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Some identities involving modified degenerate tangent numbers and polynomials

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Abstract

In this paper we give some properties, explicit formulas, several identities, a connection with modified degenerate tangent numbers and polynomials, and some integral formulas.

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1. Introduction

Recently, many mathematicians have studied in the area of the degenerate Bernoulli numbers, degenerate Euler numbers, degenerate Genocchi numbers, and degenerate tangent numbers(see [1-8]). Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The modified degenerate tangent polynomials $\mathbf{T}_n(x)$ are defined by the generating function:

$$\frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{xt/\lambda} = \sum_{n=0}^{\infty} \mathbf{T}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.1}$$

When $x = 0$, $\mathbf{T}_{n,\lambda}(0) = \mathbf{T}_{n,\lambda}$ are called the modified degenerate tangent numbers (see [7]). Note that $(1+\lambda)^{t/\lambda}$ tends to e^t as $\lambda \to 0$. Numerous properties of tangent numbers and polynomials are known. More studies and results in this subject we may see references [4], [5], [6], [7]. About extensions for the tangent numbers can be found in [5, 7, 8].

By (1), we get

$$\begin{aligned} 2 &= ((1+\lambda)^{2t/\lambda}+1)\frac{2}{(1+\lambda)^{2t/\lambda}+1} \\ &= \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{2t/\lambda} + \frac{2}{(1+\lambda)^{2t/\lambda}+1} \\ &= \sum_{n=0}^{\infty} \mathbf{T}_{n,\lambda}(2) + \mathbf{T}_{n,\lambda}\frac{t^n}{n!}. \end{aligned} \tag{1.2}$$

By comparing the coefficients on both sides of (1.2), we have the following theorem.

Theorem 1.1. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,\lambda}(2) + \mathbf{T}_{n,\lambda} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

The following elementary properties of the modified degenerate tangent polynomials $\mathbf{T}_n(x)$ are readily derived from (2.1). We, therefore, choose to omit the details involved. More studies and results in this subject we may see reference [7].

Theorem 1.2. Let $n \in \mathbb{Z}_+$. Then we have

$$\mathbf{T}_{n,\lambda}(x) = \sum_{l=0}^{n} \binom{n}{l} \mathbf{T}_{n-l,\lambda} \left(\frac{\log(1+\lambda)}{\lambda}\right)^l x^l.$$

Theorem 1.3. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,\lambda}(x) = (-1)^n \mathbf{T}_{n,\lambda}(2-x).$$

Theorem 1.4. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,\lambda}(x+y) = \sum_{l=0}^{n} \binom{n}{l} \left(\frac{\log(1+\lambda)}{\lambda}\right)^l x^l \mathbf{T}_{n-l,\lambda}(x).$$

By (1.1), we see that

$$\frac{d}{dx}\mathbf{T}_{n,\lambda}(x) = n\mathbf{T}_{n-1,\lambda}(x)\frac{\log(1+\lambda)}{\lambda}, \quad (n \geq 1). \tag{1.3}$$

By (1.3) we get

$$\begin{aligned} \int_0^x \frac{d}{dt}\left(\frac{\mathbf{T}_{n+1,\lambda}(t)}{n+1}\right) dt &= \int_0^x \frac{\log(1+\lambda)}{\lambda}\mathbf{T}_{n,\lambda}(t)dt \\ &= \frac{\mathbf{T}_{n+1,\lambda}(x) - \mathbf{T}_{n+1,\lambda}}{n+1}, \ (n \geq 1). \end{aligned} \tag{1.4}$$

By (1.4), we have the following theorem.

Theorem 1.5. For $n \in \mathbb{Z}_+$, we have

$$\frac{\mathbf{T}_{n+1,\lambda}(x) - \mathbf{T}_{n+1,\lambda}}{n+1} = \int_0^x \frac{\log(1+\lambda)}{\lambda} \mathbf{T}_{n,\lambda}(t)dt,$$

In particular,

$$\frac{\mathbf{T}_{n+1,\lambda}(1) - \mathbf{T}_{n+1,\lambda}}{n+1} = \int_0^1 \frac{\log(1+\lambda)}{\lambda} \mathbf{T}_{n,\lambda}(x)dx.$$

For $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$, we have

$$\begin{aligned} 2\sum_{l=0}^{d-1}(-1)^l(1+\lambda)^{2lt/\lambda} &= \frac{2}{(1+\lambda)^{2t/\lambda}+1} + \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{2dt/\lambda} \\ &= \sum_{n=0}^{\infty}\left(\mathbf{T}_{n,\lambda}(2d) + \mathbf{T}_{n,\lambda}\right)\frac{t^n}{n!}. \end{aligned} \tag{1.5}$$

Also, we see that

$$\begin{aligned} 2\sum_{l=0}^{d-1}(-1)^l(1+\lambda)^{2lt/\lambda} &= 2\sum_{l=0}^{d-1}\left(\sum_{n=0}^{\infty}(-1)^l\left(\frac{\log(1+\lambda)}{\lambda}\right)^n (2l)^n\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(2\sum_{l=0}^{d-1}(-1)^l 2^n l^n \left(\frac{\log(1+\lambda)}{\lambda}\right)^n\right)\frac{t^n}{n!}. \end{aligned} \tag{1.6}$$

By (1.5) and (1.6), we have the following theorem.

Theorem 1.6. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,\lambda}(2d) + \mathbf{T}_{n,\lambda} = \sum_{l=0}^{d-1}(-1)^l 2^{n+1} l^n \left(\frac{\log(1+\lambda)}{\lambda}\right)^n.$$

The beta integral is defined for $\text{Re}(x) > 0$, $\text{Re}(y) > 0$ by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{1.7}$$

For $\text{Re}(x) > 0$, the gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt. \tag{1.8}$$

The above integral for $\Gamma(x)$ is sometimes called the Eulerian integral of the second kind. Thus, by (1.7) and (1.8), we have

$$\Gamma(x+1) = x\Gamma(x), \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.9}$$

Our aim in this paper is to give some properties, explicit formulas, several identities, a connection with the modified degenerate tangent numbers and polynomials, and some integral formulas.

2. Identities involving modified degenerate tangent numbers and polynomials

In this section, we obtain several new and interesting identities involving modified degenerate tangent numbers and polynomials.

By Theorem 1.2, we get

$$\begin{aligned}\int_0^1 \mathbf{T}_{n,\lambda}(x)x^n dx &= \sum_{k=0}^{n}\binom{n}{k}\mathbf{T}_{n-k,\lambda}\left(\frac{\log(1+\lambda)}{\lambda}\right)^k \int_0^1 x^{k+n}dx. \\ &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\log(1+\lambda)}{\lambda}\right)^k \frac{\mathbf{T}_{n-k,\lambda}}{n+k+1}.\end{aligned} \tag{2.1}$$

By Theorem 1.4, we note that

$$\begin{aligned}\int_0^1 y^n \mathbf{T}_{n,\lambda}(x+y)dy &= \int_0^1 y^n \sum_{l=0}^{n}\binom{n}{l}\mathbf{T}_{n-l,\lambda}(x)y^l dy \\ &= \sum_{l=0}^{n}\binom{n}{l}\mathbf{T}_{n-l,\lambda}(x)\left(\frac{\log(1+\lambda)}{\lambda}\right)^l \int_0^1 y^{n+l}dy \\ &= \sum_{l=0}^{n}\binom{n}{l}\mathbf{T}_{n-l,\lambda}(x)\left(\frac{\log(1+\lambda)}{\lambda}\right)^l \frac{1}{n+l+1}.\end{aligned} \tag{2.2}$$

By Theorem 1.3, (1.7), and (2.2), we note that

$$\begin{aligned}&\int_0^1 y^n \mathbf{T}_{n,\lambda}(x+y)dy \\ &= (-1)^n \int_0^1 y^n \mathbf{T}_{n,\lambda}(2-(x+y))dy \\ &= \sum_{l=0}^{n}\binom{n}{l}(-1)^n\mathbf{T}_{n-l,\lambda}(1-x)\left(\frac{\log(1+\lambda)}{\lambda}\right)^l \int_0^1 y^n(1-y)^l dy \\ &= \sum_{l=0}^{n}\binom{n}{l}(-1)^n\mathbf{T}_{n-l,\lambda}(1-x)\left(\frac{\log(1+\lambda)}{\lambda}\right)^l B(n+1,l+1)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \mathbf{T}_{n-l,\lambda}(1+x) \left(\frac{\log(1+\lambda)}{\lambda} \right)^l \frac{\Gamma(n+1)\Gamma(l+1)}{\Gamma(n+1+l+1)} \\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \mathbf{T}_{n-l,\lambda}(1+x) \left(\frac{\log(1+\lambda)}{\lambda} \right)^l \frac{l!n!}{(n+l+1)!} \qquad (2.3)\\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{\mathbf{T}_{n-l,\lambda}(1+x)}{n+l+1} \left(\frac{\log(1+\lambda)}{\lambda} \right)^l \binom{n+l}{l}^{-1}.
\end{aligned}
$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\sum_{l=0}^{n} \binom{n}{l} \frac{\mathbf{T}_{n-l,\lambda}(x)}{n+l+1} \left(\frac{\log(1+\lambda)}{\lambda} \right)^l \\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{\mathbf{T}_{n-l,\lambda}(1+x)}{n+l+1} \left(\frac{\log(1+\lambda)}{\lambda} \right)^l \binom{n+l}{l}^{-1}.
\end{aligned}
$$

By (1.3), we note that

$$
\begin{aligned}
&\frac{\log(1+\lambda)}{\lambda} \int_0^1 y^n \mathbf{T}_{n,\lambda}(x+y)dy \\
&= y^n \frac{\mathbf{T}_{n+1,\lambda}(x+y)}{n+1} \Big|_0^1 - \int_0^1 ny^{n-1} \frac{\mathbf{T}_{n+1,\lambda}(x+y)}{n+1} dy \\
&= \frac{\mathbf{T}_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1} \int_0^1 y^{n-1} \mathbf{T}_{n+1,\lambda}(x+y)dy \\
&= \frac{\mathbf{T}_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1} \int_0^1 (-1)^{n+1} y^{n-1} \mathbf{T}_{n+1,\lambda}(2-x-y)dy \\
&= \frac{\mathbf{T}_{n+1,\lambda}(x+1)}{n+1} \\
&\quad - \frac{n}{n+1} \int_0^1 (-1)^{n+1} y^{n-1} \sum_{l=0}^{n+1} \binom{n+1}{l} \mathbf{T}_{n+1-l,\lambda}(1-x)(1-y)^l dy \qquad (2.4) \\
&= \frac{\mathbf{T}_{n+1,\lambda}(x+1)}{n+1} \\
&\quad - \frac{n}{n+1} (-1)^{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} \mathbf{T}_{n+1-l,\lambda}(1-x) \int_0^1 y^{n-1}(1-y)^l dy \\
&= \frac{\mathbf{T}_{n+1,\lambda}(x+1)}{n+1} \\
&\quad + \frac{n}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^{1-l} \mathbf{T}_{n+1-l,\lambda}(x+1) B(n,l+1).
\end{aligned}
$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{N}$, we have

$$\begin{aligned}\frac{\mathbf{T}_{n+1,\lambda}(x+1)}{n+1} &= \sum_{l=0}^{n}\binom{n}{l}(-1)^{l}\frac{\mathbf{T}_{n-l,\lambda}(1+x)}{n+l+1}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{l+1}\binom{n+l}{l}^{-1} \\ &\quad + \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}(-1)^{l}\mathbf{T}_{n+1-l,\lambda}(x+1)B(n,l+1).\end{aligned}$$

By (1.3) and (1.4), we see that

$$\begin{aligned}&\int_0^1 y^n \mathbf{T}_{n,\lambda}(x+y)dy \\ &= y^{n+1}\frac{\mathbf{T}_{n,\lambda}(x+y)}{n+1}\Big|_0^1 - \int_0^1 ny^{n+1}\frac{\mathbf{T}_{n-1,\lambda}(x+y)}{n+1}\left(\frac{\log(1+\lambda)}{\lambda}\right)dy \\ &= \frac{\mathbf{T}_{n,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\left(\frac{\log(1+\lambda)}{\lambda}\right)\int_0^1 y^{n+1}\mathbf{T}_{n-1,\lambda}(x+y)dy \\ &= \frac{\mathbf{T}_{n,\lambda}(x+1)}{n+1} - \frac{n\mathbf{T}_{n-1,\lambda}(x+1)}{(n+1)(n+2)}\left(\frac{\log(1+\lambda)}{\lambda}\right) \\ &\quad + (-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\left(\frac{\log(1+\lambda)}{\lambda}\right)^2\int_0^1 y^{n+2}\mathbf{T}_{n-2,\lambda}(x+y)dy \\ &= \frac{\mathbf{T}_{n,\lambda}(x+1)}{n+1} + (-1)\frac{n\mathbf{T}_{n-1,\lambda}(x+1)}{(n+1)(n+2)}\left(\frac{\log(1+\lambda)}{\lambda}\right) \\ &\quad + (-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\frac{\mathbf{T}_{n-2,\lambda}(x+1)}{n+3}\left(\frac{\log(1+\lambda)}{\lambda}\right)^2 \\ &\quad + (-1)^3\frac{n}{n+1}\frac{n-1}{n+2}\frac{n-2}{n+3}\left(\frac{\log(1+\lambda)}{\lambda}\right)^3\int_0^1 y^{n+3}\mathbf{T}_{n-3,\lambda}(x+y)dy \\ &= \frac{\mathbf{T}_{n,\lambda}(x+1)}{n+1} + (-1)\frac{n\mathbf{T}_{n-1,\lambda}(x+1)}{(n+1)(n+2)}\left(\frac{\log(1+\lambda)}{\lambda}\right) \\ &\quad + (-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\frac{\mathbf{T}_{n-2,\lambda}(x+1)}{n+3}\left(\frac{\log(1+\lambda)}{\lambda}\right)^2 \\ &\quad + (-1)^3\frac{n}{n+1}\frac{n-1}{n+2}\frac{n-2}{n+3}\frac{\mathbf{T}_{n-3,\lambda}(x+1)}{n+4}\left(\frac{\log(1+\lambda)}{\lambda}\right)^3 \\ &\quad + (-1)^4\frac{n}{n+1}\frac{n-1}{n+2}\frac{n-2}{n+3}\frac{n-3}{n+4}\left(\frac{\log(1+\lambda)}{\lambda}\right)^4\int_0^1 y^{n+4}\mathbf{T}_{n-4,\lambda}(x+y)dy\end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned}\int_0^1 y^n \mathbf{T}_{n,\lambda}(x+y)dy &= \frac{\mathbf{T}_{n,\lambda}(x+1)}{n+1} \\ &+ \sum_{m=1}^{n-1} \frac{n(n-1)\cdots(n-m+1)(-1)^m}{(n+1)(n+2)\cdots(n+m+1)} \left(\frac{\log(1+\lambda)}{\lambda}\right)^m \mathbf{T}_{n-m,\lambda}(x+1) \\ &+ (-1)^n \frac{n!}{(n+1)(n+2)\cdots(2n)} \left(\frac{\log(1+\lambda)}{\lambda}\right)^n \int_0^1 y^{2n}\mathbf{T}_{0,\lambda}(x+y)dy\end{aligned} \tag{2.5}$$

Hence, by (2.2) and (2.5), we have the following theorem.

Theorem 2.3. For $n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{l=0}^{n} \binom{n}{l} \left(\frac{\log(1+\lambda)}{\lambda}\right)^l \frac{\mathbf{T}_{n-l,\lambda}(x)}{n+l+1} &= \frac{\mathbf{T}_{n,\lambda}(x+1)}{n+1} \\ &+ \sum_{m=1}^{n-1} \frac{n(n-1)\cdots(n-m+1)(-1)^m}{(n+1)(n+2)\cdots(n+m+1)} \left(\frac{\log(1+\lambda)}{\lambda}\right)^m \mathbf{T}_{n-m,\lambda}(x+1) \\ &+ (-1)^n \frac{n!}{(n+1)(n+2)\cdots(2n)(2n+1)} \left(\frac{\log(1+\lambda)}{\lambda}\right)^n.\end{aligned}$$

By Theorem 2.2 and Theorem 2.3, we have the following corollary.

Corollary 2.4. For $n \in \mathbb{N}$, we have

$$\begin{aligned}&\frac{\mathbf{T}_{n,\lambda}(x+1)}{n+1} \left(\frac{\log(1+\lambda)}{\lambda}\right) \\ &\quad+ \sum_{m=1}^{n-1} \frac{n(n-1)\cdots(n-m+1)(-1)^m}{(n+1)(n+2)\cdots(n+m+1)} \left(\frac{\log(1+\lambda)}{\lambda}\right)^{m+1} \mathbf{T}_{n-m,\lambda}(x+1) \\ &\quad+ (-1)^n \frac{n!}{(n+1)(n+2)\cdots(2n)(2n+1)} \left(\frac{\log(1+\lambda)}{\lambda}\right)^{n+1} \\ &= \frac{\mathbf{T}_{n+1,\lambda}(x+1)}{n+1} + \frac{n}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^{1-l} \mathbf{T}_{n+1-l,\lambda}(x+1) B(n, l+1).\end{aligned}$$

From Theorem 1.3, we have $(-1)^n \mathbf{T}_{n,\lambda} = \mathbf{T}_{n,\lambda}(2)$. Putting $x = 1$ in Theorem 2.3 gives the identity

$$\begin{aligned}
&\sum_{l=0}^{n}\binom{n}{l}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{l}\frac{\mathbf{T}_{n-l,\lambda}(1)}{n+l+1}\\
&=\frac{\mathbf{T}_{n,\lambda}(2)}{n+1}\\
&+\sum_{m=1}^{n-1}\frac{n(n-1)\cdots(n-m+1)(-1)^{m}}{(n+1)(n+2)\cdots(n+m+1)}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{m}\mathbf{T}_{n-m,\lambda}(2)\\
&+(-1)^{n}\frac{n!}{(n+1)(n+2)\cdots(2n)(2n+1)}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n}.
\end{aligned}$$

Hence we have the following corollary.

Corollary 2.5. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbf{T}_{n,\lambda} &= \sum_{l=0}^{n}(-1)^{n}\binom{n}{l}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{l}\frac{\mathbf{T}_{n-l,\lambda}(1)}{n+l+1}\\
&-\sum_{m=1}^{n-1}\frac{n(n-1)\cdots(n-m+1)}{(n+2)\cdots(n+m+1)}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{m}\mathbf{T}_{n-m,\lambda}\\
&-\frac{n!}{(n+2)\cdots(2n)(2n+1)}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n}.
\end{aligned}$$

Now we observe that

$$\begin{aligned}
&\int_0^2 \mathbf{T}_{n,\lambda}(x)\mathbf{T}_{m,\lambda}(x)dx\\
&=\int_0^2\sum_{l=0}^{n}\binom{n}{l}\mathbf{T}_{l,\lambda}x^{n-l}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-l}(-1)^{m}\mathbf{T}_{m,\lambda}(2-x)dx\\
&=\int_0^2\sum_{l=0}^{n}\binom{n}{l}\mathbf{T}_{l,\lambda}x^{n-l}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-l+m-k}(-1)^{m}\sum_{k=0}^{m}\binom{m}{k}\mathbf{T}_{k,\lambda}(2-x)^{m-k}dx\\
&=\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}\mathbf{T}_{l,\lambda}\mathbf{T}_{k,\lambda}(-1)^{m}2^{n+m-l-k+1}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-l+m-k}\\
&\qquad\times B(n-l+1,m-k+1)
\end{aligned}$$

$$= \sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}\mathbf{T}_{l,\lambda}\mathbf{T}_{k,\lambda}(-1)^m 2^{n+m-l-k+1}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-l+m-k}$$
$$\times\frac{\Gamma(n-l+1)\Gamma(m-k+1)}{\Gamma(n+m-l-k+2)}$$
$$= \sum_{l=0}^{n}\sum_{k=0}^{m}\frac{\binom{n}{l}\binom{m}{k}(-1)^m 2^{n+m-l-k+1}}{\binom{m+n-l-k}{n-l}}\frac{\mathbf{T}_{l,\lambda}\mathbf{T}_{k,\lambda}}{(n+m-l-k+1)}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-l+m-k}. \tag{2.6}$$

For $m, n \in \mathbb{N}$, we have

$$\int_0^2 \mathbf{T}_{n,\lambda}(x)\mathbf{T}_{m,\lambda}(x)dx$$
$$= \mathbf{T}_{m,\lambda}(x)\frac{\mathbf{T}_{n+1,\lambda}(x)}{n+1}\frac{\lambda}{\log(1+\lambda)}\bigg|_0^2 - \int_0^2 m\mathbf{T}_{m-1,\lambda}(x)\frac{\mathbf{T}_{n+1,\lambda}(x)}{n+1}dx$$
$$= -\frac{m}{n+1}\int_0^2 \mathbf{T}_{m-1,\lambda}(x)\mathbf{T}_{n+1,\lambda}(x)dx$$
$$= (-1)^2\frac{m(m-1)}{(n+1)(n+2)}\int_0^2 \mathbf{T}_{m-2,\lambda}(x)\mathbf{T}_{n+2,\lambda}(x)dx$$

Continuing this process, we get

$$\int_0^2 \mathbf{T}_{n,\lambda}(x)\mathbf{T}_{m,,\lambda}(x)dx$$
$$= (-1)^m\frac{m(m-1)\cdots 3\cdot 2\cdot 1}{(n+1)(n+2)\cdots(n+m)}\int_0^2 \mathbf{T}_{n+m,\lambda}(x)\mathbf{T}_{0,\lambda}(x)dx \tag{2.7}$$
$$= (-1)^{m+1}\frac{m(m-1)\cdots 3\cdot 2\cdot 1}{(n+1)(n+2)\cdots(n+m)}\frac{2\mathbf{T}_{n+m+1,\lambda}}{n+m+1}\frac{\lambda}{\log(1+\lambda)}$$

By (2.6) and (2.7), we have the following theorem.

Theorem 2.6. For $m, n \in \mathbb{N}$, we have

$$\frac{m!\mathbf{T}_{n+m+1,\lambda}}{(n+1)(n+2)\cdots(n+m)(n+m+1)}$$
$$= -\sum_{l=0}^{n}\sum_{k=0}^{m}\frac{\binom{n}{l}\binom{m}{k}2^{n+m-l-k}}{\binom{m+n-l-k}{n-l}}\frac{\mathbf{T}_{l,\lambda}\mathbf{T}_{k,\lambda}}{(n+m-l-k+1)}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n+m+1-l-k}.$$

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