_2 = 1.35$.

The following numerical results of the HSNGS and FSNGS iterative methods have been summarized in Tables 1, 2 and 3.

TABLE 1. Comparison of number of iterations (Iter), execution time in seconds (Time) and maximumabsolute errors (MAE) for the iterative methods using Examples 1 and 2.

M	Method	Example 1			Example 2		
		Iter	Time	MAE	Iter	Time	MAE
64	**FSNGS**	3835	2. 38	2. 76E-08	1720	1. 13	2. 03E-05
	HSNGS	1065	0. 16	6. 16E-09	489	0. 20	2. 03E-05
128	**FSNGS**	13678	7. 50	1. 22E-07	6034	4. 06	2. 02E-05
	HSNGS	3835	0. 86	2. 75E-08	1720	1. 07	2. 03E-05
256	**FSNGS**	48395	38. 58	5. 33E-07	20907	27. 03	2. 00E-05
	HSNGS	13678	5. 62	1. 22E-07	6034	6. 45	2. 02E-05
512	**FSNGS**	169693	252. 94	2. 10E-06	71385	287. 34	1. 93E-05
	HSNGS	48395	38. 22	5. 33E-07	20907	43. 75	2. 00E-05
1024	**FSNGS**	587031	1712. 49	7. 62E-06	239975	1741. 01	1. 72E-05
	HSNGS	169693	274. 28	2. 10E-06	71385	304. 92	1. 93E-05

TABLE 2. Comparison of number of iterations (Iter), execution time in seconds (Time) and maximum absolute errors (MAE) for the iterative methods using Examples 3 and 4.

M	Method	Example 3			Example 4		
		Iter	Time	MAE	Iter	Time	MAE
64	**FSNGS**	1344	1. 17	8. 39E-05	2015	1. 26	2. 88E-06
	HSNGS	386	0. 17	8. 38E-05	562	0. 23	2. 65E-06
128	**FSNGS**	4824	2. 84	8. 39E-05	7082	4. 90	2. 90E-06
	HSNGS	1344	0. 75	8. 39E-05	2015	1. 23	2. 88E-06
256	**FSNGS**	17308	20. 03	8. 39E-05	24325	45. 42	2. 71E-06
	HSNGS	4824	4. 71	8. 39E-05	7082	7. 37	2. 90E-06
512	**FSNGS**	61658	270. 11	8. 40E-05	81729	354. 79	1. 86E-06
	HSNGS	17308	33. 05	8. 39E-05	24325	50. 33	2. 71E-06
1024	**FSNGS**	218147	2008. 35	8. 43E-05	265698	2293. 23	3. 33E-06
	HSNGS	61658	227. 65	8. 40E-05	81729	332. 37	1. 86E-06

TABLE 3. Reduction in percentages by the HSNGS iterative method

M	Number of iterations				Computational time in seconds			
	Example 1	Example 2	Example 3	Example 4	Example 1	Example 2	Example 3	Example 4
64	72. 23%	71. 57%	71. 28%	72. 11%	93. 28%	82. 30%	85. 47%	81. 75%
128	71. 96%	71. 49%	72. 14%	71. 55%	88. 53%	73. 65%	73. 59%	74. 90%
256	71. 74%	71. 14%	72. 13%	70. 89%	85. 43%	76. 14%	76. 49%	83. 77%
512	71. 48%	70. 71%	71. 93%	70. 24%	84. 89%	84. 77%	87. 76%	85. 81%
1024	71. 09%	70. 25%	71. 74%	69. 24%	83. 98%	82. 49%	88. 66%	85. 51%

Conclusion

Throughout this paper, the applicability of the proposed HSNGSmethod as compared to the FSNGS method for solving 1D nonlinear PME problems isexamined. The numerical results that are demonstrated by Tables 1 and 2 showed that the HSNGS method requires lesser number of iterations and computational time than the FSNGS method. The HSNGS method has reduced number of iterations approximately 69. 24%-72. 23% and computational time approximately 73. 59%-93. 28%, see in Table 3. Overall, the two tested numerical techniques have a good agreement in term of accuracy. From the numerical results, it can be stated that the HSNGSiterative method can be a good prospective technique to solve various kind of nonlinear differential equations. Effort to improve the rate of convergence in the proposed technique will become the subject of interest in future. Therefore, this study will look forward into the application of the Half-Sweep concept with the family of successive over relaxation method.

Acknowledgments

Authors gratefully acknowledge the financial support received from Sabah State Government Scholarship.

References

[1] Gurtin, M. E., andMaccamy, R. C., 1977, “On the diffusion of biological populations, ” Math. Biosci., 33, pp. 35-49.

[2] Lacey, A. A., Ockendon, J. R., and Tayler, A. B., 1982, “Waiting-time solutions of a nonlinear diffusion equation, ” SIAM J. Appl. Math., 42(6), pp. 1252-1264.

[3] Sadighi, A., and Ganji, D. D., 2007, “Exact solutions of nonlinear diffusion equations by variational iteration method, ”Comput. Math. Appl., 54, pp. 1112-1121.

[4] Wazwaz, A. M., 2007, “The variational iteration method: a powerful scheme for handling linear and nonlinear diffusion equations, ”Comput. Math. Appl., 54, pp. 933-939.
[5] Wazwaz, A. M., 2001, “Exact solutions to nonlinear diffusion equations obtained by the decomposition method, ” Appl. Math. Comput., 123, pp. 109-122.
[6] Pamuk, S., 2005, “On the solution of the porous media equation by decomposition method: a review, ” Phys. Lett. A., 344, pp. 184-188.
[7] Chun, C. B., Jafari, H., and Kim, Y. I., 2009, “Numerical method for the wave and nonlinear diffusion equations with the homotopy perturbation method, ”Comput. Math. Appl., 57, pp. 1226-1231.
[8] Saad, Y., 2003, Iterative methods for sparse linear systems, 2nd edition, Society for Industrial and Applied Mathematics, Philadelphia, US.
[9] Young, D. M., 1954, “Iterative methods for solving partial difference equations of elliptic type, ” T. Am. Math. Soc., 76, pp. 92-111.
[10] Young, D. M., 2014, Iterative solution of large linear systems, Academic Press, Florida, US.
[11] Abdullah, A. R., 1991, “The four point Explicit Decoupled Group (EDG) method: a fast Poisson solver, ” Int. J. Comput. Math., 38, pp. 61-70.
[12] Muthuvalu, M. S., andSulaiman, J., 2009, “Half-sweep arithmetic mean method with high order Newton-Cotes quadrature schemes to solve linear second kind Fredholm equations, ” J. Fund. Sci., 5(1), pp. 7-16.
[13] Muthuvalu, M. S., and Sulaiman, J., 2011, “Half-sweep arithmetic mean method with composite trapezoidal scheme for solving linear Fredholm integral equation, ” Appl. Math. Comput., 217, pp. 5442-5448.
[14] Aruchunan, E., andSulaiman, J., 2011, “Half-sweep conjugate gradient method for solving first order linear Fredholmintegro-differential equations, ” Aust. J. Basic Appl. Sci., 5(2), pp. 38-43.
[15] Akhir, M. K. M., Othman, M., Sulaiman, J., Majid, Z. A., and Suleiman, M., 2011, “Half-Sweep Modified Successive OverRelaxation for solving two-dimensional Helmholtz equations, ” Aust. J. Basic Appl. Sci., 5(12), pp. 3033-3039.
[16] Fauzi, N. I. M., and Sulaiman, J., 2012, “Half-Sweep Modified Successive Over Relaxation method for solving second order two-point boundary value problems using cubic spline, ” Int. J. Contemp. Math. Sci., 7(32), pp. 1579-1589.
[17] Saudi, A., andSulaiman, J., 2012, “Path planning for indoor mobile robot using Half-Sweep SOR via nine-point Laplacian (HSSOR9L), ” IOSR J. Math., 3(2), pp. 1-7.
[18] Polyanin, A. D., and Zaitsev, V. F., 2004, Handbook of Nonlinear Partial Differential Equation, Chapman and Hall, Boca Raton, US.