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Completely Monotone and some Related Functions in Hypercomplex Systems

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Abstract

This paper is devoted to give the main properties of completely monotone functions defined in a hypercomplex system $L_1(Q, m)$. We prove that a continuous function with compact support ψ is completely alternating if and only if $exp(-\alpha\psi)$ is completely monotone for each $\alpha > 0$. Moreover, we will give integral representations of for completely monotone and some harmonic related functions in hypercomplex system $L_1(Q, m)$.

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1. Introduction

A function f is completely monotonic if for all n, $(-1)^n f^{(n)}(x) \geq 0$ on $(0, \infty)$; see Widder [19], Ismail et al. [11], Feller [6], Choquet [5] and Berg [1] for properties of completely monotonic functions. Bernstien's theorem asserts that f is completely monotonic if and only if $f(x) = \int_{\mathbb{R}} e^{-xt} d\mu(t)$ where μ is a positive measure supported on a subset of $[0, \infty)$. Harmonic analysis in hypercomplex system dates back to J. Delsarte's and B. M. Levitan's work during the 1930s and 1940s, but the substantial development had to wait till the 1990s when Berezansky and Kondratiev [2] put hypercomplex system in the right setting for harmonic analysis. A central idea in harmonic analysis in various settings has been the existence of a product, usually called convolution, for functions and measures. In some cases, an investigation begins with a convolution algebra of measures as the primitive object upon which to build a theory; this is the case of the analysis of the objects called generalized hypergroups "hypercomplex system" which are the generalizations of the convolution algebra of Borel measures on a group. A hypercomplex system with a locally compact basis Q (see [2], [3] or [18]) is a set $L_1(Q, m)$ with generalized convolution, which can be defined in terms of a structure measure $c(A, B, r), \quad A, B \subset Q, r \in Q$. One important reason that explain why the harmonic analysts did not attracted to study Fourier algebra over hypercomplex system is that, the product of two continuous positive definite functions in a hypercomplex system is not necessarily positive definite in general. Consider the space $L_1(Q, m) = L_1$ of functions on Q integrable with respect to the multiplicative measure m i.e., a regular Borel measure m positive on open sets such that

$$\int c(A, B, r)dm(r) = m(A)m(B) \quad (A, B \in B_0(Q))$$

It is some-times convenient to consider, together with the measure $c(A, B, r)$, its extension to the sets form $Q \times Q$. For this purpose, we fix r and, for any $(A, B \in (Q))$, put $m_r(A \times B) = c(A, B, r)$. The space $L_1(Q, m)$ with the convolution

$$(f * g)(r) = \int\int f(p)g(q)dm_r(p, q)$$

is called a hypercomplex system with basis Q. The rule played by the generalized translation R_p acting upon functions of a point $q \in Q$ and satisfying

$$(R_p\chi(., \lambda))(q) = \chi(p, \lambda)\chi(q, \lambda)$$

A continuous bounded function $\phi(r) \quad (r \in Q)$ is called positive definite if

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j (R_{r_i} * \phi)(r_j) \geq 0$$

for all $r_1, r_2, \ldots, r_n \in Q, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and $n \in \mathbb{N}$. As pointed in [2], every continuous positive definite function $\phi \in P(Q)$ admits a unique representation in the form of an integral

$$\phi(r) = \int_{X_h} \chi(r)d\mu(\chi) \quad (r \in Q)$$

where μ is a nonnegative finite regular measure on the space of continuous bounded characters X_h. A locally bounded measurable function q is called a quadratic form if

$$q(r * s) + q(r * s^*) = 2q(r) + 2q(s)$$

for all $r, s \in Q$ and additive if $q(r * s) = 2q(r) + 2q(s)$ for all $r, s \in Q$. In the case Q is hermitian, that when Q carries the identity involution, then every quadratic form is an additive function and every negative definite function is real. A continuous bounded function $\phi(r) \quad (r \in Q)$ will be called negative definite if

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j (R_{r_i} * \phi)(r_i) \leq 0$$

for all $r_1, r_2, \ldots, r_n \in Q$, $n \in \mathbb{N}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ that satisfying $\sum_{i=1}^{n} \lambda_i = 0$.

A key result in the study of negative definite functions in hypercomplex systems is the following Levy-Khinchin representation (see, [14])

$$\psi(r) = \psi(e) + q(r) + \int_{X_h \backslash \{1\}} (1 - Re(\chi(r))) d\mu(\chi)$$

for all $r \in Q$ where q is a nonnegative quadratic form on Q and $\mu \in M_+(X_h \backslash \{1\})$. Both q and the integral part $\psi(r) - \psi(e) - q(r)$ belong to the set of negative definite functions on Q and the pair (q, μ) is uniquely determined by ψ with q being given by

$$q(r) = lim \left\{ \frac{\psi(r^{*n})}{n^2} + \frac{\psi((r * r)^{*n})}{2n} \right\}$$

2. τ-positive functions

Let $\mathbb{A}$ be a maximal algebra in $L_1(Q, m)$. A linear functional $L : \mathbb{A} \to \mathbb{C}$ is called τ-positive, where $\tau \subset \mathbb{A}$ is admissible, if

$$L(a) \geq 0 \quad for \quad all \quad a \in algspan^+(\tau)$$

This holds if and only if

$$L(a_1 * \cdots * a_n) \geq 0 \quad for \quad all \quad finite \quad sets \quad \{a_1, \ldots, a_n\} \subseteq \tau$$

Let $\alpha : L_1(Q, m) \to \mathbb{R}_+$ be an absolute value such that $\alpha(a) \geq 0$ for all $a \in L_1(Q, m)$. For $\sigma \in \mathbb{C}$, $a \in L_1(Q, m)$, we define

$$\Omega_{\sigma,a} = \frac{1}{2} \left(I + \frac{\sigma}{2\alpha(a)} R_a + \frac{\bar{\sigma}}{2\alpha(a^-)} R_{a^-} \right)$$

As a direct application of the above definition of $\tau-$ positive definite function, we can easily state the following two Lemmas:

Lemma 2.1. The sum and the pointwise limit of τ-positive functions in hypercomplex system $L_1(Q, m)$ is also τ-positive.

Lemma 2.2. Every τ-positive function in hypercomplex system $L_1(Q,m)$ is positive definite in $L_1(Q,m)$.

Theorem 2.3. A continuous τ-positive function $\phi : L_1(Q,m) \to \mathbb{C}$, where $\tau = \{\Omega_{\sigma,a}; \sigma \in \{\pm 1, \pm i\}, a \in X_h\}$ is positive definite and has integral representation

$$\phi(x) = \int_{X_h} \chi(x) d\mu(\chi)$$

where $\mu \in M_+^b(X_h)$ is concentrated on the compact set of τ-positive characters.

Proof. Let Γ denote the set of τ-positive multiplicative linear functionals on $\mathbb{A}$, which are not identically zero. Clearly Γ is a compact subset of the set of τ-positive linear functionals in $L_1(Q,m)$. By [1, Theorem 4.5.4] the linear functional L corresponding to τ-positive function ϕ has a representation

$$L(T) = \int_{\Gamma} \delta(T) d\tilde{\mu}(\delta), \quad T \in \mathbb{A},$$

where $\tilde{\mu} \in M_+(\Gamma)$. For $\delta \in \Gamma$ the function $x \to \delta(R_x)$ is a τ-positive character, and the mapping $j : \Gamma \to X_h$ given by $j(\delta)(x) = \delta(R_x)$ is a homeomorphism of Γ onto the compact set $j(\Gamma)$ of τ-positive characters. The image measure $\mu := \tilde{\mu}^j$ of $\tilde{\mu}$ under j is a Radon measure on X_h with compact support contained in $j(\Gamma)$, and replacing T by R_x we get

$$\phi(x) = \int_{X_h} \chi(x) d\mu(\chi), \quad x \in L_1(Q,m).$$

The main task in our previous works [13–16] was to give "the necessary and sufficient conditions guarantees that the product of two positive definite functions defined on hypergroup H is also positive definite on H". In this paper we will complete our study in hypercomplex system $L_1(Q,m)$. ∎

3. Completely monotone functions

Suppose that $L^{\infty}_{loc}(Q)$ denotes the set of locally bounded measurable functions on Q, and $L_1^c(Q)$ the space of integrable functions on Q with compact support. Similar to the pointed out result for hypergroups in [4] and semigroups in [14], we have

$$L_1^c(Q) * L_{\infty}^{loc}(Q) \subset C(Q)$$

For each $r \in Q$, we define the translation operator R_s by $R_s\phi(r) = \phi(r * s)$ for all $r, s \in Q$ and $\phi \in \mathbb{C}^Q$. The complex span $\mathbb{A}$ of all such operators is a commutative algebra with identity $R_1 = I$ and involution $(\sum \alpha_i R_{r_i})^- = \sum \overline{\alpha_i} R_{r_i^-}$. For real valued $\phi \in L_{\infty}^{loc}(Q)$ and $r \in Q$ we define $\nabla_r \phi : Q \to \mathbb{R}$ by

$$(\nabla_r \phi)(s) := (I - R_r)(\phi)(s)$$

We call ϕ completely monotone if $\phi \geq 0$ and

$$\nabla_{r_1}\nabla_{r_2}\cdots\nabla_{r_n}\phi \geq 0$$

for all $n \in \mathbb{N}$ and $r_1, r_2, \ldots, r_n \subset Q$. The function ϕ is said to be completely alternating if

$$\nabla_{r_1}\nabla_{r_2}\cdots\nabla_{r_n}\phi \leq 0$$

for all $n \in \mathbb{N}$ and $r_1, r_2, \ldots, r_n \subset Q$. With $\Delta_r\psi := -\nabla_r\psi$ we see from

$$\nabla_{r_1}\nabla_{r_2}\cdots\nabla_{r_n}(\Delta_x\psi) = -\nabla_{r_1}\nabla_{r_2}\cdots\nabla_{r_n}\nabla_x\psi$$

that $\psi \in L_\infty^{loc}(Q)$ is completely alternating if and only if $\Delta_r\psi$ is completely monotone for each $r \in Q$. The set of completely monotone (res. alternating) functions is denoted $\mathbb{M}(Q)$ (resp. $\mathbb{A}(Q)$). It is clear that $\mathbb{M}(Q)$ and $\mathbb{A}(Q)$ are closed convex cones in $\mathbb{R}^Q$. For more details about completely monotone and some related functions the reader can see [7–10].

Theorem 3.1. A continuous function $\phi : Q \longrightarrow \mathbb{R}$ is completely monotone if and only if there exists a measure $\mu \in M_+^b(X_{h+})$ such that for all $r \in Q$

$$\phi(r) = \int_{X_{h+}} \chi(r)d\mu(\chi)$$

Proof. Firstly we will prove that every completely monotone in $L_1(Q, m)$ is τ-positive. For $u_1, \ldots, u_n, r_1, \ldots, r_m \in Q$ we have

$$(I - R_{u_1})\cdots(I - R_{u_n})R_{r_1}\cdots R_{r_m}\phi(e) = \nabla_{u_1}\cdots\nabla_{u_n}\phi(r_1 * \cdots * r_m)$$

so every completely monotone in $L_1(Q, m)$ is τ-positive. Then the necessary condition follows from Theorem 3.2. Suppose $\phi : Q \longrightarrow \mathbb{R}$ has the integral representation

$$\phi(r) = \int_{X_{h+}} \chi(r)d\mu(\chi).$$

Since $0 \leq \chi(r) \leq 1$ for all $r \in Q$, then $\phi(r) \geq 0$ and since

$$\nabla_{u_1}\cdots\nabla_{u_n}\chi(r) = \chi(r)\prod_{i=1}^{n}[1 - \chi(u_i)]$$

so,

$$\nabla_{u_1}\cdots\nabla_{u_n}\phi(r) = \int_{X_{h+}} \chi(r)\prod_{i=1}^{n}[1 - \chi(u_i)]d\mu(\chi) \geq 0,$$

hence $\phi : Q \longrightarrow \mathbb{R}$ is completely monotone. ■

Corollary 3.2. Every completely monotone function $\phi : Q \longrightarrow \mathbb{R}$ in the the space $L_1(Q, m)$ is positive definite.

Proof. Let $\phi : Q \longrightarrow \mathbb{R}$ be a completely monotone function, so as pointed out from the above Theorem ϕ has the integral representation

$$\phi(r) = \int_{X_{h+}} \chi(r) d\mu(\chi).$$

This implies

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda_j} (R_{r_i} * \phi)(r_j) = \sum_{i,j=1}^{n} \lambda_i \bar{\lambda_j} \int_{X_{h+}} \chi(r_i * r_j) d\mu(\chi)$$
$$= \int_{X_{h+}} \sum_{i=1}^{n} |\lambda_i \chi(r_i)|^2 d\mu(\chi) \geq 0$$

This proves the required. ■

Corollary 3.3. The sum and the point wise limit of completely monotone functions in hypercomplex system $L_1(Q, m)$ is also completely monotone.

Proof. As a direct application of the above Theorem we can easily get the first part of the Corollary. For the proof of the second part, let $\phi_1, \phi_2, \ldots, \phi_m$ are a sequence of completely monotone functions in hypercomplex system $L_1(Q, m)$ and suppose that $\phi_0 = lim_{m\to\infty}\phi_m$, then

$$\nabla_{u_1} \cdots \nabla_{u_n} \phi_0 = \nabla_{u_1} \cdots \nabla_{u_n} lim_{m\to\infty}\phi_m = lim_{m\to\infty} \nabla_{u_1} \cdots \nabla_{u_n} \phi_m \geq 0$$

This implies the required.

Theorem 3.4. The cone $\mathbb{M}(Q)$ is an extreme subset of $\mathbb{P}_b(Q)$, the set of bounded positive definite functions on Q, and $\mathbb{M}^1(Q)$ is a Bauer simplex with $ex(\mathbb{M}^1(Q)) = X_{h+}$. For $\phi_1, \phi_2 \in \mathbb{M}(Q)$ also $\phi_1.\phi_2 \in \mathbb{M}(Q)$. A function $\phi \in \mathbb{P}_b(Q)$ is completely monotone if and only if the representing measure μ is concentrated on X_{h+}.

Proof. As pointed out of Theorem 3.1. $\mathbb{M}(Q) \subseteq \mathbb{P}_b(Q)$. Suppose that $\phi = \phi_1 + \phi_2 \in \mathbb{M}(Q)$ and $\phi_i \in \mathbb{P}_b(Q)$ with

$$\phi_i(x) = \int_{X_{h+}} \chi(x) d\mu_i(\chi), \quad i = 1, 2.$$

then $\mu = \mu_1 + \mu_2$, where $\mu \in M(X_{h+})$ is the representing measure for ϕ. Hence, μ_1, μ_2 are concentrated on X_{h+} so $\phi_1, \phi_2 \in \mathbb{M}(Q)$, So $\mathbb{M}(Q)$ is an extreme subset of $\mathbb{P}_b(Q)$. By transitivity of extremality an extreme point of $\mathbb{M}^1(Q)$ is also an extreme point of $\mathbb{P}_b^1(Q)$, hence $ex(\mathbb{M}^1(Q)) \subseteq X_{h+}$, and in fact there is equality since $X_{h+} \subseteq$

$\mathbb{M}^1(Q) \cap ex(\mathbb{P}^1_b(Q))$. Recalling Lemma 3.2., then for any $\mu, \nu \in M(\hat{X}_+)$ we have $supp(\mu * \nu) \subseteq X_{h+}$, and it follows that $\mathbb{M}(Q)$ is a stable under multiplication. By unicity of the representing measure for $\phi \in \mathbb{P}_b(Q)$ it follows that $\mathbb{M}^1(X_{h+})$ is a simplex, and that the representing measure for ϕ is concentrated on X_{h+} if $\phi \in \mathbb{P}_b(Q)$ is completely monotone. ■

Lemma 3.5. A bounded measurable function $\phi \in C_c(X)$ is positive definite if and only if there exists a ψ in $L^2(X)$ such that $\phi = \psi \bullet \tilde{\psi}$, where

$$f \bullet \tilde{g}(x) = \int_X f(x * y)\overline{g(y)}d\eta(y).$$

for all $f, g \in C_c(X)$.

Proof. The proof is as in Pederson [13, Lemma 7.2.4]. ■

Theorem 3.6. Let ϕ_1 and ϕ_2 belongs to $C_c(Q)$ then the product $\phi_1.\phi_2$ is completely monotone on Q if and only if ϕ_1 and ϕ_2 are completely monotone on X.

Proof. From the above lemma there exists $\psi_1, \psi_2 \in L^2(Q)$ such that $\phi_1 = \psi_1 \bullet \tilde{\psi}_1$ and $\phi_2 = \psi_2 \bullet \tilde{\psi}_2$, so

$$\begin{aligned}
\phi_1.\phi_2(x) &= (\psi_1 \bullet \tilde{\psi}_1(x)).(\psi_2 \bullet \tilde{\psi}_2(x)) \\
&= \int_{X_h} \psi_1(x * y)\overline{\psi_1(y)}d\eta(y) \int_{X_h} \psi_2(x * z)\overline{\psi_2(z)}d\eta(z) \\
&= \int_{X_h}\int_{X_h} \psi_1(x * y)\psi_2(x * z)\overline{\psi_1(y)\psi_2(z)}d\eta(y)d\eta(z) \\
&= \int_{X_h}\int_{X_h} \psi_1.\psi_2(x * y, x * z)\overline{\psi_1.\psi_2(y, z)}d\eta(y)d\eta(z) \\
&= \int_{X_h}\int_{X_h} \psi_1.\psi_2(x * (y, z))\overline{\psi_1.\psi_2(y, z)}d\eta(y)d\eta(z)
\end{aligned}$$

Applying Fubini's theorem to the right hand side we get

$$\phi_1.\phi_2(x) = \int_{X_h \times X_h} \psi_1.\psi_2(x * (y, z))\overline{\psi_1.\psi_2(y, z)}d\nu(y, z)$$

This implies

$$\phi_1.\phi_2(x) = \psi_1.\psi_2 \bullet \widetilde{\psi_1.\psi_2}(x).$$

Corollary 3.7. Let $\phi \in C_c(Q)$ be completely monotone such that $|\phi(x * x^-)| < \chi$ for all $x \in Q$. Then if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in $\{z \in \mathbb{C}; z < \chi\}$ and $a_n \geq 0$ for all $n \geq 0$, the composed kernel $f \circ \phi$ is a gain completely monotone. In particular, if $\phi \in C_c(Q)$ is completely monotone, then so is $exp(\phi)$.

4. Completely alternating functions

Theorem 4.1. A continuous function $\psi : Q \to \mathbb{R}$ is completely alternating if and only if there exists an additive continuous function $h : Q \to \mathbb{R}_+$ and a unique measure $\mu \in M_+(X_{h+}\backslash\{1\})$ such that for all $r \in Q$

$$\psi(r) = \psi(e) + h(r) + \int_{X_{h+}\backslash\{1\}} (1 - \chi(r))d\mu(\chi)$$

Proof. It follows from the definition that ψ is lower bounded by $\psi(e)$, firstly, we assume that $\psi(e) = 0$. Let $S \supseteq Q$ be a minimal semigroup containing the hypercomplex basis Q. Introducing

$$\Delta_s\psi(r) := \frac{1}{2}[\psi(r * s) + \psi(r * s^*)] - \psi(r); \quad r, s \in Q,$$

and as pointed in [1, Proposition 4.3.11] $\Delta_s\psi$ is bounded and positive definite on Q. Therefore appealing to Bochner's theorem for hypercomplex systems ([12], Theorem 12.3B)

$$\Delta_s\psi(r) = \int_{X_{h+}} \rho_\chi(r)d\sigma_s(\chi)$$

for some $\sigma_s \in M_+^b(X_{h+})$, where we denote the canonical extension of $\chi \in X_{h+}$ to a function on Q by ρ_χ. A simple calculation implies

$$\begin{aligned} -\Delta_t\Delta_s\psi(r) &= \int_{X_{h+}} \rho_\chi(r)[1 - Re\rho_\chi(t)]d\sigma_s(\chi) \\ &= \int_{X_{h+}} \rho_\chi(r)[1 - Re\rho_\chi(s)]d\sigma_t(\chi) \end{aligned}$$

for $r, s, t \in Q$, implying

$$[1 - Re\rho_\chi(t)]d\sigma_s(\chi) = [1 - Re\rho_\chi(s)]d\sigma_t(\chi)$$

by the uniqueness of the Fourier transform ([12], Theorem 12.2A). Noting that the $\{\chi \in X_{h+}; Re\chi(s) \leq 1\}$ are open sets in X_{h+} with union(over s)given by $X_{h+}\backslash\{1\}$, we can find a unique Radon measure μ on $X_{h+}\backslash\{1\}$ such that for every $s \in S$

$$[1 - Re\rho_\chi(s)]d\mu(\chi) = d\sigma_s(\chi), \quad \text{on} \quad X_{h+}$$

The set $\hat{S}$ of all bounded semigroup characters on the semigroup S is a compact Hausdorff space with respect to the topology of point wise convergence. The canonical mapping $\zeta : X_{h+} \to \hat{S}$, $\zeta(\chi) := \rho_\chi$ is continuous, and obviously $\Delta_s\psi$ is the Laplace transform of σ_s^ζ(the image measure of σ_s under ζ) for each $s \in S$ hence from [1, Lemma 4.3.12, Definition 4.3.13 and Theorem 4.6.7] there exists an additive continuous function $h : Q \to \mathbb{R}_+$ on S such that for all $r \in S$

$$\begin{aligned}\psi(r) &= h(r) + \int_{\hat{S}_+\backslash\{1\}} (1 - \rho(r))d\mu^{\zeta}(\rho) \\ &= h(r) + \int_{X_{h+}\backslash\{1\}} (1 - \chi(r))d\mu(\chi)\end{aligned}$$

where $\mu \in M_+(X_{h+}\backslash\{1\})$. ■

Corollary 4.2. A continuous function with compact support $\psi \in \mathbb{A}(Q)$ if and only if $exp(-t\psi) \in \mathbb{M}(Q)$ for each $t > 0$.

Proof. suppose that $exp(-t\psi) \in \mathbb{M}(Q)$ for all $t > 0$ then $1 - exp(-t\psi) \in \mathbb{A}(Q)$, so $\dfrac{1 - exp(-t\psi)}{t} \in \mathbb{A}(Q)$. For the converse it suffices to prove that $exp(-\psi) \in \mathbb{M}(Q)$ for $\psi \in \mathbb{A}(Q)$ with the representation

$$\psi(x) = \psi(e) + h(x) + \int_{X_{h+}\backslash\{1\}} (1 - \chi(x))d\mu(\chi)$$

Since $\mathbb{M}(Q)$ is closed under multiplication and $exp(-h) \in X_{h+}$, it suffices to prove that

$$x \to exp[-\int_{X_{h+}\backslash\{1\}} (1 - \chi(x))d\mu(\chi)]$$

belongs to $\mathbb{M}(Q)$. This in fact true because $\chi \in \mathbb{M}(Q)$ and

$$exp(c\chi) = \sum_{n=0}^{\infty} \frac{1}{n!} c^n \chi^n, \quad c \geq 0.$$

■

Remark 4.3. In [13], depending on the results given by Pederson [17, lemma 7.2.4], we were proved the stability of the set of continuous positive definite functions with compact support on hypergroups. In the same direction, the second part of Theorem 4.2. ensure that, the product of two completely monotone functions not only positive definite, but also completely monotone on $L_1(Q, m)$.

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