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Projective lag synchronization of the chaotic complex nonlinear systems with uncertain parameters and its applications in secure communications

Emad E. Mahmoud[1,2] and **S. Abdel-Khalek**[1,2]

[1] *Department of Mathematics, Faculty of Sciences,Taif University, Taif, KSA*
[2] *Department of Mathematics, Faculty of Sciences, Sohag University, Sohag, Eygpt.*

Abstract

The main aim of this research is to find an analytical and numerical study to investigate the projective Lag synchronization (PLS) of the chaotic complex nonlinear systems with uncertain parameters. We can be used this study to achieve secure communication between these systems. Based on the adaptive control technique and the Lyapunov function a scheme is designed to achieve PLS of chaotic attractors of these systems. The PLS of two identical complex Chen systems are taken as an example to verify the feasibility of the presented scheme. Numerical simulations are calculated to demonstrate the effectiveness of the proposed synchronization scheme and verify the theoretical results. The results will provide theoretical foundation for the secure communication applications based on the proposed scheme. In this secure communication scheme, synchronization between transmitter and receiver is achieved and message signals are recovered. The encryption and restoration of the signals are simulated numerically.

Keywords: Projective lag synchronization, Secure communication, Lyapunov function, Complex systems.

Introduction

Since the concept of synchronization introduced by Pecora and Carroll [1], chaos synchronization, as a very active topic in nonlinear science, has attracted increasing attention. To date, chaos synchronization has been developed extensively due to its various applications [2–5]. A great many synchronization methods have been proposed, such as complete synchronization (CS) [1], generalized synchronization (GS) [6], phase synchronization (PhS) [7], anti-synchronization (AS) [8], lag synchronization (LS) [9],projective synchronization (PS) and projective lag synchronization (PLS) etc. In 1999, projective synchronization (PS) has been first reported by Mainieri and Rehacek [10] in partially linear systems that the drive and response systems synchronize up to a constant scaling factor δ. Later there is many searches in PS whether the nonlinear systems with real or complex variables [10-12]. Complete synchronization and anti-synchronization are the special cases of PS where the scaling factor $\delta=1$ and $\delta=-1$, respectively.

Recently, PLS is studied in nonlinear systems with real variables [13, 14]. PLS means the state of the response system at time t is asymptotically synchronous with the drive system at time $t-\tau$, namely, $\lim_{t\to\infty}\|x(t)-\mu y(t-\tau)\|=0$, where x(t) and y(t) are the states of the response and drive systems, respectively. Thus, PLS is the more general definition of PS.

In engineering applications, LS and PLS always affect the dynamical behaviors of chaotic systems. For example, in the telephone communication system, the voice one hears on the receiver side at time t is the voice from the transmitter side at time $t-\tau$. So, strictly speaking, it is not reasonable to require the slave system to synchronize the master system at exactly the same time [15, 16].

Since Fowler et al.[17] introduced the complex Lorenz equations, complex systems have played an important role in many branches of physics [18], especially for chaos-communication, where the complex variables (doubling the number of variables) increase the contents and security of the transmitted information [19]. The main idea of chaos communication is to utilize the chaotic signals as carriers for information transmission, and at the receiver end chaos synchronization is employed to recover the information signal. Hence, the synchronization of complex chaotic systems has attracted great attention in the last few decades [18-25].

A dynamical system is called chaotic if it is deterministic, has long-term a periodic behavior, and exhibits sensitive dependence on the initial conditions. If the system has one positive Lyapunov exponent then the system is called chaotic [19].

The motivation of this search is to prepare analytical and numerical study to investigate the definition of PLS when the drive and response systems (with complex variables) are identical with uncertain parameters. We hope to state new scheme to achieve this study and exploit this scheme to secure communication between these systems. The PLS of two identical complex Chen systems is taken as an example to verify the feasibility of the presented scheme. This chaotic complex system appears in several applications in physics, engineering and other applied sciences [19].

A chaotic complex nonlinear system

A complex dynamical system is called chaotic if it is deterministic, has long-term a periodic behavior, and exhibits sensitive dependence on the initial conditions. A chaotic complex attractor is defined as a complex chaotic attractor with one positive Lyapunov exponents. The sum of Lyapunov exponents must be negative to ensure that system is dissipative. It is even more complicated than chaotic real systems and has more unstable manifolds. Due to chaotic complex systems with characteristics of high capacity, high security and high efficiency, it has a broadly applied potential in nonlinear circuits, secure communications, lasers, neural networks, biological systems and so on. Therefore, research on chaotic complex nonlinear systems is extremely important nowadays [14].

Consider the chaotic complex nonlinear system as follow:

$$\dot{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x})\mathbf{A} + \mathbf{f}(\mathbf{x}), \qquad (1)$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ is a state complex vector, $\mathbf{x} = \mathbf{x}^r + j\mathbf{x}^i$, $\mathbf{x}^r = (u_1, u_3, ..., u_{2n-1})^T$, $\mathbf{x}^i = (u_2, u_4, ..., u_{2n})^T$, $j = \sqrt{-1}$, T denotes transpose, $\mathbf{\Phi}(\mathbf{x})$ is $n \times n$ complex matrix and the elements of this matrix are state complex variables, $\mathbf{A}$ is $n \times 1$ vector of system parameters, $\mathbf{f} = (f_1, f_2, ..., f_n)^T$ is a vector of linear or nonlinear complex functions and superscripts r and i stand for the real and imaginary parts of the state complex vector $\mathbf{x}$.

In this paper we study the definition of LS of two identical systems of the form (1) with unknown parameters by designing a control scheme.

Remark 1. Most of chaotic complex system can be described by (1), such as complex Lorenz, Chen and Lü systems.

In order to show the results of our scheme of two identical systems of the form (1) we choose, as an example, the chaotic complex Chen systems which have been introduced and studied recently in our work [6].

The chaotic complex Chen system is:

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= (c - a)x - xz + \gamma y, \qquad (2) \\ \dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - bz, \end{aligned}$$

where $\mathbf{x} = (x_1, x_2, x_3)^T = (x, y, z)^T$, a, c, b are positive parameters, $x = u_1 + ju_2, y = u_3 + ju_4$ are complex functions, and $u_l \quad (l = 1, ..., 4), z = u_5$ is real function. Dots represent derivatives with respect to time and an overbar denotes complex conjugate variables.

The chaotic complex Chen system is a 5-dimensional continuous real autonomous system. System (2) has trivial and non-trivial fixed points.

System (2) exhibits chaotic behavior when $a = 42, c = 26$ and $4 < b < 6$, for more detail see [6]. In [6] we calculated numerically, by using the Lyapunov exponents, the parameters values at which these chaotic attractors exist see Fig. 1.

In this system the main variables participating in the dynamics are complex. Clearly, if the variables of the system are complex the equations involve twice as many variables and control parameters, thus making it that much harder for a hostile agent to intercept and decipher the coded message. System (2) is used to describe and simulate the physics of detuned lasers and thermal convection of liquid flows.

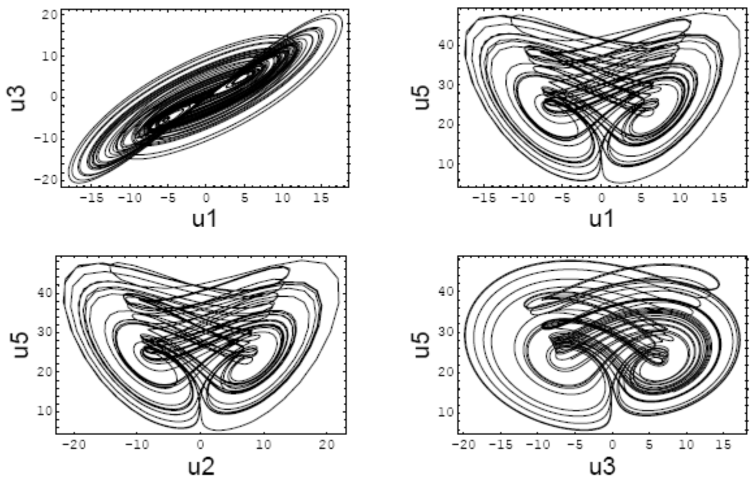

Figure 1: Chaotic attractors of chaotic complex Chen system in some plans.

A scheme for design a complex controller of adaptive PLS

We consider two identical chaotic complex nonlinear systems of the form (1), one is the master system (we denote the master system with the subscript m) as:

$$\dot{\mathbf{x}}_m = \dot{\mathbf{x}}_m^r + j\dot{\mathbf{x}}_m^i = \boldsymbol{\Phi}(\mathbf{x}_m)\mathbf{A} + \mathbf{f}(\mathbf{x}_m), \qquad (3)$$

and the second is the controlled slave system (with subscript s) as:

$$\dot{\mathbf{x}}_s = \dot{\mathbf{x}}_s^r + j\dot{\mathbf{x}}_s^i = \boldsymbol{\Phi}(\mathbf{x}_s)\mathbf{A} + \mathbf{f}(\mathbf{x}_s) + \mathbf{L}, \qquad (4)$$

where the additive complex controller $\mathbf{L} = (L_1, L_2, ..., L_n)^T = \mathbf{L}^r + j\mathbf{L}^i$, $\mathbf{L}^r = (v_1, v_3, ..., v_{2n-1})^T$, $\mathbf{L}^i = (v_2, v_4, ..., v_{2n})^T$.

Definition. Two complex dynamical systems coupled in a master-slave configuration can exhibit PLS if there exists a vector of the complex error function δ define such as:

$$\boldsymbol{\delta} = \boldsymbol{\delta}^r + j\boldsymbol{\delta}^i = \lim_{t\to\infty}\|\mathbf{x}_s(t) - \mathrm{M}\mathbf{x}_m(t-\tau)\| = \mathbf{0}, \qquad (5)$$

where $\boldsymbol{\delta} = (\delta_1, \delta_2, ..., \delta_n)^T$, $\mathbf{x}_m(t)$ and $\mathbf{y}_s(t)$ are the state complex vectors of the master and slave systems, respectively, $\boldsymbol{\delta}^r = \lim_{t\to\infty}\|\mathbf{x}_s^r(t) - \mathbf{x}_m^r(t-\tau)\| = 0$ and $\boldsymbol{\delta}^i = \lim_{t\to\infty}\|\mathbf{x}_s^i(t) - \mathbf{x}_m^i(t-\tau)\| = 0$, $\boldsymbol{\delta}^r = (\delta_{u_1}, \delta_{u_3}, ..., \delta_{u_{2n-1}})^T$, $\boldsymbol{\delta}^i = (\delta_{u_2}, \delta_{u_4}, ..., \delta_{u_{2n}})^T$, τ is the positive time lag and M=diag(μ, μ,…, μ) where m is real number.

Remark 2. When $\tau = 0$ in Eq.(5) we define projective synchronization between systems (3) and (4).

Remark 3. If we define $\delta = \lim_{t\to\infty}\|\mathbf{x}_s(t) - \mathrm{M}\mathbf{x}_m(t-\tau)\|$, $\mu = 1$ we get lag synchronization of systems (3) and (4), while if $\mu = -1$ we obtain anti lag synchronization of the same systems.

Theorem 1. If nonlinear controller is designed as:

$$\begin{aligned}\mathbf{L} = \mathbf{L}^r + j\mathbf{L}^i &= -\boldsymbol{\Phi}(\mathbf{x}_s(t))\hat{\mathbf{A}} - \boldsymbol{f}(\mathbf{x}_s(t)) + \mathrm{M}\boldsymbol{\Phi}(\mathbf{x}_m(t-\tau))\hat{\mathbf{A}} + \mathrm{M}\boldsymbol{f}(\mathbf{x}_m(t-\tau)) - \mathbf{K}\boldsymbol{\delta} \\ &= -\boldsymbol{\Phi}^r(\mathbf{x}_s(t))\hat{\mathbf{A}} - \boldsymbol{f}^r(\mathbf{x}_s(t)) + \mathrm{M}\boldsymbol{\Phi}^r(\mathbf{x}_m(t-\tau))\hat{\mathbf{A}} + \mathrm{M}\boldsymbol{f}^r(\mathbf{x}_m(t-\tau)) - \mathbf{K}\boldsymbol{\delta}^r \\ &\quad + j[-\boldsymbol{\Phi}^i(\mathbf{x}_s(t))\hat{\mathbf{A}} - \boldsymbol{f}^i(\mathbf{x}_s(t)) + \mathrm{M}\boldsymbol{\Phi}^i(\mathbf{x}_m(t-\tau))\hat{\mathbf{A}} + \mathrm{M}\boldsymbol{f}^i(\mathbf{x}_m(t-\tau)) - \mathbf{K}\boldsymbol{\delta}^i],\end{aligned} \tag{6}$$

and the adaptive laws of parameters are selected as:

$$\begin{cases} \dot{\hat{\mathbf{A}}} = (\boldsymbol{\Phi}^r(\mathbf{x}_s(t)))^T\boldsymbol{\delta}^r + (\boldsymbol{\Phi}^i(\mathbf{x}_s(t)))^T\boldsymbol{\delta}^i + \\ (-\mathrm{M}\boldsymbol{\Phi}^r(\mathbf{x}_m(t-\tau)))^T\boldsymbol{\delta}^r + (-\mathrm{M}\boldsymbol{\Phi}^i(\mathbf{x}_m(t-\tau)))^T\boldsymbol{\delta}^i - \boldsymbol{\Lambda}\tilde{\mathbf{A}}, \end{cases} \tag{7}$$

then the slave system (4) projective lag synchronize the master system (3) asymptotically, where $\mathbf{K} = diag(k_1, k_2, ..., k_n)$, $\boldsymbol{\Lambda} = diag(\zeta_1, \zeta_2, ..., \zeta_n)$, k_l, ζ_l are positive constants , $l = 1, 2, ..., n$. The parameters of vectors $\hat{\mathbf{A}}$ is the parameters estimation of vectors $\mathbf{A}$, $\tilde{\mathbf{A}} = \hat{\mathbf{A}} - \mathbf{A}$.

Proof: From the definition of PLS:

$$\boldsymbol{\delta} = \boldsymbol{\delta}^r + j\boldsymbol{\delta}^i = \mathbf{x}_s(t) - \mathrm{M}\mathbf{x}_m(t-\tau). \tag{8}$$

So,

$$\begin{aligned}\dot{\boldsymbol{\delta}} = \dot{\boldsymbol{\delta}}^r + j\dot{\boldsymbol{\delta}}^i &= \dot{\mathbf{x}}_s(t) - \mathrm{M}\dot{\mathbf{x}}_m(t-\tau) \\ &= \dot{\mathbf{x}}_s^r(t) - \mathrm{M}\dot{\mathbf{x}}_m^r(t-\tau) + j[\dot{\mathbf{x}}_s^i(t) - \mathrm{M}\dot{\mathbf{x}}_m^i(t-\tau)].\end{aligned} \tag{9}$$

From chaotic complex systems (3) and (4), we get the error complex dynamical system as follows:

$$\begin{aligned}\dot{\boldsymbol{\delta}} = \dot{\boldsymbol{\delta}}^r + j\dot{\boldsymbol{\delta}}^i &= \boldsymbol{\Phi}^r(\mathbf{x}_s(t))\mathbf{A} + \boldsymbol{f}^r(\mathbf{x}_s(t)) - \mathrm{M}\boldsymbol{\Phi}^r(\mathbf{x}_m(t-\tau))\mathbf{A} - \mathrm{M}\boldsymbol{f}^r(\mathbf{x}_m(t-\tau)) + \mathbf{L}^r \\ &\quad + j[\boldsymbol{\Phi}^i(\mathbf{x}_s(t))\mathbf{A} + \boldsymbol{f}^i(\mathbf{x}_s(t)) - \mathrm{M}\boldsymbol{\Phi}^i(\mathbf{x}_m(t-\tau))\mathbf{A} - \mathrm{M}\boldsymbol{f}^i(\mathbf{x}_m(t-\tau)) + \mathbf{L}^i].\end{aligned} \tag{10}$$

Thus, substituting from equation (6) about $\mathbf{L}^r$, $\mathbf{L}^i$ in (10) we obtain:

$$\begin{aligned}\dot{\boldsymbol{\delta}} = \dot{\boldsymbol{\delta}}^r + j\dot{\boldsymbol{\delta}}^i &= \boldsymbol{\Phi}^r(\mathbf{x}_s(t))(\mathbf{A} - \hat{\mathbf{A}}) - \mathrm{M}\boldsymbol{\Phi}^r(\mathbf{x}_m(t-\tau))(\mathbf{A} - \hat{\mathbf{A}}) - \mathbf{K}\boldsymbol{\delta}^r \\ &\quad + j[\boldsymbol{\Phi}^i(\mathbf{x}_s(t))(\mathbf{A} - \hat{\mathbf{A}}) - \mathrm{M}\boldsymbol{\Phi}^i(\mathbf{x}_m(t-\tau))(\mathbf{A} - \hat{\mathbf{A}}) - \mathbf{K}\boldsymbol{\delta}^i],\end{aligned} \tag{11}$$

where vector of the parameters errors is defined as $\tilde{\mathbf{A}} = \hat{\mathbf{A}} - \mathbf{A}$. By separating the real and the imaginary parts in Eq. (11), the error complex system is written as:

$$\begin{cases} \dot{\boldsymbol{\delta}}^r = \boldsymbol{\Phi}^r(\mathbf{x}_s(t))(\mathbf{A} - \hat{\mathbf{A}}) - \mathrm{M}\boldsymbol{\Phi}^r(\mathbf{x}_m(t-\tau))(\mathbf{A} - \hat{\mathbf{A}}) - \mathbf{K}\boldsymbol{\delta}^r, \\ \dot{\boldsymbol{\delta}}^i = \boldsymbol{\Phi}^i(\mathbf{x}_s(t))(\mathbf{A} - \hat{\mathbf{A}}) - \mathrm{M}\boldsymbol{\Phi}^i(\mathbf{x}_m(t-\tau))(\mathbf{A} - \hat{\mathbf{A}}) - \mathbf{K}\boldsymbol{\delta}^i. \end{cases} \tag{12}$$

For positive parameters, we may now define a Lyapunov function for this system by the following positive definite quantity:

$$\begin{aligned}V(t) &= \tfrac{1}{2}[(\boldsymbol{\delta}^r)^T\boldsymbol{\delta}^r + (\boldsymbol{\delta}^i)^T\boldsymbol{\delta}^i + (\hat{\mathbf{A}} - \mathbf{A})^T(\hat{\mathbf{A}} - \mathbf{A})] \\ &= \tfrac{1}{2}\left(\sum_{l=1}^{n}\delta_{u_{2l-1}}^2 + \sum_{l=1}^{n}\delta_{u_{2l}}^2 + \tilde{\mathbf{A}}^T\tilde{\mathbf{A}}\right).\end{aligned} \tag{13}$$

Note now that the total time derivative of $V(t)$ along the trajectory of the error system (12) is as follows:

$$\begin{aligned}\dot V(t) &= (\dot{\boldsymbol\delta}^r)^T\boldsymbol\delta^r + (\dot{\boldsymbol\delta}^i)^T\boldsymbol\delta^i + \tilde{\mathbf{A}}^T\dot{\tilde{\mathbf{A}}}\\ &= (\boldsymbol\Phi^r(\mathbf{x}_s(t))(\mathbf{A}-\hat{\mathbf{A}}) - \mathrm{M}\boldsymbol\Phi^r(\mathbf{x}_m(t-\tau))(\mathbf{A}-\hat{\mathbf{A}}) - \mathbf{K}\delta^r)^T\boldsymbol\delta^r \\ &\quad +(\boldsymbol\Phi^i(\mathbf{x}_s(t))(\mathbf{A}-\hat{\mathbf{A}}) - \mathrm{M}\boldsymbol\Phi^i(\mathbf{x}_m(t-\tau))(\mathbf{A}-\hat{\mathbf{A}}) - \mathbf{K}\delta^i)^T\boldsymbol\delta^i \\ &\quad +\tilde{\mathbf{A}}^T\dot{\hat{\mathbf{A}}},\end{aligned} \tag{14}$$

where $\dot{\hat{\mathbf{A}}} = \dot{\tilde{\mathbf{A}}}$. By substituting from Eq. (7) about $\dot{\hat{\mathbf{A}}}$ in Eq. (14) we obtain:

$$\begin{aligned}\dot V(t) &= [\boldsymbol\Phi^r(\mathbf{x}_s(t))(-\tilde{\mathbf{A}}) - \mathrm{M}\boldsymbol\Phi^r(\mathbf{x}_m(t-\tau))(-\tilde{\mathbf{A}}) - \mathbf{K}\delta^r]^T\boldsymbol\delta^r \\ &\quad +[\boldsymbol\Phi^i(\mathbf{x}_s(t))(-\tilde{\mathbf{A}}) - \mathrm{M}\boldsymbol\Phi^i(\mathbf{x}_m(t-\tau))(-\tilde{\mathbf{A}}) - \mathbf{K}\delta^i]^T\boldsymbol\delta^i \\ &\quad +\tilde{\mathbf{A}}^T[\boldsymbol\Phi^r(\mathbf{x}_s(t)))^T\boldsymbol\delta^r + (\boldsymbol\Phi^i(\mathbf{x}_s(t)))^T\boldsymbol\delta^i + \\ &\quad (-\mathrm{M}\boldsymbol\Phi^r(\mathbf{x}_m(t-\tau)))^T\boldsymbol\delta^r + (-\mathrm{M}\boldsymbol\Phi^i(\mathbf{x}_m(t-\tau)))^T\boldsymbol\delta^i - \Lambda\tilde{\mathbf{A}}], \\ &= -[(\mathbf{K}\delta^r)^T\boldsymbol\delta^r + (\mathbf{K}\delta^i)^T\boldsymbol\delta^i] - \tilde{\mathbf{B}}^T(\boldsymbol\Lambda\tilde{\mathbf{B}}) - \tilde{\mathbf{A}}^T(\boldsymbol\Lambda\tilde{\mathbf{A}}), \\ &= -\left(\sum_{l=1}^n k_l\delta^2_{u_{2l-1}} + \sum_{l=1}^n k_l\delta^2_{u_{2l}}\right) - \tilde{\mathbf{B}}^T(\boldsymbol\Lambda\tilde{\mathbf{B}}) - \tilde{\mathbf{A}}^T(\boldsymbol\Lambda\tilde{\mathbf{A}}).\end{aligned} \tag{15}$$

Since $V(t)$ is a positive definite function and its derivative is negative definite, thus according to the well-known Lyapunov theorem, the complex error system (10) is asymptotically stable, which means that $\delta_{u_{2l}}$ and $\delta_{u_{2l-1}}$ tend to zero as $t\to\infty$, $l=1,2,\ldots,n$. Consequently, the states of the slave system and the master system will be globally anti-synchronized asymptotically with lag in time. This completes the proof.

PLS between two complex Chen systems

Analytical formula of controller

Let us now investigate the PLS of two identical chaotic complex Chen systems with uncertain parameters as an example for Section 3. The master and the slave systems are thus defined, respectively, as follows:

$$\begin{aligned}\dot x_m &= a(y_m - x_m),\\ \dot y_m &= (c-a)x_m - x_m z_m + c y_m,\\ \dot z_m &= 1/2(\bar x_m y_m + x_m\bar y_m) - b z_m,\end{aligned} \tag{17}$$

and

$$\begin{aligned}\dot x_s &= a(y_s - x_s) + L_1,\\ \dot y_s &= (c-a)x_s - x_s z_s + c y_s + L_2,\\ \dot z_s &= 1/2(\bar x_s y_s + x_s\bar y_s) - b z_s + L_3,\end{aligned} \tag{18}$$

where $L_1 = v_1 + jv_2$, $L_2 = v_3 + jv_4$ and $L_3 = v_5$, $L_4 = v_7$ are complex and real control functions, respectively, which are to be determined.

The complex systems (17) and (18) can be formed respectively as:

$$\begin{pmatrix}\dot x_m\\ \dot y_m\\ \dot z_m\end{pmatrix} = \begin{pmatrix} y_m - x_m & 0 & 0\\ -x_m & x_m + y_m & 0\\ 0 & 0 & -z_m\end{pmatrix}\begin{pmatrix}a\\ c\\ b\end{pmatrix} + \begin{pmatrix}0\\ -x_m z_m\\ 1/2(\bar x_m y_m + x_m\bar y_m)\end{pmatrix}, \tag{19}$$

and

$$\begin{pmatrix}\dot{x}_s\\ \dot{y}_s\\ \dot{z}_s\end{pmatrix}=\begin{pmatrix}y_s-x_s & 0 & 0\\ -x_s & x_s+y_s & 0\\ 0 & 0 & -z_s\end{pmatrix}\begin{pmatrix}a\\ c\\ b\end{pmatrix}+\begin{pmatrix}0\\ -x_s z_s\\ 1/2\left(\bar{x}_s y_s+x_s\bar{y}_s\right)\end{pmatrix}+\begin{pmatrix}L_1\\ L_2\\ L_3\end{pmatrix}. \qquad (20)$$

So, by comparing the complex systems (19) and (20) with the form of systems (3) and (4) respectively we find:

$$\boldsymbol{\Phi}(\mathbf{x}_m)=\begin{pmatrix}y_m-x_m & 0 & 0\\ -x_m & x_m+y_m & 0\\ 0 & 0 & -z_m\end{pmatrix},\ \boldsymbol{\Phi}(\mathbf{x}_s)=\begin{pmatrix}y_s-x_s & 0 & 0\\ -x_s & x_s+y_s & 0\\ 0 & 0 & -z_s\end{pmatrix},$$

$$\mathbf{A}=\begin{pmatrix}a\\ c\\ b\end{pmatrix},\ \mathbf{f}(\mathbf{x}_m)=\begin{pmatrix}0\\ -x_m z_m\\ 1/2\left(\bar{x}_m y_m+x_m\bar{y}_m\right)\end{pmatrix},\ \mathbf{f}(\mathbf{x}_s)=\begin{pmatrix}0\\ -x_s z_s\\ 1/2\left(\bar{x}_s y_s+x_s\bar{y}_s\right)\end{pmatrix}.$$

According to Theorem 1, the controller is designed as:

$$\mathbf{L}=-\boldsymbol{\Phi}(\mathbf{x}_s(t))\hat{\mathbf{A}}-\boldsymbol{f}(\mathbf{x}_s(t))+\mathrm{M}\boldsymbol{\Phi}(\mathbf{x}_m(t-\tau))\hat{\mathbf{A}}+\mathrm{M}\boldsymbol{f}(\mathbf{x}_m(t-\tau))-\mathbf{K}\boldsymbol{\delta},$$

$$\begin{pmatrix}L_1\\ L_2\\ L_3\end{pmatrix}=\begin{pmatrix}-\hat{a}\left(y_s(t)-x_s(t)\right)+\mu\hat{a}\left(y_m(t-\tau)-x_m(t-\tau)\right)-k_1\delta_1\\ -\hat{c}(y_s(t)+x_s(t))+\hat{a}x_s(t)+x_s(t)z_s(t)+\mu\phi_1-k_2\delta_2\\ \hat{b}z_s(t)-1/2\left(\bar{x}_s(t)y_s(t)+x_s(t)\bar{y}_s(t)\right)+\mu\phi_2-k_3\delta_3\end{pmatrix},$$

where $\phi_1=[\hat{c}(y_m(t-\tau)+x_m(t-\tau))-\hat{a}x_m(t-\tau)-x_m(t-\tau)z_m(t-\tau)],$

$\phi_2=[-\hat{b}z_m(t-\tau)+1/2\left(\bar{x}_m(t-\tau)y_m(t-\tau)+x_m(t-\tau)\bar{y}_m(t-\tau)\right)]$

$$\mathbf{L}=\begin{pmatrix}-\hat{a}(u_{3s}-u_{1s}-\mu u_{3m}+\mu u_{1m})-k_1\delta_{u_1}\\ -\hat{c}(u_{3s}+u_{1s})+\hat{a}u_{1s}+u_{1s}u_{5s}+\mu[\hat{c}(u_{3m}+u_{1m})-\hat{a}u_{1m}-u_{1m}u_{5m}]-k_2\delta_{u_3}\\ \hat{b}(u_{5s}-\delta u_{5m})-u_{1s}u_{3s}+\mu u_{1m}u_{3m}-u_{2s}u_{4s}+\delta u_{2m}u_{4m}-k_3\delta_{u_5}\end{pmatrix}$$

$$+j\begin{pmatrix}-\hat{a}(u_{4s}-u_{2s}-\mu u_{4m}+\mu u_{2m})-k_1\delta_{u_2}\\ -\hat{c}(u_{4s}+u_{2s})+\hat{a}u_{2s}+u_{2s}u_{5s}+\mu[\hat{c}(u_{4m}+u_{2m})-\hat{a}u_{2m}-u_{2m}u_{5m}]-k_2\delta_{u_4}\\ 0\end{pmatrix}, \qquad (21)$$

where $\delta_{u_l}=u_{ls}-\mu u_{lm}$, $l = 1,2,3,4,5,7.$

Since $\mathbf{A}=(a,\ c,\ b)^T$ we can calculate the adaptive laws of parameters by using (7) as:

$$\dot{\hat{\mathbf{A}}}=\begin{pmatrix}\dot{\hat{a}}\\ \dot{\hat{c}}\\ \dot{\hat{b}}\end{pmatrix}=\begin{pmatrix}-\delta_{u1}^2-\delta_{u2}^2+\zeta_1\tilde{a}\\ (\delta_{u_3}+\delta_{u_1})\delta_{u_3}+(\delta_{u_4}+\delta_{u_2})\delta_{u_2}+\zeta_2\tilde{c}\\ \delta_{u_5}^2+\zeta_3\tilde{b}\end{pmatrix}. \qquad (22)$$

Numerical results

To verify and demonstrate the feasibility of the proposed scheme, we discuss the simulation results of the projective lag synchronization between two identical chaotic complex Chen systems (17) and (18). Systems (17) and (18) with the controller (21) are solved numerically, and the parameters are chosen as $a=42, b=4, c=26$. The initial condition of the master system state vector, the initial value of the slave system state vector and the diagonal constant matrices are taken as $(x_m(0), y_m(0), z_m(0))^T = (1+2j, 3+4j, 5)^T$, $(x_s(0), y_s(0), z_s(0))^T = (6+8j, 3+4j, 9)^T$, $\mathbf{K} = diag(12,8,10)$, $\zeta = diag(6,2,5)$ and □=0.1. The initial values of estimate for unknown parameters vector are considered as $(\hat{a}(0), \hat{c}(0), \hat{b}(0))^T = (3,2,1)^T$. The results are depicted in Figures 1, 2.

In Fig.1 the solutions of (17) and (18) are plotted subject to different initial conditions and show that projective lag synchronization is indeed achieved. In Fig. 1a we select $\mu = -1$ and the attractors in (u_1, u_3, u_5) space of master system (17) and slave system (18) have the same size but opposite shape with time lag. But when we choose $\mu = -2$ in Fig. 1b the attractors of (17) and (18) have the opposite shape in (u_1, u_3, u_5) space, but the size of the attractor of the slave system is twice as big as of the master system with time lag. Figure 2 shows that the estimated values of the unknown parameters $\hat{a}(t)$ $\hat{c}(t)$ $\hat{b}(t)$ converge to $42, 26, 4$ respectively. These results ensure, our scheme is suitable for effecting adaptive PLS of two identical chaotic complex nonlinear systems.

The application in secure communications

In this section, secure communications scheme based on PLS between two identical chaotic complex Chen systems is investigated. We consider the two chaotic complex Chen systems as transmitter and receiver systems. For one thing, we choose arbitrarily the information signal as $r(t) = 1+\cos t$. Take $\breve{r}(t) = r(t) + u_{1m}$ and suppose that $\breve{r}(t)$ is added to the variable u_{2m}. Numerical results of application to secure communication are shown in Figs. 5. The information signal $r(t)$ and the transmitted signal $\breve{r}(t)$ are shown in Fig. 5(a) and (b), respectively. The recovered information signal, which is denoted by $r^*(t) = \breve{r}(t) - u_{1s} - u_{2s}$, is shown in Fig. 5(c). Fig. 5(d) displays the error signal between the original information signal and the recovered one. From Fig. 5(d), it is easy to find that the information signal $r(t)$ is recovered exactly after a very short transient.

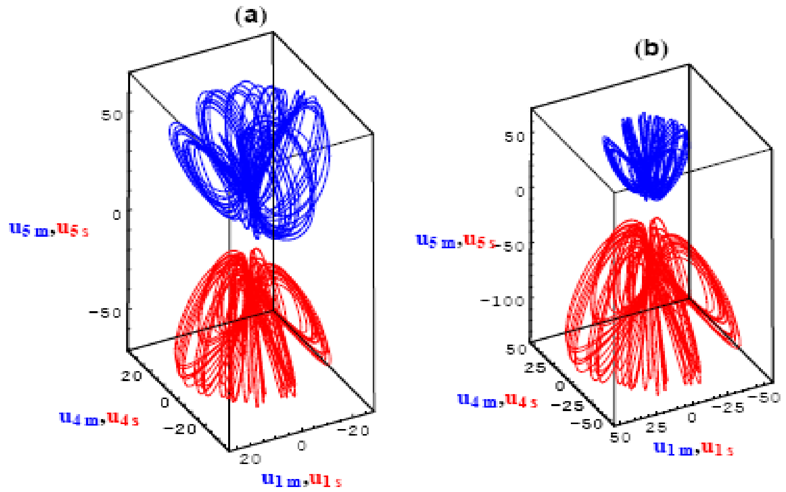

Figure 2: Chaotic attractors of the Master system (17) (blue color) and the slave system (18) (red color): (a) when $\mu = -1$. (b) when $\mu = -2$.

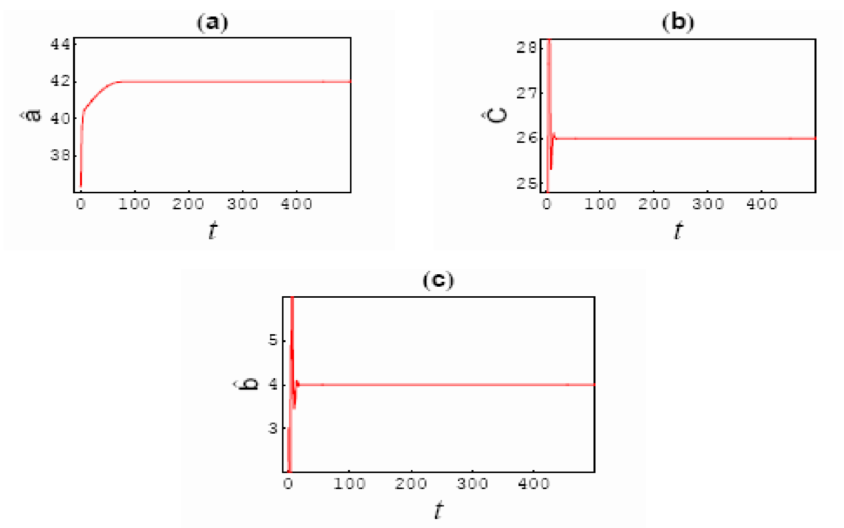

Figure 3: Adaptive parameters estimation laws $\hat{a}(t), \hat{c}(t), \hat{b}(t)$ of the master and slave systems.

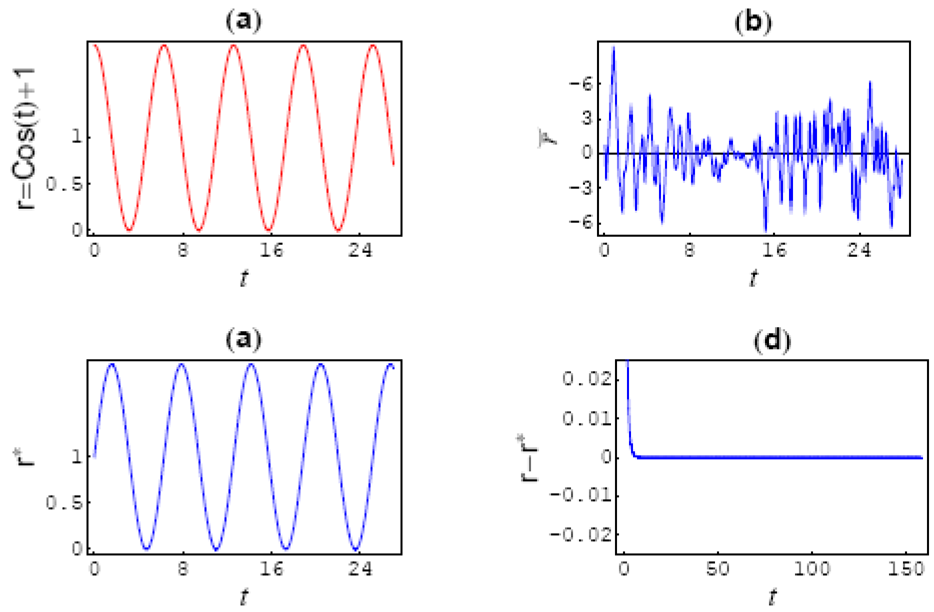

Figure 4: Simulation results of secure communication using PLS of two identical chaotic complex Chen systems.

Conclusion

Synchronization and control are important topics which have been studied to date primarily on dynamical systems described by real variables in applied nonlinear sciences. There also, exist, however, interesting cases of dynamical systems where the main variables participating in the dynamics are complex as, for example, when amplitudes of electromagnetic fields are involved. Our goal in this paper is to study and investigate projective lag synchronization of chaotic attractors of complex systems with uncertain parameters. A scheme is designed to achieve projective lag synchronization of two identical or different chaotic complex nonlinear systems with uncertain parameters based on Lyapunov functions. Through this scheme we determined analytically the control complex functions and adaptive laws of parameters to achieve projective lag synchronization. Illustrative examples are given to verify the correctness of our scheme. The secure communications by using projective lag synchronization in two chaotic complex Chen system is implemented.

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