Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 3 (2016), pp. 2631–2634
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http://www.ripublication.com/gjpam.htm

Domination in the Unitary Addition Cayley Graphs

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Abstract

Let $(\Gamma, *)$ be a finite group and e be its identity. Let A be a generating set of Γ such that $e \notin A$ and $a^{-1} \in A$ for all $a \in A$. The Cayley graph is defined by $G = (V(G), E(G))$, where $V(G) = \Gamma$ and $E(G) = \{(x, xa)|x \in V(G), a \in A\}$, denoted by $Cay(\Gamma, A)$. If Γ is an abelian group and B is a subset of Γ, then the addition Cayley graph $G^{'} = Cay^{+}(\Gamma, B)$ is the graph having the vertex set $V(G^{'}) = \Gamma$ and the edge set $E(G^{'}) = \{ab : a + b \in B\}$, where $a, b \in \Gamma$. For a positive integer $n > 1$, the unitary addition Cayley graph G_n is the graph whose vertex set is $\mathbb{Z}_n$, the integers modulo n and if U_n denotes the set of all units of the ring $\mathbb{Z}_n$, then two vertices a and b are adjacent if and only if $a + b \in U_n$. In this paper, we attempt to find the domination number on the Unitary Addition Cayley graphs G_n for some n.

AMS subject classification: 05C69.
Keywords: Dominating sets, domination number, Unitary Cayley Graphs, Unitary Addition Cayley graphs.

1. Introduction

All through the study, we only consider finite, undirected and simple graphs $G = (V(G), E(G))$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph G is called the order of G and is denoted by $n = n(G)$ and $m = m(G)$ is the number of edges or rather the size of G. The open neighborhood $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the degree of v. The closed neighborhood of a vertex $v \in V$ is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. A regular graph is a graph whose vertices have all the same degree. If $d(x) = r$ for all $x \in V(G)$, we call G $r - regular$. The graph definitions, terminologies, and notations, unless otherwise indicated, are taken from the books by Harary in [5] and by Chartrand and Lesniak in [2].

In a graph G, any $S \subseteq V(G)$ is a *dominating set* in G if $\bigcup_{v \in S} N_G[v] = V(G)$. A dominating set in G is also called a γ-set in G. The minimum cardinality $\gamma(G)$ of a γ-set in G is the *domination number* of G. Any γ-set in G of cardinality $\gamma(G)$ is referred to as the minimum γ-set in G.

Let Γ be a finite group with e as the identity. A generating set of the group Γ is a subset A such that every element of Γ can be expressed as the product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$. The Cayley graph $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, x * a) | x \in V(G), a \in A\}$, and it is denoted by $Cay(\Gamma, A)$. For any positive integer n, let Z_n denotes the additive cyclic group of integers modulo n. If we represent the elements of Z_n by $0, 1, \ldots, n-1$, then $U_n = \{a \in \mathbb{Z}_n | gcd(a, n) = 1\}$ is a subset of Z_n of order $\phi(n)$, where $\phi(n)$ is the Euler's ϕ function. The Cayley graph $Cay(\mathbb{Z}_n, U_n)$ is known as unitary Cayley graph. Hence, the unitary Cayley graph X_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring $\mathbb{Z}_n$, then two vertices a and b are adjacent if and only if $a - b \in U_n$. The unitary Cayley graph X_n is also defined as, $X_n = Cay(\mathbb{Z}_n, U_n)$ [6].

Let Γ be an abelian group and B be a subset of Γ. The addition Cayley graph $G^{'} = Cay^{+}(\Gamma, B)$ is the graph having the vertex set $V(G^{'}) = \Gamma$ and the edge set $E(G^{'}) = \{ab : a + b \in B\}$, where $a, b \in \Gamma$. Now, for a positive integer $n > 1$, the unitary addition Cayley graph G_n is the graph whose vertex set is $\mathbb{Z}_n$, the integers modulo n and if U_n denotes the set of all units of the ring $\mathbb{Z}_n$, then two vertices a and b are adjacent if and only if $a + b \in U_n$. The unitary addition Cayley graph G_n is also defined as, $G_n = Cay^{+}(\mathbb{Z}_n, U_n)$. Unitary addition Cayley graphs are motivated from the definition of addition Cayley graphs [3].

In this paper, we attempt to find the domination number in the Unitary Addition Cayley graphs G_n for some n.

2. Preliminaries

We now give some results which are essentials and important in this study.

2.1. Dominating Set and Domination Number

Clearly $1 \leq \gamma(G) \leq n$ for any graph G on n vertices. For graphs without isolated vertices, the upper bound was improved considerably by Ore, who first published results about domination in graphs.

Theorem 2.1. (Ore [7],1962) If G is a graph without isolated vertices, then $\gamma(G) \leq n(G)/2$.

Theorem 2.2. Tarr provided the following in [8].

i. For a complete graph K_p with p vertices, $\gamma(K_p) = 1$;

ii. For a path P_n with n vertices, $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$;

iii. For a cycle C_p with p vertices, $\gamma(C_p) = \left\lceil \frac{p}{3} \right\rceil$.

2.2. Unitary Cayley graphs

Let $X_n = Cay(\mathbb{Z}_n, U_n)$ be a unitary Cayley graph.

Theorem 2.3. [1] X_n is $\phi(n)$-regular for all n.

Theorem 2.4. [4] X_n, $n \geq 2$, is bipartite if and only if n is even.

2.3. Unitary addition Cayley graphs

Theorem 2.5. [3] The unitary addition Cayley graph G_n is connected for all n.

Theorem 2.6. [3] The unitary addition Cayley graph G_n is isomorphic to the unitary Cayley graph X_n if and only if n is even.

3. Domination in the Unitary Addition Cayley Graphs

We now give some results of the domination in the unitary addition Cayley graphs for some n.

Theorem 3.1. If n is prime, then $\gamma(G_n) = 1$.

Proof. Suppose n is prime. Let v be a vertex of the unitary addition Cayley graph G_n. Since $|U_n| = \phi(n)$, let $U_n = \{1, v_2, \ldots, v_{\phi(n)-1}, v_{\phi(n)}\}$. By the definition of unitary addition Cayley graph, $0 \in V(G_n)$ is adjacent to a vertex $v \in V(G_n)$ if and only if $(0 + v, n) = 1$. This implies that $(v, n) = 1$. That means, $v \in U_n$. Hence, $d(0) = \phi(n)$. Since n is prime, then $d(0) = n - 1$. Thus, 0 dominates G_n and $\gamma(G_n) = 1$. ■

Theorem 3.2. Let n be an even integer such that $\phi(n) = 2$. Then $\gamma(G_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof. Suppose $\phi(n) = 2$. Then $|U_n| = 2$. Since n is even, then G_n is isomorphic to the unitary Cayley graph X_n from Theorem 2.6. Hence, G_n is $2 - regular$ by Theorem 2.3. Since G_n is connected for all n by Theorem 2.5, we must have $G_n \cong C_n$, a cycle with n vertices. Therefore, by Theorem 2.2 (iii), $\gamma(G_n) = \left\lceil \frac{n}{3} \right\rceil$. ■

Theorem 3.3. If n is an integer such that $n = 2^r$, $r \geq 2$, then $\gamma(G_n) = 2$.

Proof. Let $n = 2^r$, $r \geq 2$. Then U_n consists all the odd vertices of $\mathbb{Z}_n$. Since n is even, then no two even labeled vertices are adjacent. This implies that even labeled vertices and odd labeled vertices form a bipartition of the vertex set. Since n is even, then G_n is $\phi(n) - regular$ by Theorem 2.6 and Theorem 2.3. Now let D be a $\gamma-$set of G_n

and $i \in D$. Without loss of generality, let $i \in U_n$, that is, i is an odd labeled vertex. Then i can not be adjacent to any $k \in U_n \setminus \{i\}$ since $i + k \notin U_n$. Also, $d(i) = \phi(n) = 2^r - 2^{r-1} = 2^{r-1}(2-1) = 2^{r-1} = 2^r \cdot 2^{-1} = \frac{2^r}{2} = \frac{n}{2}$. Hence, i is adjacent to any $j \in \mathbb{Z}_n \setminus U_n$, that is, i dominates all the even vertices of G_n. In a similar manner, j is adjacent to any $l \in U_n$, that is, j dominates all the odd vertices of G_n. Thus, $D = \{i, j\}$ and D is a minimum dominating set of G_n. Therefore, $\gamma(G_n) = 2$. ■

Theorem 3.4. If $n = 2p$ where p is prime, then $\gamma(G_n) = 2$.

Proof. Suppose $n = 2p$. If $p = 2$, then by Theorem **??**, the statement follows. Suppose, p is an odd prime. Then $\phi(n) = p - 1$. Since n is even, then due to Theorem 2.6 and Theorem 2.4, G_n, $n \geq 2$, is bipartite, i.e., even labeled vertices and the odd labeled vertices form a bipartition of the vertex set. Let D be a $\gamma - set$ of G_n. Now, let $i \in D$ such that i is an odd labeled vertex in G_n. Since n is even, then $deg(i) = p - 1$ by Theorem 2.6 and Theorem 2.3. Since the number of even labeled vertices is p, then there exists an even labeled vertex say j, such that i and j are not adjacent. Thus, i dominates itself and all even labeled vertices except j. Similarly, we can show that j dominates itself and all odd labeled vertices except i. Thus, $D = \{i, j\}$ and D is a minimum dominating set of G_n. Therefore, $\gamma(G_n) = 2$. ■

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