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A comparison study on the harmonic balance method and rational harmonic balance method for the *Duffing-harmonic* oscillator

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Abstract

The *Duffing-harmonic* oscillator is a common model in nonlinear sciences and engineering. In the present paper, the harmonic balance method and rational harmonic balance method have been introduced to derive the approximate periods of strongly nonlinear *Duffing-harmonic* oscillator. The comparison of two methods is made to demonstrate that the rational harmonic balance method (RHBM) gives almost similar results to next higher-order approximation results of harmonic balance method (HBM). It is highly remarkable that the solution procedure in both methods are simple and takes less computational effort for determining approximate periods and shows a good agreement compared with the exact ones.

Keywords:- *Duffing-harmonic* oscillator; Harmonic balance method; Nonlinear algebraic equations; Power series solution; Rational Harmonic balance method

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1. Introduction

Considerable attention has been directed towards the study of nonlinear oscillations which are appearing mathematically in the form of nonlinear differential equations (NDEs). Obtaining exact solutions for NDEs have many difficulties. It is very difficult to solve nonlinear problems and in general it is often more difficult to get an analytic approximation than a numerical one. A few nonlinear systems can be solved explicitly, and numerical methods especially the most popular Runge-Kutta fourth order method are frequently used to calculated approximate solutions. However, numerical schemes do not always give accurate results especially the class of stiff differential equations, chaotic differential equation, which present a more serious challenge to numerical analysis. In this situation, many researchers have been showed an intensifying interest in the field of analytical approximate techniques. Popular method for solving NDEs associated with oscillatory systems is Perturbation Method [1-3], which is the most versatile tools available in nonlinear analysis of engineering problems and they are constantly being developed and applied to ever more complex problems. However, for the strongly nonlinear regime perturbation method can not yields desired results.

Taking into account the above exchanges, we ought to present new advancement of all the more effective methods for finding approximate analytical solutions to complicated nonlinear problems in distinctive fields of study especially nonlinear oscillatory system has recently attracted the attention of researchers, where existing methods have not been fruitful up to now.

As a result, due to conquer these weak-points, in recent years, a number of researchers have devoted their time and effort to find potent approaches for investigating of the nonlinear phenomena. As the earliest effort, they developed a large variety of approximate methods commonly used for nonlinear oscillatory systems especially for strongly nonlinear oscillators including He's Homotopy Perturbation Method [4], Max-Min Approach Method [5], Parameter Expansion Method [6], Nonlinear Time Transformation Method [7], Algebraic Method [8], Energy Balance Method [9-11], He's Energy Balance Method [12], Rational Energy Balance Method [13], He's Frequency-Amplitude Formulation [14], Enhanced Cubication Method [15], Residue Harmonic Balance Method [16], Iteration Method [17] and so on. However, most of these papers only the first-order approximation is considered. In addition, the aforementioned methods also do not have this ability to gain the solution in high precision. Furthermore, the solution procedures are tremendously difficult task and cumbersome especially for obtaining higher order approximation. In this situation, we will see that the Harmonic Balance Method [18-26] and Rational Harmonic Balance Method [27-30] considered in this study can be applied to strongly nonlinear *Duffing-harmonic* oscillator. The RHBM discussed by Mickens R.E. and Semwogerere [27], for instance, has rarely been applied to the determination of periodic solutions of nonlinear problems. In fact, to the best of our knowledge, recently Belendez A. et al. [28-29] as well as Yamgoue S.B. et al. [30] used it to solve a simple-term oscillator equation of plasma physics in a completely analytic fashion. Generally, a set of troublesome nonlinear algebraic equations are found when HBM and RHBM is imposed. Sometime analytical solutions of these algebraic equations fail especially for large amplitude. In present study, this limitation is removed. Approximate solutions of the same equations are found in which the nonlinear algebraic equations are solved by a new parameter. Using RHBM, the second-order approximate period derived here is more accurate and closer to the next higher-order than obtained by HBM. Considering the interesting property that the proposed technique not only provides accurate results but also it is more convenient and effective for solving more complex nonlinear problems.

2. Application

2.1. Solution approaches of HBM to the *Duffing-harmonic* oscillator

Let us consider the *Duffing-harmonic* oscillator and initial conditions

$$\ddot{x}+x^3/(1+x^2)=0\,,\ \ x(0)=a_0\,,\ \ \dot{x}(0)=0. \tag{1}$$

Eq. (1) can be rewrite another form as

$$\ddot{x}+x^3-x^5+x^7-\cdots=0 \tag{2}$$

Let us consider a two-term solution, *i.e.*, $x=a_0(\rho\cos(\omega t)+u\cos(3\omega t))$ for the Eq. (2). Substituting this solution along with $\rho=1-u$ into Eq. (2), it reduces to

$$\begin{aligned}(1-u)\,\omega^2\cos(\omega t)+9u\omega^2\cos(3\omega t) &=(3a_0^2/4-5a_0^4/8-3a_0^2u/2+9a_0^2u^2/4+\cdots)\cos(\omega t)\\ &+(a_0^2/4-5a_0^4/16+3a_0^2u/4-9a_0^2u^2/4+\cdots)\cos(3\omega t)]/4+HOH,\end{aligned} \tag{3}$$

where HOH stands for higher-order harmonics terms.

Now comparing the coefficients of equal harmonic trems, the following relations are:

$$\begin{aligned}(1-u)\,\omega^2 &=3a_0^2/4-5a_0^4/8-3a_0^2u/2+25a_0^4u/16+9a_0^2u^2/4-15a_0^4u^2/4+\cdots\\ 9u\omega^2 &= a_0^2/4-5a_0^4/16+3a_0^2u/4-5a_0^4u/16-9a_0^2u^2/4+5a_0^4u^2/2-63a_0^6u^2/32+\cdots\end{aligned} \tag{4}$$

From the first equation of Eq. (4), it can easily written as

$$\omega^2=(3a_0^2/4-5a_0^4/8-3a_0^2u/2+25a_0^4u/16+9a_0^2u^2/4-15a_0^4u^2/4+\cdots)/(1-u) \tag{5}$$

Applying Eq. (5) into second equation of Eq. (4) with simplification, we get the following nonlinear algebraic equation of u:

$$\begin{aligned}&a_0^2/4-5a_0^4/16+21a_0^6/64-25a_0^2u/4+45a_0^4u/8-21a_0^6u/4+21a_0^2u^2/2-45a_0^4u^2/4\\ &+189a_0^6u^2/16-16a_0^2u^3+25a_0^4u^3-1995a_0^6u^3/64+23a_0^2u^4/2-675a_0^4u^4/16\\ &+2275a_0^6u^4/32+355a_0^4u^5/8-2009a_0^6u^5/16-85a_0^4u^6/4+2499a_0^6u^6/16\\ &-3844a_0^6u^7/32+1365a_0^6u^8/32=0\end{aligned} \tag{6}$$

For the *Duffing-harmonic* oscillator, the power series solution of u presented in Eq. (6) is invalid. Herein u is substituted by $u_0+u_2a_0^2+u_4a_0^4+\cdots$ into Eq. (6) and then equating the coefficients of $a_0^2,\ a_0^4,\cdots$ yields

$$
\begin{aligned}
&1-25u_0+42u_0^2-64u_0^3+46u_0^4=0,\\
&-5/4+45u_0/2-45u_0^2+100u_0^3-675u_0^4/4+355u_0^5/2-85u_0^6-25u_2+84u_0u_2\\
&-192u_0^2u_2+184u_0^3u_2=0,\\
&21/16-21u_0+189u_0^2/4-1995u_0^3/16+2275u_0^4/8-2009u_0^5/4+2499u_0^6/4\\
&-3843u_0^7/8+1365u_0^8/8+45u_2/2-90u_0u_2+300u_0^2u_2-675u_0^3u_2+1775u_0^4u_2/2\\
&-510u_0^5u_2+42u_2^2-192u_0u_2^2+276u_0^2u_2^2-25u_4+84u_0u_4-192u_0^2u_4+184u_0^3u_4=0,\\
&\cdots
\end{aligned}
\tag{7}
$$

It can clear be seen that the coefficients of u_0, u_2, u_4, respectively in the three equations of Eq. (7) are the same constant i.e. 25. Therefore, by choosing $\lambda = 1/25$, in the three equations of Eq. (7), the equations of u_0 , u_2 ,u_4, can be written as:

$$
\begin{aligned}
&u_0=\lambda(1+42u_0^2-64u_0^3+46u_0^4),\\
&u_2=\lambda(-5/4+45u_0/2-45u_0^2+100u_0^3-675u_0^4/4+355u_0^5/2-85u_0^6+84u_0u_2\\
&-192u_0^2u_2+184u_0^3u_2),\\
&u_4=\lambda(21/16-21u_0+189u_0^2/4-1995u_0^3/16+2275u_0^4/8-2009u_0^5/4+2499u_0^6/4\\
&-3843u_0^7/8+1365u_0^8/8+45u_2/2-90u_0u_2+300u_0^2u_2-675u_0^3u_2+1775u_0^4u_2/2\\
&-510u_0^5u_2+42u_2^2-192u_0u_2^2+276u_0^2u_2^2+84u_0u_4-192u_0^2u_4+184u_0^3u_4).
\end{aligned}
\tag{8}
$$

The power series solution of Eq. (8) in terms of λ is

$$
\begin{aligned}
&u_0=\lambda+42\lambda^3-64\lambda^4+3574\lambda^5-13440\lambda^6+394320\lambda^7-2391424\lambda^8+\cdots,\\
&u_2=-\frac{5}{4}\lambda+\frac{45}{2}\lambda^2-150\lambda^3+3175\lambda^4-\frac{107795}{4}\lambda^5+\frac{1009705}{2}\lambda^6-5099980\lambda^7+\cdots,\\
&u_4=\frac{21}{16}\lambda-\frac{393}{8}\lambda^2+\frac{6735}{8}\lambda^3-\frac{221163}{16}\lambda^4+\frac{1888951}{8}\lambda^5-\frac{28175843}{8}\lambda^6+\cdots.
\end{aligned}
\tag{9}
$$

Now substituting the value of $u = u_0 + u_2 a^2 + u_4 a^4$ where u_0, u_2, u_4 are calculated by Eq. (9) into Eq. (5), the second-order approximate period of *Duffing-harmonic* oscillator is

$$T_2 = 7.40158/a_0 + 2.97549\,a_0 + \cdots. \tag{10}$$

In same manipulation discussed above, the method can be used to determine higher-order approximations. In this study, a third-order approximate solution is

$$x(t) = a_0\cos(\omega t) + a_0 u(\cos(3\omega t) - \cos(\omega t) + a_0 v(\cos(5\omega t) - \cos(\omega t)). \tag{11}$$

Substituting Eq. (11) into the Eq. (2) and equating the coefficients of $\cos(\omega t)$, $\cos(3\omega t)$ and $\cos(5\omega t)$ equal to zero the following equations are yields

$$\begin{aligned}
&(1-u-v)\,\omega^2 = 3a_0^2/4-5a_0^4/8+35a_0^6/64-3a_0^2u/2+25a_0^4u/16-49a_0^6u/32\\
&+9a_0^2u^2/4-15a_0^4u^2/4+147a_0^6u^2/32-3a_0^2u^3/2+25a_0^4u^3/4-175a_0^6u^3/16\\
&-25a_0^4u^4/4-9a_0^2v/4+45a_0^4v/16-49a_0^6v/16+9a_0^2uv/2-10a_0^4uv+231a_0^6uv/16\\
&-3a_0^2u^2v+75a_0^4u^2v/4+\cdots\\
&9u\omega^2 = a_0^2/4-5a_0^4/16+21a_0^6/64+3a_0^2u/4-5a_0^4u/16-9a_0^2u^2/4+5a_0^4u^2/2\\
&-63a_0^6u^2/32+2a_0^2u^3-25a_0^4u^3/4+525a_0^6u^3/64+125a_0^4u^4/16+5a_0^4v/16 \qquad (12)\\
&-21a_0^6v/32-3a_0^2uv/2+5a_0^4uv/2-63a_0^6uv/32+3a_0^2u^2v/2-75a_0^4u^2v/8+\cdots\\
&25v\omega^2 = -a_0^4/16+7a_0^6/64+3a_0^2u/4-15a_0^4u/16+7a_0^6u/8-3a_0^2u^2/4\\
&+5a_0^4u^2/2-231a_0^6u^2/64-25a_0^4u^3/8+525a_0^6u^3/64+25a_0^4u^4/16+3a_0^2v/2\\
&-25a_0^4v/16+91a_0^6v/64-9a_0^2uv/2+35a_0^4uv/4-189a_0^6uv/16+15a_0^2u^2v/4\\
&-75a_0^4u^2v/4+315a_0^6u^2v/8+\cdots
\end{aligned}$$

Another form of the first equation of Eq. (12),

$$\begin{aligned}
&\omega^2 = (3a_0^2/4-5a_0^4/8+35a_0^6/64-3a_0^2u/2+25a_0^4u/16-49a_0^6u/32\\
&+9a_0^2u^2/4-15a_0^4u^2/4+147a_0^6u^2/32-3a_0^2u^3/2+25a_0^4u^3/4-175a_0^6u^3/16\\
&-25a_0^4u^4/4-9a_0^2v/4+45a_0^4v/16-49a_0^6v/16+9a_0^2uv/2-10a_0^4uv+231a_0^6uv/16\\
&-3a_0^2u^2v+75a_0^4u^2v/4+\cdots)/(1-u-v)
\end{aligned} \qquad (13)$$

By omitting ω^2 from second and third equations of Eq. (12) with the help of Eq. (13) and simplification, the following nonlinear algebraic equation of u and v are:

$$\begin{aligned}
&-a_0^2/4+5a_0^4/16-21a_0^6/64+25a_0^2u/4-45a_0^4u/8+21a_0^6u/4-21a_0^2u^2/2\\
&+45a_0^4u^2/4-189a_0^6u^2/16+16a_0^2u^3-25a_0^4u^3+1995a_0^6u^3/64-23a_0^2u^4/2\\
&+675a_0^4u^4/16+a_0^2v/4-5a_0^4v/8+63a_0^6v/64-18a_0^2uv+365a_0^4uv/16\\
&-105a_0^6uv/4+141a_0^2u^2v/4-605a_0^4u^2v/8+3507a_0^6u^2v/32-47a_0^2u^3v/2\\
&+\cdots=0, \qquad (14)\\
&a_0^4/16-7a_0^6/64-3a_0^2u/4+7a_0^4u/8-49a_0^6u/64+3a_0^2u^2/2-55a_0^4u^2/16\\
&+287a_0^6u^2/64-3a_0^2u^3/4+45a_0^4u^3/8-189a_0^6u^3/16-75a_0^4u^4/16+315a_0^6u^4/16\\
&+69a_0^2v/4-113a_0^4v/8+791a_0^6v/64-123a_0^2uv/4+445a_0^4uv/16-1547a_0^6uv/64\\
&+189a_0^2u^2v/4-255a_0^4u^2v/4+3843a_0^6u^2v/64-135a_0^2u^3v/4+\cdots=0.
\end{aligned}$$

In the case for *Duffing-harmonic* oscillator, the power series solutions of u and v presented in Eq. (14) are invalid. Herein u is substituted by $u_0+u_2a_0^2+u_4a_0^4+\cdots$ and v is substituted by $v_0+v_2a_0^2+v_4a_0^4+\cdots$ into Eq. (14) and then equating the coefficients of $a_0^2,\ a_0^4,\cdots$ yields

$$1-25u_0+42u_0^2-64u_0^3+46u_0^4-v_0+72u_0v_0-141u_0^2v_0+94u_0^3v_0-3v_0^2$$
$$-117u_0v_0^2+147u_0^2v_0^2+5v_0^3+70u_0v_0^3-2v_0^4=0,$$
$$3u_0-6u_0^2+3u_0^3-69v_0+123u_0v_0-189u_0^2v_0+135u_0^3v_0+207v_0^2-405u_0v_0^2$$
$$+270u_0^2v_0^2-354v_0^3+426u_0v_0^3+216v_0^4=0, \tag{15}$$

$$5/16-45u_0/8+45u_0^2/4-25u_0^3+675u_0^4/16-355u_0^5/8+85u_0^6/4$$
$$+25u_2/4-21u_0u_2+48u_0^2u_2-46u_0^3u_2-5v_0/8+365u_0v_0/16-605u_0^2v_0/8$$
$$+1125u_0^3v_0/8-285u_0^4v_0/2+965u_0^5v_0/16-18u_2v_0+141u_0u_2v_0/2$$
$$-141u_0^2u_2v_0/2-5v_0^2/16-265u_0v_0^2/4+855u_0^2v_0^2/4-290u_0^3v_0^2+2465u_0^4v_0^2/16$$
$$+117u_2v_0^2/4-147u_0u_2v_0^2/2+25v_0^3/8+465u_0v_0^3/4-1125u_0^2v_0^3/4$$
$$+1545u_0^3v_0^3/8-35u_2v_0^3/2-95v_0^4/16-895u_0v_0^4/8+1175u_0^2v_0^4/8+5v_0^5$$
$$+715u_0v_0^5/16-25v_0^6/16+v_2/4-18u_0v_2+141u_0^2v_2/4-47u_0^3v_2/2+3v_0v_2/2$$
$$+117u_0v_0v_2/2-147u_0^2v_0v_2/2-15v_0^2v_2/4-105u_0v_0^2v_2/2+2v_0^3v_2=0, \tag{16}$$

$$1/16+7u_0/8-55u_0^2/16+45u_0^3/8-75u_0^4/16+3u_0^5/2+u_0^6/16-3u_2/4$$
$$+3u_0u_2-9u_0^2u_2/4-113v_0/8+445u_0v_0/16-255u_0^2v_0/4+915u_0^3v_0/8$$
$$-1005u_0^4v_0/8+981u_0^5v_0/16-123u_2v_0/4+189u_0u_2v_0/2-405u_0^2u_2v_0/4$$
$$+495v_0^2/8-1665u_0v_0^2/8+3105u_0^2v_0^2/8-1585u_0^3v_0^2/4+1355u_0^4v_0^2/8$$
$$+405u_2v_0^2/4-135u_0u_2v_0^2-395v_0^3/2+625u_0v_0^3-1675u_0^2v_0^3/2$$
$$+3525u_0^3v_0^3/8-213u_2v_0^3/2+5785v_0^4/16-835u_0v_0^4+9015u_0^2v_0^4/16$$
$$-2833v_0^5/8+6971u_0v_0^5/16+569v_0^6/4+69v_2/4-113u_0v_2/4+189u_0^2v_2/4$$
$$-135u_0^3v_2/4-207v_0v_2/2+405u_0v_0v_2/2-135u_0^2v_0v_2+531v_0^2v_2/2$$
$$-639u_0v_0^2v_2/2-216v_0^3v_2=0, \tag{17}$$

$\cdots$

In Eqs. (15)-(17) the equations of u_0, v_0, u_2, v_2 can be written in another form as

$$u_0=\lambda(1+42u_0^2-64u_0^3+46u_0^4-v_0+72u_0v_0-141u_0^2v_0+94u_0^3v_0+\cdots), \tag{18}$$

$$v_0=\mu(u_0-2u_0^2+u_0^3+41u_0v_0-63u_0^2v_0+45u_0^3v_0+69v_0^2-135u_0v_0^2+\cdots), \tag{19}$$

$$u_2=\lambda(-5/4+45u_0/2-45u_0^2+100u_0^3-675u_0^4/4+355u_0^5/2-85u_0^6$$
$$+84u_0u_2-192u_0^2u_2+184u_0^3u_2+5v_0/2-365u_0v_0/4+605u_0^2v_0/2+\cdots), \tag{20}$$

$$v_2=\mu(-1/12-7u_0/6+55u_0^2/12-15u_0^3/2+25u_0^4/4-2u_0^5-u_0^6/12$$
$$+u_2-4u_0u_2+3u_0^2u_2+113v_0/6-445u_0v_0/12+85u_0^2v_0/2+\cdots), \tag{21}$$

where λ is defined in Eq. (8) and $\mu=1/23$. The algebraic relation between λ and μ is

$$\mu=25\lambda/23 \tag{22}$$

Now solving Eq. (18) and Eq. (19) and then Eq. (20) and Eq. (21) simultaneously in terms of λ :

$$u_0 = \lambda + \frac{941}{23}\lambda^3 + \frac{378}{23}\lambda^4 + \frac{1626871}{529}\lambda^5 + \cdots$$

$$v_0 = \frac{25}{23}\lambda^2 - \frac{50}{23}\lambda^3 + \frac{49725}{529}\lambda^4 - \frac{128400}{529}\lambda^5 + \cdots$$

$$u_2 = -\frac{5}{4}\lambda + \frac{6236}{276}\lambda^2 - \frac{41725}{276}\lambda^3 + \frac{17309425}{6348}\lambda^4 - \frac{110254165}{6348}\lambda^5 + \cdots \quad (23)$$

$$v_2 = -\frac{25}{276}\lambda - \frac{725}{276}\lambda^2 + \frac{337625}{6348}\lambda^3 - \frac{954025}{1587}\lambda^4 + \frac{355128650}{36501}\lambda^5 + \cdots$$

Substituting the values of $u = u_0 + u_2 a^2 + \cdots$ and $v = v_0 + v_2 a^2 + \cdots$ where u_0, u_2, and v_0, v_2, are calculated by Eq. (23) into Eq. (13), the third-order approximate period of the *Duffing-harmonic* oscillator is:

$$T_3 = 7.415647 / a_0 + 2.935536\, a_0 + \cdots \quad (24)$$

2.2. Solution approaches of RHBM to the *Duffing-harmonic* oscillator

Let us consider a two-term solution, *i.e.*, $x(t) = \dfrac{A_1 \cos(\omega t)}{1 + u\cos(2\omega t)}$ and substituting into Eq. (2), it reduce to

$$\begin{aligned}
&\omega^2 (1 + 3u - 13u^2/2 - 13u^3/2 - 123u^4/8 - 41u^5/8 - 23u^6/16)\cos(\omega t) \\
&+ \omega^2 u(-1 + 23u/4 - 33u^2/2 - 5u^3/4 - 61u^4/8 - 33u^5/64)\cos(3\omega t) \\
&= A_1^2[3/4 - 5A_1^2/8 + 35A_1^4/64 + 2u - 15A_1^2 u/16 + 21u^2/8 - 13A_1^2u^2/32 \\
&+ 3u^3/2 + 11u^4/32)\cos(\omega t) + (1/4 - 5A_1^2/16 + 21A_1^4/64 + 3u/2 - 11A_1^2 u/16 \\
&+ 15u^2/8 - 5A_1^2u^2/16 + 5u^3/4 + 9u^4/32)\cos 3\varphi] + HOH,
\end{aligned} \quad (25)$$

where HOH represents the higher-order harmonic terms.

Now from Eq. (25), we comparing the coefficients of equal harmonic terms, the following equations are:

$$\begin{aligned}
&\omega^2 (1 + 3u - 13u^2/2 - 13u^3/2 - 123u^4/8 - 41u^5/8 - 23u^6/16) \\
&= A_1^2 (3/4 - 5A_1^2/8 + 35A_1^4/64 + 2u - 15A_1^2 u/16 + 21u^2/8 - 13A_1^2u^2/32 \\
&\qquad + 3u^3/2 + 11u^4/32)
\end{aligned} \quad (26)$$

$$\begin{aligned}
&\omega^2 u(-1 + 23u/4 - 33u^2/2 - 5u^3/4 - 61u^4/8 - 33u^5/64) \\
&= A_1^2 (1/4 - 5A_1^2/16 + 21A_1^4/64 + 3u/2 - 11A_1^2 u/16 + 15u^2/8 \\
&\qquad - 5A_1^2u^2/16 + 5u^3/4 + 9u^4/32)
\end{aligned} \quad (27)$$

After applying initial conditions Eqs. (26)-(27) take the following form

$$\begin{aligned}
&\omega^2(1+3u-13u^2/2-13u^3/2-123u^4/8-41u^5/8-23u^6/16)\\
&=a_0^2(3/4-5a_0^2/8+35a_0^4/64+7u/2-55a_0^2u/16+105a_0^4u/32+59u^2/8\\
&-253a_0^2u^2/32+525a_0^4u^2/64+35u^3/4-39a_0^2u^3/4+175a_0^4u^3/16+191u^4/32\\
&-109a_0^2u^4/16+525a_0^4u^4/64+35u^5/16-41a_0^2u^5/16+105a_0^4u^5/32+11u^6/32\\
&-13a_0^2u^6/32+35a_0^4u^6/64)
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\omega^2u(-1+23u/4-33u^2/2-5u^3/4-61u^4/8-33u^5/64)\\
&=a_0^2(1/4-5a_0^2/16+21a_0^4/64+2u-31a_0^2u/16+63a_0^4u/32+41u^2/8-79a_0^2u^2/16\\
&+315a_0^4u^2/64+13u^3/2-53a_0^2u^3/8+105a_0^4u^3/16+149u^4/32-79a_0^2u^4/16\\
&+315a_0^4u^4/64+29u^5/16-31a_0^2u^5/16+63a_0^4u^5/32+9u^6/32-5a_0^2u^6/16\\
&+21a_0^4u^6/64)
\end{aligned} \tag{29}$$

Eq. (28) can be written another form as

$$\begin{aligned}
&\omega^2=[a_0^2(3/4-5a_0^2/8+35a_0^4/64+7u/2-55a_0^2u/16+105a_0^4u/32+59u^2/8\\
&-253a_0^2u^2/32+525a_0^4u^2/64+35u^3/4-39a_0^2u^3/4+175a_0^4u^3/16+191u^4/32\\
&-109a_0^2u^4/16+525a_0^4u^4/64+35u^5/16-41a_0^2u^5/16+105a_0^4u^5/32+11u^6/32\\
&-13a_0^2u^6/32+35a_0^4u^6/64)]/(1+3u-13u^2/2-13u^3/2-123u^4/8-41u^5/8\\
&-23u^6/16)
\end{aligned} \tag{30}$$

By elimination of ω^2 from the Eq. (29) with the help of Eq. (30) and simplification, the following nonlinear algebraic equation of u is:

$$\begin{aligned}
&-1/4+5a_0^2/16-21a_0^4/64-7u/2+7a_0^2u/2-7a_0^4u/2-139u^2/16\\
&+137a_0^2u^2/16-2261a_0^4u^2/256-55u^3/8+337a_0^2u^3/64-609a_0^4u^3/128\\
&+31u^4/32-185a_0^2u^4/64+3395a_0^4u^4/512+35u^5/8+15a_0^2u^5/32\\
&+119a_0^4u^5/128+1963u^6/256+2743a_0^2u^6/512-71631a_0^4u^6/4096\\
&+1927u^7/128-1409a_0^2u^7/1024-45437a_0^4u^7/2048+7055u^8/512\\
&-9643a_0^2u^8/2048-71953a_0^4u^8/4096+1145u^9/256+59a_0^2u^9/256\\
&-14595a_0^4u^9/1024+245u^{10}/2048+1769a_0^2u^{10}/1024-31633a_0^4u^{10}/4096\\
&+305u^{11}/1024+33a_0^2u^{11}/1024-2765a_0^4u^{11}/2048+465u^{12}/2048-491a_0^2u^{12}/2048\\
&+777a_0^4u^{12}/4096=0
\end{aligned} \tag{31}$$

Here mentioned that, the *Duffing-harmonic* oscillator, the power series solution of u presented in Eq. (31) is invalid. Herein u is substituted by $u_0+u_2a_0^2+u_4a_0^4+\cdots$ into Eq. (31) and then equating the coefficients of $a_0^2,\ a_0^4,\cdots$ yields

$$\begin{aligned}
&-1/4-7u_0/2-139u_0^2/16-55u_0^3/8+31u_0^4/32+35u_0^5/8+1963u_0^6/256\\
&+1927u_0^7/128+7055u_0^8/512+1145u_0^9/256+245u_0^{10}/2048+305u_0^{11}/1024\\
&+465u_0^{12}/2048=0,
\end{aligned} \tag{32}$$

$$\begin{aligned}&5/16+7u_0/2+137u_0^2/16+337u_0^3/64-185u_0^4/64+15u_0^5/32+2743u_0^6/512\\&-1409u_0^7/1024-9643u_0^8/2048+59u_0^9/256+1769u_0^{10}/1024+33u_0^{11}/1024\\&-491u_0^{12}/2048-7u_2/2-139u_0u_2/8-165u_0^2u_2/8+31u_0^3u_2/8+175u_0^4u_2/8\\&+5889u_0^5u_2/128+13489u_0^6u_2/128+7055u_0^7u_2/64+10305u_0^8u_2/256\\&+1225u_0^9u_2/1024+3355u_0^{10}u_2/1024+1395u_0^{11}u_2/512=0,\end{aligned} \tag{33}$$

$$\begin{aligned}&-21/64-7u_0/2-2261u_0^2/256-609u_0^3/128+3395u_0^4/512+119u_0^5/128\\&-71631u_0^6/4096-45437u_0^7/2048-71953u_0^8/4096-14595u_0^9/1024\\&-31633u_0^{10}/4096-2765u_0^{11}/2048+777u_0^{12}/4096+7u_2/2+137u_0u_2/8\\&+1011u_0^2u_2/64-185u_0^3u_2/16+75u_0^4u_2/32+8229u_0^5u_2/256-9863u_0^6u_2/1024\\&-9643u_0^7u_2/256+531u_0^8u_2/256+8845u_0^9u_2/512+363u_0^{10}u_2/1024\\&-1473u_0^{11}u_2/512-139u_2^2/16-165u_0u_2^2/8+93u_0^2u_2^2/16+175u_0^3u_2^2/4\\&+29445u_0^4u_2^2/256+40467u_0^5u_2^2/128+49385u_0^6u_2^2/128+10305u_0^7u_2^2/64\\&+11025u_0^8u_2^2/2048+16775u_0^9u_2^2/1024+15345u_0^{10}u_2^2/1024-7u_4/2-139u_0u_4/8\\&-165u_0^2u_4/8+31u_0^3u_4/8+175u_0^4u_4/8+5889u_0^5u_4/128+13489u_0^6u_4/128\\&+7055u_0^7u_4/64+10305u_0^8u_4/256+1225u_0^9u_4/1024+3355u_0^{10}u_4/1024\\&+1395u_0^{11}u_4/512=0.\end{aligned} \tag{34}$$

From Eq. (32) it can be written as

$$\begin{aligned}u_0=\lambda(&-1-139u_0^2/4-55u_0^3/2+31u_0^4/8+35u_0^5/2+1963u_0^6/64+1927u_0^7/32\\&+7055u_0^8/128+1145u_0^9/64+245u_0^{10}/512+305u_0^{11}/256+465u_0^{12}/512),\end{aligned} \tag{35}$$

where $\lambda=1/14$.

The power series solution of Eq. (35) in terms of λ is

$$u_0=-\lambda-\frac{139}{4}\lambda^3+\frac{55}{2}\lambda^4-\frac{9645}{4}\lambda^5+\frac{38085}{8}\lambda^6-\frac{1689953}{8}\lambda^7+\frac{11076869}{16}\lambda^8+\cdots \tag{36}$$

Substituting Eq. (36) into Eqs. (33)-(34), it takes the form of first order algebraic equations of u_2 and u_4 and the values are:

$$u_2=0.02971859 \text{ and } u_4=-0.01153767 \tag{37}$$

Now substituting the value of $u=u_0+u_2a^2+u_4a^4$ where u_0,u_2,u_4 are calculated by Eq. (36) and Eq. (37) into Eq. (30), the second-order approximate period of *Duffing-harmonic* oscillator is

$$T_2=7.41671/a_0+2.952566\,a_0+\cdots. \tag{38}$$

3. Results and discussions

We outline the exactness of estimated approximate periods obtained by HBM and RHBM. Comparing all the periods with existing periods previously obtained by several authors and the exact period T_{ex}. For *Duffing-harmonic* oscillator, the exact period is

$$T_{ex}=7.4163\cdots/a_0+2.93048\cdots a_0+\cdots$$

which is stated in Belendez A. *et al.* [4].
The second- and third-order approximate periods obtained by HBM are the following:

$T_2 = 7.40158 / a_0 + 2.97549\, a_0 + \cdots,$

$T_3 = 7.415647 / a_0 + 2.935536\, a_0 + \cdots$.

and in RHBM

$T_2 = 7.41671 / a_0 + 2.952566\, a_0 + \cdots$

In Ref. [4] approximately solved (8) using Homotopy Perturbation method in two different cases. They calculated the following approximate period of oscillation in orders

$T_a = 7.2552 / a_0 + 2.7207\, a_0 + \cdots,$

$T_b = 7.2552 / a_0 + 3.0230\, a_0 + \cdots.$

In Ref. [9] approximately solved (8) using energy balance method that incorporates salient feathers of frequency-amplitude relation. They calculated the following approximate period of oscillation as:

$T = 7.255197 / a_0 + 2.821465\, a_0 + \cdots.$

Also in Ref. [21] approximately solved (8) using a classical harmonic balance method. They determined the following approximate period of oscillation

$T_2 = 7.2551978 / a_0 + 3.022998\, a_0 + \cdots$

Comparing all the approximate periods with corresponding exact period, it can be seen that the approximate periods obtained by this study show an excellent agreement and is better than those obtained previously by Belendez A. *et al.* [4], Ozis T. *et al.* [9] and Lim C.W. *et al.* [21]. It has been mentioned that, using RHBM, the second-order approximate period is almost similar to third-order approximate period obtained by HBM. Interesting issue is here that, most of the existing methods including Belendez A. *et al.* [4], Hailing W. *et al.* [7], Ozis T. *et al.* [9], Zuniga A.E. *et al.* [15], Hu H. [17], Mickens R.E. [20], Lim C.W. *et al.* [21] and Hu H. *et al.* [25] have considered only first-order approximation which leads low accuracy. Moreover, the solution procedures are cumbersome especially for obtaining higher approximations. The advantages of this method include its simplicity and computational efficiency, and the ability to objectively find better agreement than several existing results.

4. Conclusion

Using HBM and RHBM to obtained approximate periods for strongly nonlinear *Duffing-harmonic* oscillator. The solution procedure of the introduced methods are very simple, easy and straightforward. In *Duffing-harmonic* oscillator, the approximate periods obtained using introduced methods show much better agreement with the corresponding exact period than the periods of the other existing techniques. High accuracy of the approximate periods obtained from *Duffing-harmonic* oscillator reveals the versatility of the introduced methods in solving highly nonlinear class of problems. To entirety up, it can say that the introduced methods is a better and efficient alternative than the existing.

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