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Finitness of the point spectrum of transport operator with matricial 2×2 potential

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Abstract

The transport operator where potential is a matrix 2×2 with determinant equal to zero is considered. The elements of the matrix are complex-valued exponentialy decreasing functions in $(-\infty, \infty)$.

Some sufficient conditions of finitness of the point spectrum of transport operator are given.

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1. Introduction

There are many works concerning the different types of transport, as exemple only we indicate the works $[1-2]$. In the work [4] the operator

$$L_u^1 = -i\mu\frac{\partial u}{\partial x} + c(x)\int_{-1}^{1} u(x,\mu')d\mu',\ x \in \mathbb{R} \tag{1.0}$$

in the space $L^2(D)$ where $D = \mathbb{R}\times[-1,1]$, was considred using the Friedrichs' model under the condition on the potential

$$|c(x)| \leq M\exp(-\in|x|),\ x \in \mathbb{R}$$

Some conditions of finitness of the point spectrum was proposed in the work [4] the same quetion was considred for the operator

$$L'u(x,u) = -i\mu\frac{\partial u}{\partial x} + c(x)\int_{-1}^{1} k(x,u,\mu')d\mu',\ \left(x,\mu'\right) \in D$$

where $k(x,u,\mu') = \sum_{j=1}^{n} a_j(x)c_j(u)c_j\left(\mu'\right)$, if $n = 1$ the point spectrum ifinite then asymptotic behaviour the solution of the corresponding evolution equation was obtained.

In our article we consider the case where the potential $c(x)$ is a matrix function 2×2.

2. Statment of the problem

We use the space $\mathbb{C}^2$ with the norm

$$|Z|^2 = |Z_1|^2 + |Z_2|^2,$$

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in \mathbb{C}^2.$$

Let $\mathbb{R} =]-\infty,\infty[$ and $D = \mathbb{R}\times[-1,1]$. We denote the element $u \in L^2\left(D,\mathbb{C}^2\right)$ by

$$u(x,\mu) = \begin{pmatrix} u_1(x,\mu) \\ u_2(x,\mu) \end{pmatrix}, \qquad x \in \mathbb{R},\ \mu \in [-1,1] \tag{1.1}$$

We consider the operator $L : L^2\left(D,\mathbb{C}^2\right) \to L^2\left(D,\mathbb{C}^2\right)$ defined by the expression

$$Lu = -i\mu\frac{\partial u}{\partial x} + c(x)\int_{-1}^{1} u(x,\mu')d\mu', \tag{1.2}$$

with maximal domain of definition, where

$$c(x) = \begin{pmatrix} c_{11}(x) & c_{12}(x) \\ c_{21}(x) & c_{22}(x) \end{pmatrix}. \tag{1.3}$$

We suppose that

$$\det c(x) = 0 \tag{1.4}$$

and

$$|c_{ik}(x)| \le M \exp\left(- \in |x|\right), \; x \in \mathbb{R}, \; i, k = 1, 2 \tag{1.5}$$

for some $M > 0$ and $\epsilon > 0$.Note that from (1.1) it results that the matrix $c(x)$ has the form

$$c(x) = \begin{pmatrix} p_1(x)\, q_1(x) & p_1(x)\, q_2(x) \\ p_2(x)\, q_1(x) & p_2(x)\, q_2(x) \end{pmatrix}. \tag{1.6}$$

Really let

$$|c_{11}(x)| = \max_{i,k} |c_{ik}(x)|$$

then we can pose

$$p_1(x) = \sqrt{|c_{11}(x)|}, q_1(x) = \frac{c_{11}(x)}{p_1(x)}.$$

So

$$c_{11}(x) = p_1(x)\, q_1(x),$$

later

$$q_2(x) = \frac{c_{12}(x)}{p_1(x)},$$

$$p_1(x) = \frac{c_{21}(x)}{q_1(x)}$$

and from (1.4) we obtain

$$c_{22}(x) = p_2(x)\, q_2(x)$$

according to (1.5) . We suppose that

$$|p_i(x)|, |q_i(x)| < M \exp\left(-\frac{\varepsilon}{2}|x|\right), x \in \mathbb{R}. \tag{1.7}$$

We denote by A^* the adjoint of some matrix $A : \mathbb{C}^2 \to \mathbb{C}^2$. We will use the matrix

$$C_1(x) = \begin{pmatrix} \overline{p_1(x)} & \overline{P_2(x)} \\ 0 & 0 \end{pmatrix}, \; C_2(x) = \begin{pmatrix} q_1(x) & q_2(x) \\ O & 0 \end{pmatrix}, \tag{1.8}$$

the equality (1.6) signifies the factorization

$$C(x) = C_1(x)^* C_2(x) \tag{1.9}$$

using (1.1) , (1.3) we can rewrite the operator $L : L^2\left(D, \mathbb{C}^2\right) \to L^2\left(D, \mathbb{C}^2\right)$ $(see\ (1.2))$ as:

$$\begin{cases} (Lu_1) = -i\mu\dfrac{\partial u_1}{\partial x} + \displaystyle\int_{-1}^{1} c_{11}(x)u_1(x,\mu') + c_{12}(x)u_2(x,\mu')d\mu' \\ (Lu_2) = -i\mu\dfrac{\partial u_1}{\partial x} + \displaystyle\int_{-1}^{1} c_{21}(x)u_1(x,\mu') + c_{22}(x)u_2(x,\mu')d\mu' \end{cases} \tag{1.10}$$

3. Friedrichs' model

We will use the results from the work [3] and at the begining we recall the correspênding notions. Denote F the Fourier transformation inthe space $L^2(\mathbb{R})$ defined by :

$$Fu(s) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-ist} u(\tau)\, d\tau, \qquad s \in \mathbb{R},$$

and in the space $L^2\left(D, \mathbb{C}^2\right)$ by

$$F\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(s) = \begin{pmatrix} Fu_1(s) \\ Fu_2(s) \end{pmatrix}.$$

We denote by H_1 the Hilbert space of scalar functions $\varphi(s,\mu)$, $(s,\mu) \in \mathbb{R} \times [-1, 1]$ with the norm

$$\|\varphi\|^2_{H_1} = \int\limits_{\mathbb{R}} \int\limits_{-1}^{1} |\varphi(s,\mu)|^2 \frac{1}{|\mu|} ds\, d\mu.$$

(We keep the symbol H for our case of vector functions $\varphi(s,\mu)$).

Let $F_0 : L^2(D) \to H_1$, be the operator defined by the relation: $F_0 u(s,\mu) = u\left(\dfrac{s}{\mu}, \mu\right)$, and introduce an unitary operator $U = F_0 F : L^2(D) \to H_1$.

In the work [3] it was proved that integral operator $v : L^2(D) \to L^2(D)$ given an (0.1) namely

$$vf(x,\mu) = c(x) \int\limits_{-1}^{1} f\left(x, \mu'\right) d\mu',$$

has the following transformation $V : H_1 \to H_1$ if $c(x) = \overline{c_1(x)} c_2(x)$ then: $V\varphi = UvU^{-1}\varphi = A^* B\varphi$, $\varphi = Uf$. Where $A, B \colon H_1 \to L^2(\mathbb{R})$

$$A\varphi(x) = c_1(x)\, W\varphi(x), \qquad B\varphi(x) = c_2(x)\, W\varphi(x)$$

and

$$W\varphi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \int\limits_{-1}^{1} \varphi(s,\mu)\, e^{-ix\frac{s}{\mu}} \frac{1}{|\mu|} ds\, d\mu \tag{2.1}$$

Friedrichs' model was $UL^1U^{-1} = S + A^*B$, where $S\varphi(\tau, \mu)$.

Now we come back to the operator (1.1) .We introduce the Hilbert space H of vector function $\varphi(x, \mu)$ with values in $\mathbb{C}^2$ and the norm

$$\|\varphi\|_H^2 = \int_{\mathbb{R}} \int_{-1}^{1} |\varphi(s, \mu)|_{\mathbb{C}^2}^2 \frac{1}{|\mu|} ds d\mu.$$

We consider the space $L^2(D, \mathbb{C}^2)$ as direct sum $L^2(D, \mathbb{C}) \oplus L^2(D, \mathbb{C})$ and by the same way $H = H_1 \oplus H_1$, as $L^2(D) = L^2(D, \mathbb{C})$ we can define unitary operator $U : L^2(D, \mathbb{C}^2) \to H$ using unitary operator $U : L^2(D) \to H_1$.

Denote the operators $A_i, B_k : H_1 \to L^2(\mathbb{R})$ by relations

$$A_i\varphi(x) = \overline{p_i(x)} W\varphi(x), \; B_k\varphi(x) = q_k(x) W(x), \; i, k = \overline{1, 2} \; (see\, (2.1))$$

and we introduce the matrix operators $A, B : H \to L^2(\mathbb{R}, \mathbb{C}^2)$ as

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

Lemma 3.1. Let $T = ULU^{-1} : H \to H$ then $T = S + V$, $V = A^*B$ where

$$S\varphi(\tau) = \begin{pmatrix} \tau\varphi_1(\tau) \\ \tau\varphi_2(\tau) \end{pmatrix},$$

$$\varphi(\tau) = \begin{pmatrix} \varphi_1(\tau) \\ \varphi_2(\tau) \end{pmatrix}.$$

Proof. Using unitary operator $U : L^2(D, \mathbb{C}^2) \to H$, we transform the system (2.10) . Where $c_{i,k}(x) = p_i(x) q_k(x)$ in the system

$$\begin{cases} (T\varphi_1) = S\varphi_1 + A_1^*B_1\varphi_1 + A_1^*B_2\varphi_2 \\ (T\varphi_1) = S\varphi_2 + A_2^*B_1\varphi_1 + A_2^*B_2\varphi_2 \end{cases}$$

where $\varphi_i = Uu_i$, $i = \overline{1, 2}$. According to (2.2) we have

$$\begin{pmatrix} A_1^*B_1 & A_1^*B_2 \\ A_2^*B_1 & A_2^*B_2 \end{pmatrix} = \begin{pmatrix} A_1^* & 0 \\ A_2^* & 0 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} = A^*B$$

what proves lemma (3.1). ■

4. Operator $K(\xi)$ and the spectrum in the domain $\xi \neq 0$

Denote $S_\xi = (S-\xi)^{-1}$, $\mathrm{Im}\,\xi \neq 0$ and $T_\xi = (T-\xi)^{-1}$, $\xi \in \rho(T)$. We are looking for the presentation T_ξ by S_ξ. Let $(T-\xi)\psi = \varphi$ then (*see* lemma (3.1)) $(S-\xi)\psi + A^*B\psi = \varphi$ or $\psi + S_\xi A^*B\psi = S_\xi\varphi$. Multiplying by B we fined $B\psi + BS_\xi A^*B\psi = BS_\xi\varphi$ or: $K(\xi)B\psi = BS_\xi\varphi$, where

$$K(\xi) = 1 + BS_\xi A^* \tag{3.1}$$

Substituting $B\psi = K(\xi)^{-1}BS_\xi\varphi$, we obtain

$$T_\xi\varphi = S_\xi\varphi - S_\xi A^*K(\xi)^{-1}BS_\xi\varphi \tag{3.2}$$

every eigenvalue λ, $\mathrm{Im}\,\lambda \neq 0$ of the operator T is a pole of the function $K(\xi)^{-1}$ i.e the operator $K(\lambda)$ must be non-invertible. Therefore the equation

$$K(\xi)h = 0 \tag{3.3}$$

must have unique solution $h = 0$ the transformation $(3.1) - (3.2)$ obviously hold not in the space H_1 only, but in the space H too.

Theorem 4.1. Let the operator $K(\xi) : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ be defined by (2.2), (3.1) then 1) the operator $K(\xi)-1$, $\xi \notin \mathbb{R}$ is completely continuous and $\|K(\xi)-1\| \to 0$, $r \to \infty$ uniformly in the domain $|\xi| > r$, $|\mathrm{Im}\,\xi| > \gamma_0$ for every $\gamma_0 > 0$. 2) the operator $K(\xi) - 1$ admits an analytical prolongation over half -axis $(-\infty, 0)$, $(0, \infty)$ and $\|K_\pm(\xi) - 1\| \to 0$, $|\xi| \to \infty$ uniformly in the domain $|\mathrm{Im}\,\xi| < \epsilon_1$ for every $\epsilon_1 < \dfrac{\epsilon}{2}$. Where $K_\pm(\xi)$ denote the prolongation respectively in the domain $\mathrm{Im}\,\xi > 0$ $(\mathrm{Im}\,\xi < 0)$.

Proof. We repeate the proof from [3] for the space H_1 really:

$$\begin{aligned}
K(\xi) - 1 &= BS_\xi A^* \\
&= \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S_\xi & 0 \\ 0 & S_\xi \end{pmatrix}\begin{pmatrix} A_1^* & 0 \\ A_2^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} B_1S_\xi A_1^* + B_2S_\xi A2^* & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} B_1S_\xi A_1^* + B_2S_\xi A_2^* & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

and the operator $B_1S_\xi A_1^*$ considred in [3], theorem is proved. ∎

It results from the theorem (3.1) that the point spectrum can have only $\xi = 0$ as a point of accummulation.

5. Study of the spectrum in a neighborhood of $\xi = 0$

Let us consider the operator $K(\xi)$ more detaied following the work [3] we denote

$$l(\tau,\xi) = \int_{-1}^{1} \frac{d\mu}{\tau\mu - \xi},\ \xi \notin [-|\tau|,|\tau|]$$

and

$$I(u,\xi) = \int_{\mathbb{R}} l(\tau,\xi)\, e^{i\tau u} d\tau = \gamma(\xi) + I_0(u,s) \tag{4.1}$$

where

$$\gamma(\xi) = -\pi i\ signv\ \ln\xi,\ \xi = \alpha + iv \tag{4.2}$$

and $\ln\xi$ denotes the branch in the domain $\xi \notin [0,\infty)$ such that $\ln(-1) = \pi i$. The function $I_0(u,\xi)$ is such integral operator with kermel $\exp\left(\frac{-\epsilon}{4}(|x|+|y|)\right)|I_0(x-y,\xi)|$ is uniformement bounded in the domain $|\xi| < \delta$ where $\delta > 0$ is some constant. Denote:

$$n(x,y) = p_1(x)\, q_1(y) + p_2(x)\, q_2(y) \tag{4.3}$$

Lemma 5.1. Let $K(\xi) : L^2\left(\mathbb{R},\mathbb{C}^2\right) \to L^2\left(\mathbb{R},\mathbb{C}^2\right)$, then the equation

$$K(\xi)\, h = 0,\ |\xi| < \delta,\ h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{4.4}$$

is equivalent to the system

$$\begin{cases} h_1 + \dfrac{\gamma(\xi)}{2\pi}\left[(h_1,\overline{p_1})\, q_1 + (h_1,\overline{p_2})\, q_2\right] + Q(\xi)\, h_1 = 0 \\ h_2 = 0 \end{cases} \tag{4.5}$$

where the operator $Q(\xi)$ is uniformly bounded if $|\xi| < \delta$.

Proof. By analogy to the relation (3.5) from [3] we have

$$\begin{aligned}(K(\xi)-1)\, h(y) &= \left(C_2(y)\, F^{-1} l(.,s)\, F C_1^*(x)\, h\right)(y) \\ &= \int_{\mathbb{R}} k(x,y,\xi)\, h(x)\, dx.\end{aligned}$$

where

$$k(x,y,\xi) = \frac{1}{2\pi} C_2(y)\, C_1^*(x)\, I(y-x,\xi)$$

According to (1.6) and (4.3)

$$C_2(y)\, C_1^*(x) = \begin{pmatrix} q_1(y) & q_2(y) \\ O & 0 \end{pmatrix}\begin{pmatrix} p_1(x) & 0 \\ p_2(x) & 0 \end{pmatrix} = \begin{pmatrix} n(x,y) & 0 \\ 0 & 0 \end{pmatrix}$$

So the equation (4.4) becomes the system

$$\begin{cases} h_1(y) + \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} n(x, y) \left[\gamma(\xi) + I_0(y - x, s) h_1(x) dx \right] = 0 \\ h_2(x) = 0 \end{cases}$$

and we obtain the eqution (4.5) where

$$Q(s) h_1(y) = \frac{1}{\pi} \int_{\mathbb{R}} n(x, y) I_0(y - x, s) h_1(x) dx, \qquad |s| < \delta \tag{4.6}$$

lemma is proved. ∎

Let

$$p = \begin{pmatrix} (q_1, \overline{p_1}) & (q_1, \overline{p_2}) \\ (q_2, \overline{p_1}) & (q_2, \overline{p_2}) \end{pmatrix} \tag{4.7}$$

where the notation $(q_k, \overline{p_i}) = \int_{\mathbb{R}} q_k(x) \overline{p_i}(x) dx$ signifies the scalar product in $L^2(\mathbb{R})$.

Later we introduce the condition

$$\det p \neq 0 \tag{4.8}$$

denote

$$\Gamma(\xi) = 1 + \frac{\gamma(\xi)}{2\pi} p \tag{4.9}$$

Lemma 5.2. Let the condition (4.8) holds then
1) We have for the solution of the equation (4.5)

$$\Gamma(\xi) \begin{pmatrix} (h_1, \overline{p_1}) \\ (h_1, \overline{p_2}) \end{pmatrix} + \begin{pmatrix} (Q(s) h_{1,} \overline{p_1}) \\ (Q(s) h_{1,} \overline{p_2}) \end{pmatrix} = 0 \tag{4.10}$$

where the operator $Q(s)$ is defined in (4.6).
2) There exists $\delta > 0$ such that

$$\left\| \Gamma(\xi)^{-1} \right\| \leq \frac{4\pi}{\gamma(\xi)} \left\| p_{-1} \right\| \tag{4.11}$$

Proof. 1) Multiplying (4.5) by $\overline{p_1}$ and $\overline{p_2}$ in the space $L^2(\mathbb{R})$ we obtain

$$\begin{cases} (h_1, \overline{p_1}) + \dfrac{\gamma(s)}{2\pi} [(h_1, \overline{p_1})(q_1, \overline{p_1}) + (h_1, \overline{p_2})(q_2, \overline{p_1})] + (Q(\xi) h_1, \overline{p_1}) = 0 \\ (h_1, \overline{p_2}) + \dfrac{\gamma(s)}{2\pi} [(h_1, \overline{p_2})(q_1, \overline{p_2}) + (h_1, \overline{p_2})(q_2, \overline{p_2})] + (Q(\xi) h_1, \overline{p_2}) = 0 \end{cases}$$

wath coinside with (4.10) .

2) Recall that if B is some bounded operator and

$$\|B\| < 1 \quad \text{then} \quad \left\|(1-B)^{-1}\right\| \leq (1-\|B\|)^{-1}$$

therefore if (*see* (4.1))

$$\Gamma(\xi) = \frac{\gamma(\xi)}{2\pi} p \left(\frac{2\pi}{\gamma(\xi)} p^{-1} + 1 \right)$$

and $(\gamma(\xi) \to \infty, \ \xi \to 0)$. (*see* (4.2) then choosing $\delta_1 > 0$ such that)

$$\frac{2\pi}{|\gamma(\xi)|} \left\|p^{-1}\right\| < \frac{1}{2} \qquad |\xi| < \delta_1$$

we obtain

$$\begin{aligned} \left\|\Gamma(\xi)^{-1}\right\| &= \frac{2\pi}{|\gamma(\xi)|} \left\|p^{-1}\right\| \left\| \left(\frac{2\pi}{\gamma(\xi)} p^{-1} + 1 \right)^{-1} \right\| \\ &\leq \frac{2\pi}{|\gamma(\xi)|} \left\|p^{-1}\right\| \left(1 - \left\| \frac{2\pi}{\gamma(\xi)} \right\| \left\|p^{-1}\right\| \right)^{-1} \\ &\leq \frac{2\pi}{|\gamma(\xi)|} \left\|p^{-1}\right\| \left(1 - \frac{1}{2} \right)^{-1} = \frac{4\pi}{|\gamma(\xi)|} \left\|p^{-1}\right\| \end{aligned}$$

lemma is proved. ■

Denote

$$\|p\|_\epsilon = \max_k \sup_x \left[|p_k(x)| \, e^{\frac{\epsilon}{4}|x|} \right], \ \|q\|_\epsilon = \max_k \sup_x \left[|q_k(x)| \, e^{\frac{\epsilon}{4}|x|} \right] \tag{4.12}$$

Denote by $M(\xi)$ the norm in $L^2(\mathbb{R})$ of the integral operator with kernel

$$k(x, y, \xi) = \exp\left(-\frac{\epsilon}{4} (|x| + |y|) \right) I_0(y - x, \xi) \tag{4.13}$$

and denote

$$M_0(\xi) = M(\xi) \|p\|_\epsilon \|q\|_\epsilon \left[2 \left\|p^{-1}\right\| \sqrt{\|p_1\|^2 + \|p_2\|^2} \sqrt{\|q_1\|^2 + \|p_2\|^2} + \sqrt{2} \right] \tag{4.14}$$

Theorem 5.3. Suppose that the condition (1.7) and (4.8) hold. Then the operator L (*see* (1.10)) has not a point spectrum. So, it is sufficient to prove that equation (3.3) in the domain $|\xi| < \delta_0$ defined by the condition

$$|M_0(\xi)| < 1, \ \ |\xi| < \delta_0 \tag{4.15}$$

for some $\delta_0 > 0$ $(see\ \ (4.14)\,,\ \ (4.12))$.

Proof. The operator L and T have the same point spectrum. So, it is sufficient to prove that the eqution (3.3) has unique solution $h = 0$ for some condition $|\xi| < \delta_0$. We must estimate all terms of the equation (4.5).

1) We rewrite (4.6) as

$$\begin{aligned} Q(\xi)\, h_1(y) &= \frac{1}{\pi}\int_{\mathbb{R}} n(x,y)\, I_0(y-x,\xi)\, h_1(x)\, dx \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \exp\left(-\frac{\epsilon}{4}(|x|+|y|)\right) I_0(y-x,\xi) \\ &\quad \times \exp\left(\frac{\epsilon}{4}(|x|+|y|)\right) n(x,y)\, h_1(x)\, dx \end{aligned}$$

by definition of $M(\xi)$ using inequality $(a+b)^2 \le 2\left(a^2+b^2\right)$, $a, b > 0$, we obtain

$$\begin{aligned} \|Q(\xi)\, h_1\|^2 &\le M(\xi)^2 \int_{\mathbb{R}} \exp\left(\frac{\epsilon}{2}(|x|+|y|)\right) |n(x,y)|^2\, |h_1(x)|^2\, dx \\ &\le 2M(\xi)^2\, \|p\|_\epsilon^2\, \|q\|_\epsilon^2\, \|h_1\|^2 \end{aligned}$$

So,

$$\|Q(\xi)\, h_1\| \le \sqrt{2}M(\xi)\, \|p\|_\epsilon\, \|q\|_\epsilon\, \|h_1\| \tag{4.16}$$

2) First term in (4.5) was estimate (4.16) and $(4.10) - (4.11)$:

$$\begin{aligned} \|(h_1,\overline{p_1})\, q_1 + (h_1,\overline{p_2})\, q_2\| &\le |(h_1,\overline{p_1})|\, \|q_1\| + |(h_1,\overline{p_2})|\, \|q_2\| \\ &\le \sqrt{\|q_1\|^2+\|q_2\|^2}\sqrt{|(h_1,\overline{p_1})|^2+|(h_1,\overline{p_2})|^2} \\ &\le \sqrt{\|q_1\|^2+\|q_2\|^2}\, \left\|\Gamma^{-1}(\xi)\right\| \\ &\quad \times\sqrt{|(Q(\xi)\, h_1,\overline{p_1})|^2+|(Q(\xi)\, h_1,\overline{p_2})|^2} \\ &\le \sqrt{\|q_1\|^2+\|q_2\|^2}\sqrt{\|p_1\|^2+\|p_2\|^2} \\ &\quad \times \left\|\Gamma^{-1}(\xi)\right\|\, \|Q(\xi)\|\, \|h_1\| \end{aligned}$$

Taking into account the factor $\dfrac{\gamma(\xi)}{2\pi}$ we obtain $(see\ \ (4.14))$

$$\begin{aligned} &\left\|\frac{\gamma(\xi)}{2\pi}\left[(h_1,\overline{p_1})\, q_1 + (h_1,\overline{p_2})\, q_2\right]\right\| \\ &\quad \le 2M(\xi)\, \left\|p^{-1}\right\|\, \|p\|_\epsilon\, \|q\|_\epsilon\, \sqrt{\|q_1\|^2+\|q_2\|^2}\sqrt{\|p_1\|^2+\|p_2\|^2} \end{aligned}$$

and according to (4.5) we have $\|h_1\| \leq M_0(\xi)\|h_1\|$ what is impossible (*see* (4.15)) if $\delta_0 = \min(\delta_1, \delta)$ theorem is proved. ■

In view of theorem (4.1) we have our main result: the operator L under certain condition has finite point spectrum.

Remark 5.4. Even if p_i, q_k are small the equation (4.5) is not to be small, because $\gamma(\xi) \to \infty$, $\xi \to 0$. We reduce (4.5) to the system $(h_1, \overline{q_1})$ $(h_1, \overline{p_1})$ and later we come back to (4.5) and the expression "small" appears.

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