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Some properties of special polynomials arising from p-adic integrals on $\mathbb{Z}_p$

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Abstract

In this paper, we study some properties of special polynomials associated with Korobov polynomials and give some new identities of those polynomials by using p-adic invariant integrals on $\mathbb{Z}_p$.

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1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $UD\left(\mathbb{Z}_p\right)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD\left(\mathbb{Z}_p\right)$, the p-adic invariant integral on $\mathbb{Z}_p$ is given by

$$\begin{aligned} I_0(f) &= \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [9, 11, 12, 10, 15, 13, 14, 16])}. \end{aligned} \tag{1.1}$$

From (1.1), we have

$$I_0(f_1) = I_0(f) + f'(0), \quad \text{(see [4, 14])}, \tag{1.2}$$

where

$$f_1(x) = f(x+1),\ f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}.$$

Let us assume that q is an ineterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. As is well known, Korobov polynomials are defined by the generating function to be

$$\frac{qt}{(1+t)^q - 1}(1+t)^x = \sum_{n=0}^{\infty} P_{n,q}(x)\frac{t^n}{n!}, \quad \text{(see [17, 26])}. \tag{1.3}$$

When $x = 0$, $P_n = P_n(0)$ are called Korobov numbers. It is known that the Daehee polynomials are given by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \quad \text{(see [12, 10])}.$$

When $x = 0$, $D_n = D_n(0)$ are called Daehee numbers.

In this paper, we study some properties of special polynomials associated with Korobov polynomials and give some new identities of those polynomials by using p-adic invariant integrals on $\mathbb{Z}_p$.

2. Some special polynomials

In this section, we assume that $t, q\ (\neq 0) \in \mathbb{C}_p$ such that $|t|_p < |q|_p^{-1} p^{-\frac{1}{p-1}}$. Let us take $f(y) = (1+t)^{\frac{y}{q}+x}$. Then, by (1.2), we get

$$\int_{\mathbb{Z}_p} (1+t)^{qy+x} \, d\mu_0(y) = \frac{q \log(1+t)}{(1+t)^q - 1} (1+t)^x \tag{2.4}$$
$$= \left(\frac{\log(1+t)}{t}\right)\left(\frac{qt(1+t)^x}{(1+t)^q - 1}\right).$$

Now, we observe that

$$\left(\frac{\log(1+t)}{t}\right)\left(\frac{qt(1+t)^x}{(1+t)^q - 1}\right) = \left(\sum_{l=0}^{\infty} D_l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} P_{m,q}(x) \frac{t^m}{m!}\right) \tag{2.5}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} P_{m,q}(x) D_{n-m} \binom{n}{m}\right) \frac{t^n}{n!}.$$

Therefore, by (2.4) and (2.5), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$\int_{\mathbb{Z}_p} (qy+x)_n \, d\mu_0(y) = \sum_{m=0}^{n} P_{m,q}(x) D_{n-m} \binom{n}{m},$$

where $(x)_n = x(x-1)\cdots(x-n+1)$.

The Stirling number of the first kind is given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l, \quad (n \geq 0). \tag{2.6}$$

The Stirling number of the second kind is defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0), \quad \text{(see [21, 24, 23, 22, 25, 26, 27])}.$$

By (1.2), we easily get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t - 1} e^{xt} \tag{2.7}$$
$$= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

(see [2, 1, 3, 4, 5, 6, 7, 8, 9, 11, 12, 10, 15, 13, 14, 16, 17, 18, 19, 20, 21, 24, 23, 22, 25, 26, 27]), where $B_n(x)$ are ordinary Bernoulli polynomials.

Thus, from (2.7), we have

$$\int_{\mathbb{Z}_p} (x+y)^n \, d\mu_0(y) = B_n(x), \quad (n \geq 0). \tag{2.8}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

By (2.6), we see that

$$\begin{aligned} \int_{\mathbb{Z}_p} (qy+x)_n \, d\mu_0(y) &= \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} (qy+x)^l \, d\mu_0(y) \\ &= \sum_{l=0}^{n} S_1(n,l)\, q^l \int_{\mathbb{Z}_p} \left(y + \frac{x}{q}\right)^l d\mu_0(y) \\ &= \sum_{l=0}^{n} S_1(n,l)\, q^l B_l\left(\frac{x}{q}\right). \end{aligned} \tag{2.9}$$

Therefore, by Theorem 2.1 and (2.9), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$\sum_{l=0}^{n} S_1(n,l)\, q^l B_l\left(\frac{x}{q}\right) = \sum_{m=0}^{n} P_{m,q}(x)\, D_{n-m} \binom{n}{m}.$$

Remark 2.3. Let us define Korobov and Daehee mixed-type polynomials as follows:

$$T_{n,q}(x) = \int_{\mathbb{Z}_p} (qy+x)_n \, d\mu_0(y), \quad (n \geq 0). \tag{2.10}$$

Then, by (2.10), we get

$$T_{n,q}(x) = \sum_{m=0}^{n} \binom{n}{m} P_{m,q}(x)\, D_{n-m}.$$

The generating function of $T_{n,q}(x)$ is given by

$$\sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!} = \frac{q \log(1+t)}{(1+t)^q - 1} (1+t)^x. \tag{2.11}$$

Replacing t by $e^t - 1$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{n,q}(x) \frac{\left(e^t-1\right)^n}{n!} &= \frac{qt}{e^{qt}-1} e^{xt} \\ &= \sum_{n=0}^{\infty} q^n B_n\left(\frac{x}{q}\right) \frac{t^n}{n!}, \end{aligned} \tag{2.12}$$

and

$$\sum_{n=0}^{\infty} T_{n,q}(x)\frac{\left(e^t-1\right)^n}{n!} = \sum_{n=0}^{\infty} T_{n,q}(x)\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} T_{n,q}(x)\, S_2(m,n)\right)\frac{t^m}{m!}. \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.4. For $m \geq 0$, we have

$$q^m B_m\left(\frac{x}{q}\right) = \sum_{n=0}^{m}\left(\sum_{l=0}^{n}\binom{n}{l}P_{l,q}(x)\, D_{n-l}\right)S_2(m,n),$$

and

$$\left(\sum_{n=0}^{m}\binom{m}{n}P_{n,q}(x)\, D_{m-n}\right) = \sum_{n=0}^{m} S_1(m,n)\, q^n B_n\left(\frac{x}{q}\right).$$

In view of (2.10), we define the Korobov and Daehee mixed-type polynomials of the second kind as follows:

$$\widehat{T}_{n,q}(x) = \int_{\mathbb{Z}_p}(-qy+x)_n\, d\mu_0(y), \quad (n \geq 0). \tag{2.14}$$

The generating function of $\widehat{T}_{n,q}(x)$ are given by

$$\sum_{n=0}^{\infty}\widehat{T}_{n,q}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(-qy+x)_n\, d\mu_0(y)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}(1+t)^{-qy+x}\, d\mu_0(y). \tag{2.15}$$

From (1.2), we have

$$\int_{\mathbb{Z}_p}(1+t)^{-qy}\, d\mu_0(y) = \frac{q\log(1+t)}{(1+t)^q-1}(1+t)^q.$$

Thus, we have

$$\sum_{n=0}^{\infty}\widehat{T}_{n,q}(x)\frac{t^n}{n!} = \frac{q\log(1+t)}{(1+t)^q-1}(1+t)^{q+x}. \tag{2.16}$$

From (2.14), we have

$$\widehat{T}_{n,q}(x) = \int_{\mathbb{Z}_p} (-qy+x)_n \, d\mu_0(y) \tag{2.17}$$
$$= \sum_{l=0}^{n} S_1(n,l)(-1)^l q^l \int_{\mathbb{Z}_p} \left(y - \frac{x}{q}\right)^l d\mu_0(y)$$
$$= \sum_{l=0}^{n} S_1(n,l)(-1)^l q^l B_l\left(-\frac{x}{q}\right).$$

We observe that

$$\sum_{n=0}^{\infty} B_n(-x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{-xt} = \frac{t}{1-e^{-t}}e^{-(1+x)t} \tag{2.18}$$
$$= \frac{-t}{e^t-1}e^{-t(x+1)} = \sum_{n=0}^{\infty} B_n(1+x)(-1)^n\frac{t^n}{n!}.$$

By (2.17) and (2.18), we get

$$\widehat{T}_{n,q}(x) = \sum_{l=0}^{n} S_1(n,l)\, q^l B_l\left(1+\frac{x}{q}\right). \tag{2.19}$$

By replacing t by e^t-1 in (2.16), we get

$$\sum_{n=0}^{\infty} \widehat{T}_{n,q}(x)\frac{1}{n!}\left(e^t-1\right)^n = \frac{qt}{e^{qt}-1}e^{(q+x)t} \tag{2.20}$$
$$= \sum_{n=0}^{\infty} q^n B_n\left(1+\frac{x}{q}\right)\frac{t^n}{n!},$$

and

$$\sum_{n=0}^{\infty} \widehat{T}_{n,q}(x)\frac{1}{n!}\left(e^t-1\right)^n = \sum_{n=0}^{\infty} \widehat{T}_{n,q}(x)\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \tag{2.21}$$
$$= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \widehat{T}_{n,q}(x)\, S_2(m,n)\right)\frac{t^m}{m!}.$$

Therefore, by (2.19), (2.20) and (2.21), we obtain the following theorem.

Theorem 2.5. For $m \geq 0$, we have

$$q^m B_m\left(1+\frac{x}{q}\right) = \sum_{n=0}^{m} S_2(m,n)\,\widehat{T}_{n,q}(x),$$

and

$$\widehat{T}_{m,q}(x) = \sum_{n=0}^{m} S_1(m,n)\, q^n B_n\left(1+\frac{x}{q}\right).$$

We observe that

$$\begin{aligned}(-1)^n \frac{\widehat{T}_{n,q}}{n!} &= \frac{(-1)^n}{n!}\int_{\mathbb{Z}_p}(-qy)_n\, d\mu_0(y) = \int_{\mathbb{Z}_p}\binom{qy+n-1}{n} d\mu_0(y) \qquad (2.22)\\ &= \sum_{l=0}^{n}\binom{n-1}{n-l}\int_{\mathbb{Z}_p}\binom{qy}{l} d\mu_0(y)\\ &= \sum_{l=1}^{n}\binom{n-1}{l-1}\frac{1}{l!}\int_{\mathbb{Z}_p}(qy)_l\, d\mu_0(y)\\ &= \sum_{l=1}^{n}\binom{n-1}{l-1}\frac{T_{l,q}}{l!}.\end{aligned}$$

Therefore, by (2.22), we obtain the following theorem.

Theorem 2.6. For $n \geq 1$, we have

$$(-1)^n \frac{\widehat{T}_{n,q}}{n!} = \sum_{l=1}^{n}\binom{n-1}{l-1}\frac{T_{l,q}}{l!},$$

and

$$(-1)^n \frac{T_{n,q}}{n!} = \sum_{l=1}^{n}\binom{n-1}{l-1}\frac{\widehat{T}_{l,q}}{l!},$$

where $T_{n,q} = T_{n,q}(0)$, $\widehat{T}_{n,q} = \widehat{T}_{n,q}(0)$.

For $\alpha \in \mathbb{N}$, let us consider Korobov and Daehee mixed-type polynomials of order α as follows:

$$T_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(qx_1+\cdots+qx_\alpha+x)_n\, d\mu_0(x_1)\cdots d\mu_0(x_\alpha). \qquad (2.23)$$

Then, we have

$$\begin{aligned}&T_{n,q}^{(\alpha)}(x) \qquad (2.24)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(qx_1+\cdots+qx_\alpha+x)_n\, d\mu_0(x_1)\cdots d\mu_0(x_\alpha)\\ &= \sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(qx_1+\cdots+qx_\alpha+x)^l\, d\mu_0(x_1)\cdots d\mu_0(x_\alpha)\\ &= \sum_{l=0}^{n} S_1(n,l)\, q^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(x_1+\cdots+x_\alpha+\frac{x}{q}\right)^l d\mu_0(x_1)\cdots d\mu_0(x_\alpha).\end{aligned}$$

We observe that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_\alpha+x)t} d\mu_0(x_1)\cdots d\mu_0(x_\alpha) = \left(\frac{t}{e^t-1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{2.25}$$

where $B_n^{(\alpha)}(x)$ are the Bernoulli polynomials of order α.

By (2.24) and (2.25), we get

$$T_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} S_1(n,l)\, q^l B_l^{(\alpha)}\left(\frac{x}{q}\right). \tag{2.26}$$

From (2.23), we can derive the generating function of $T_{n,q}^{(\alpha)}(x)$ as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} T_{n,q}^{(\alpha)}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (qx_1+\cdots+qx_\alpha+x)_n\, d\mu_0(x_1)\cdots d\mu_0(x_\alpha)\frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty}\binom{qx_1+\cdots+qx_\alpha+x}{n} t^n d\mu_0(x_1)\cdots d\mu_0(x_\alpha) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{qx_1+\cdots+qx_\alpha+x}\, d\mu_0(x_1)\cdots d\mu_0(x_\alpha) \\
&= (1+t)^x\left(\frac{q\log(1+t)}{(1+t)^q-1}\right)^\alpha.
\end{aligned} \tag{2.27}$$

As is known, the Daehee numbers of order α are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^\alpha = \sum_{l=0}^{\infty} D_l^{(\alpha)}\frac{t^l}{l!}, \quad \text{(see [12])}. \tag{2.28}$$

Now, we define the Korobov polynomials of order α as follows:

$$\left(\frac{qt}{(1+t)^q-1}\right)^\alpha (1+t)^x = \sum_{m=0}^{\infty} P_{m,q}^{(\alpha)}(x)\frac{t^m}{m!}. \tag{2.29}$$

From (2.28) and (2.29), we have

$$\begin{aligned}\left(\frac{q\log(1+t)}{(1+t)^q-1}\right)^{\alpha}(1+t)^x &= \left(\frac{\log(1+t)}{t}\right)^{\alpha}\left(\frac{qt}{(1+t)^q-1}\right)^{\alpha}(1+t)^x \\ &= \left(\sum_{l=0}^{\infty} D_l^{(\alpha)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} P_{m,q}^{(\alpha)}(x)\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}D_l^{(\alpha)}P_{n-l,q}^{(\alpha)}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.30}$$

Therefore, by (2.27) and (2.30), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$T_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n}\binom{n}{l}D_l^{(\alpha)}P_{n-l,q}^{(\alpha)}(x).$$

By replacing t by $e^t - 1$ in (2.27), we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(\alpha)}(x)\frac{\left(e^t-1\right)^n}{n!} &= \left(\frac{qt}{e^{qt}-1}\right)^{\alpha}e^{xt} \\ &= \sum_{n=0}^{\infty} B_n^{(\alpha)}\left(\frac{x}{q}\right)q^n\frac{t^n}{n!},\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(\alpha)}(x)\frac{\left(e^t-1\right)^n}{n!} &= \sum_{n=0}^{\infty} T_{n,q}^{(\alpha)}(x)\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} T_{n,q}^{(\alpha)}(x)S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem:

Theorem 2.8. For $m \geq 0$, we have

$$q^m B_m^{(\alpha)}\left(\frac{x}{q}\right) = \sum_{n=0}^{m}\left(\sum_{l=0}^{n}\binom{n}{l}P_{l,q}^{(\alpha)}(x)D_{n-l}^{(\alpha)}\right)S_2(m,n),$$

and

$$\left(\sum_{n=0}^{m}\binom{m}{n}P_{n,q}^{(\alpha)}(x)D_{m-n}^{(\alpha)}\right) = \sum_{n=0}^{m} S_1(m,n)q^n B_n^{(\alpha)}\left(\frac{x}{q}\right).$$

Let us consider Daehee and Korobov mixed-type polynomials of the second kind with order α as follows:

$$\widehat{T}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-qx_1 - qx_2 - \cdots - qx_\alpha + x)_n \, d\mu_0(x_1) \cdots d\mu_0(x_\alpha), \tag{2.33}$$

where $n \geq 0$.

Thus, by (2.33), we get

$$\begin{aligned} \widehat{T}_{n,q}^{(\alpha)}(x) & \\ &= \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-qx_1 - \cdots - qx_\alpha + x)^l \, d\mu_0(x_1) \cdots d\mu_0(x_\alpha) \\ &= \sum_{l=0}^{n} S_1(n,l)(-1)^l q^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(x_1 + \cdots + x_\alpha - \frac{x}{q}\right)^l d\mu_0(x_1) \cdots d\mu_0(x_\alpha) \\ &= \sum_{l=0}^{n} S_1(n,l)(-1)^l q^l B_l^{(\alpha)}\left(-\frac{x}{q}\right). \end{aligned} \tag{2.34}$$

Now, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{(\alpha)}(-x) \frac{t^n}{n!} &= \left(\frac{t}{e^t - 1}\right)^\alpha e^{-xt} \\ &= \left(\frac{-t}{e^{-t} - 1}\right)^\alpha e^{-(x+\alpha)t} \\ &= \sum_{n=0}^{\infty} B_n^{(\alpha)}(x+\alpha)(-1)^n \frac{t^n}{n!}. \end{aligned} \tag{2.35}$$

Thus, by (2.35), we get

$$B_n^{(\alpha)}(-x) = (-1)^n B_n^{(\alpha)}(x+\alpha), \quad (n \geq 0). \tag{2.36}$$

From (2.34) and (2.36), we have

$$\begin{aligned} \widehat{T}_{n,q}^{(\alpha)}(x) &= \sum_{l=0}^{n} (-1)^l S_1(n,l) q^l B_l^{(\alpha)}\left(-\frac{x}{q}\right) \\ &= \sum_{l=0}^{n} S_1(n,l) q^l B_l^{(\alpha)}\left(\alpha + \frac{x}{q}\right). \end{aligned} \tag{2.37}$$

By (2.33), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widehat{T}_{n,q}^{(\alpha)}(x)\frac{t^n}{n!} \qquad (2.38)\\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-qx_1 - \cdots - qx_\alpha + x)_n \, d\mu_0(x_1) \cdots d\mu_0(x_\alpha) \frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{-qx_1 - \cdots - qx_\alpha + x} \, d\mu_0(x_1) \cdots d\mu_0(x_\alpha) \\
&= (1+t)^x \left(\frac{q \log(1+t)}{(1+t)^q - 1} (1+t)^q \right)^\alpha \\
&= (1+t)^{x+q\alpha} \left(\frac{q \log(1+t)}{(1+t)^q - 1} \right)^\alpha .
\end{aligned}$$

Note that

$$\begin{aligned}
&\left(\frac{q \log(1+t)}{(1+t)^q - 1} \right)^\alpha (1+t)^{x+q\alpha} \qquad (2.39)\\
&= \left(\frac{\log(1+t)}{t} \right)^\alpha \left(\frac{qt}{(1+t)^q - 1} \right)^\alpha (1+t)^{x+q\alpha} \\
&= \left(\sum_{l=0}^{\infty} D_l^{(\alpha)} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} P_{m,q}^{(\alpha)}(x+q\alpha) \frac{t^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} D_l^{(\alpha)} P_{n-l,q}^{(\alpha)}(x+q\alpha) \right) \frac{t^n}{n!}.
\end{aligned}$$

Therefore, by (2.38) and (2.39), we obtain the following theorem.

Theorem 2.9. For $n \geq 0$, we have

$$\widehat{T}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} D_l^{(\alpha)} P_{n-l,q}^{(\alpha)}(x+q\alpha).$$

Replacing t by $e^t - 1$ in (2.38), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{T}_{n,q}^{(\alpha)}(x) \frac{\left(e^t - 1\right)^n}{n!} &= \left(\frac{qt}{e^{qt} - 1} \right)^\alpha e^{(x+q\alpha)t} \qquad (2.40)\\
&= \sum_{n=0}^{\infty} B_n^{(\alpha)}\left(\frac{x}{q} + \alpha \right) q^n \frac{t^n}{n!},
\end{aligned}$$

and

$$\sum_{n=0}^{\infty} \widehat{T}_{n,q}^{(\alpha)}(x) \frac{\left(e^t-1\right)^n}{n!} = \sum_{n=0}^{\infty} \widehat{T}_{n,q}^{(\alpha)}(x) \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \widehat{T}_{n,q}^{(\alpha)}(x) S_2(m,n) \right) \frac{t^m}{m!}. \tag{2.41}$$

Therefore, by (2.40) and (2.41), we obtain the following theorem.

Theorem 2.10. For $m \geq 0$, we have

$$q^m B_m^{(\alpha)} \left(\alpha + \frac{x}{q} \right) = \sum_{n=0}^{m} \sum_{l=0}^{n} \binom{n}{l} D_{n-l}^{(\alpha)} P_{l,q}^{(\alpha)}(x+q\alpha) S_2(m,n),$$

and

$$\sum_{n=0}^{m} \binom{m}{n} D_{m-n}^{(\alpha)} P_{n,q}^{(\alpha)}(x+q\alpha) = \sum_{n=0}^{m} S_1(m,n) q^n B_n^{(\alpha)} \left(\alpha + \frac{x}{q} \right).$$

Let $\widehat{T}_{n,q}^{(\alpha)} = \widehat{T}_{n,q}^{(\alpha)}(0)$, $T_{n,q}^{(\alpha)} = T_{n,q}^{(\alpha)}(0)$. Then, we easily get

$$(-1)^n \frac{\widehat{T}_{n,q}^{(\alpha)}}{n!} = \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{T_{l,q}^{(\alpha)}}{l!},$$

and

$$(-1)^n \frac{T_{n,q}^{(\alpha)}}{n!} = \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{\widehat{T}_{l,q}^{(\alpha)}}{l!},$$

where $n \geq 1$.

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