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Identities arising from differential equations for generalized Laguerre polynomials

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Abstract

The purpose of this paper is to give some new and interesting identities related to generalized Laguerre polynomials arising from linear differential equations.

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1. Introduction

For arbitrary real α, the polynomial solutions of the differential equations

$$xy'' + (\alpha + 1 - x)\, y' + ny = 0, \quad \text{(see [1, 12, 16])} \tag{1.1}$$

are called generalized Laguerre polynomials.

The generating function for the generalized Laguerre polynomials, $L_n^{(\alpha)}(x)$, $(n \geq 0)$, are given by

$$\frac{1}{(1-t)^{\alpha+1}} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n, \quad \text{(see [12])}. \tag{1.2}$$

From (1.2), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} t^k \cdot \sum_{m=0}^{\infty} \binom{k+\alpha+m}{m} t^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!} x^k \binom{n+\alpha}{n-k} \right) t^n. \end{aligned} \tag{1.3}$$

Thus, by (1.3), we get

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-1)^k}{k!} x^k, \quad (n \geq 0). \tag{1.4}$$

The first few generalized Laguerre polynomials are

$$\begin{aligned} L_0^{(\alpha)}(x) &= 1 \\ L_1^{(\alpha)}(x) &= -x + \alpha + 1 \\ L_2^{(\alpha)}(x) &= \frac{x^2}{2} - (\alpha+2)\, x + \frac{(\alpha+2)(\alpha+1)}{2} \\ L_3^{(\alpha)}(x) &= -\frac{x^3}{6} + \frac{(\alpha+3)}{2} x^2 - \frac{(\alpha+2)(\alpha+3)}{2} x + \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{6}, \ldots \end{aligned}$$

From (1.4), we note that

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} \approx \frac{n^\alpha}{\Gamma(\alpha+1)}, \quad (\text{see } [1, 16]). \tag{1.5}$$

It is well known that the Rodrigues formula for $L_n^{(\alpha)}(x)$ is given by

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right), \quad (n \geq 0), \\ &= x^{-\alpha}\frac{1}{n!}\left(\frac{d}{dx}-1\right)^n x^{n+\alpha}, \quad (\text{see } [1, 2, 5, 3, 12, 16]). \end{aligned} \tag{1.6}$$

From Laurent series and (1.2), we note that

$$L_n^{(\alpha)}(x) = \frac{1}{2\pi i}\oint_C \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\alpha+1}} t^{-n-1}dt, \tag{1.7}$$

where the contour circles the origin in a counterclockwise direction.

It is well known that the generalized Laguerre polynomials satisfy a three-term recurrence relation

$$(n+1)L_{n+1}^{(\alpha)}(x) = (2n+1+\alpha-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x).$$

Several authors have studied some interesting extensions and modifications of Laguerre polyhnomials along with related mathematical physics, combinatorial, probabilistic, and statistical applications (see [1, 2, 5, 3, 6, 8, 7, 9, 10, 11, 4, 12, 13, 14, 15, 16]).

In [7], Kim studied nonlinear differential equations arising from Frobenius-Euler numbers and polynomials.

In this paper, we study identities arising from differential equations for the generalized Laguerre polynomials. From our study for the generalized Laguerre polynomials, we derive some explicit and new identities related to those polynomials.

2. Identities arising from differential equations for generalized Laguerre polynomials

Let

$$F = F(t \mid x, \alpha) = \frac{1}{(1-t)^{\alpha+1}}e^{-\frac{tx}{1-t}}, \quad (\alpha > -1). \tag{2.8}$$

Then, by (2.8), we get

$$F^{(1)} = \frac{d}{dt}(F(t \mid x, \alpha)) = \left\{(\alpha+1)(1-t)^{-1} - x(1-t)^{-2}\right\}F, \tag{2.9}$$

$$F^{(2)} = \frac{d}{dt}F^{(1)} = \left\{(\alpha+1)(\alpha+2)(1-t)^{-2} - 2x(\alpha+2)(1-t)^{-3} + x^2(1-t)^{-4}\right\}F, \tag{2.10}$$

and

$$\begin{aligned}F^{(3)} &= \frac{d}{dt}F^{(2)} \\ &= \Big\{(\alpha+1)(\alpha+2)(\alpha+3)(1-t)^{-3} \\ &\quad -3x(\alpha+2)(\alpha+3)(1-t)^{-4} \\ &\quad +3(\alpha+3)x^2(1-t)^{-5} \\ &\quad -x^3(1-t)^{-6}\Big\}F.\end{aligned}$$

Continuing this process, we set, for $N = 0, 1, 2, \ldots$,

$$\begin{aligned}F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t \mid x, \alpha) \\ &= \left(\sum_{i=N}^{2N} b_{i-N}^{(\alpha)}(N, x)(1-t)^{-i}\right)F.\end{aligned} \tag{2.11}$$

On the one hand, by (2.11), we get

$$\begin{aligned}F^{(N+1)} &= \frac{d}{dt}\left(F^{(N)}\right) \\ &= \left(\sum_{i=N+1}^{2N+1}(\alpha+i)\, b_{i-N-1}^{(\alpha)}(N, x)(1-t)^{-i}\right)F \\ &\quad -\left(x\sum_{i=N+2}^{2N+2} b_{i-N-2}^{(\alpha)}(N, x)(1-t)^{-i}\right)F.\end{aligned} \tag{2.12}$$

On the other hand, by replacing N by $N+1$ in (2.11), we have

$$F^{(N+1)} = \left(\sum_{i=N+1}^{2N+2} b_{i-N-1}^{(\alpha)}(N+1, x)(1-t)^{-i}\right)F. \tag{2.13}$$

From (2.12) and (2.13), we can derive the following equations:

$$b_0^{(\alpha)}(N+1, x) = (\alpha+N+1)\, b_0^{(\alpha)}(N, x), \tag{2.14}$$

$$b_{N+1}^{(\alpha)}(N+1, x) = -x b_N^{(\alpha)}(N, x), \tag{2.15}$$

and

$$\begin{aligned}b_{i-N-1}^{(\alpha)}(N+1, x) &= -x b_{i-N-2}^{(\alpha)}(N, x) + (\alpha+i)\, b_{i-N-1}^{(\alpha)}(N, x), \\ &\qquad (N+2 \le i \le 2N+1).\end{aligned} \tag{2.16}$$

In addition, by (2.11), we get

$$F = F^{(0)} = b_0^{(\alpha)}(0, x)\, F. \tag{2.17}$$

By comparing the coefficients on both sides of (2.17), we have

$$b_0^{(\alpha)}(0, x) = 1. \tag{2.18}$$

From (2.9) and (2.11), we note that

$$\begin{aligned}
&\left\{(\alpha + 1)(1 - t)^{-1} - x(1 - t)^{-2}\right\} F \\
&= F^{(1)} \\
&= \left(\sum_{i=1}^{2} b_{i-1}^{(\alpha)}(1, x)(1 - t)^{-i}\right) F \\
&= \left\{b_0^{(\alpha)}(1, x)(1 - t)^{-1} + b_1^{(\alpha)}(1, x)(1 - t)^{-2}\right\} F.
\end{aligned} \tag{2.19}$$

Thus, by (2.19), we get

$$b_0^{(\alpha)}(1, x) = \alpha + 1, \quad b_1^{(\alpha)}(1, x) = -x. \tag{2.20}$$

Moreover, from (2.14) and (2.15), we have

$$\begin{aligned}
b_0^{(\alpha)}(N + 1, x) &= (\alpha + N + 1)\, b_0^{(\alpha)}(N, x) \\
&= (\alpha + N + 1)(\alpha + N)\, b_0^{(\alpha)}(N - 1, x) \\
&\vdots \\
&= (\alpha + N + 1)(\alpha + N) \cdots (\alpha + 2)\, b_0^{(\alpha)}(1, x) \\
&= (\alpha + N + 1)_{N+1},
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
b_{N+1}^{(\alpha)}(N + 1, x) &= -x b_N^{(\alpha)}(N, x) \\
&= (-x)^2\, b_{N-1}^{(\alpha)}(N - 1, x) \\
&\vdots \\
&= (-x)^N\, b_1^{(\alpha)}(1, x) \\
&= (-x)^{N+1},
\end{aligned} \tag{2.22}$$

where $(x)_n = x(x - 1) \cdots (x - n + 1)$, $(n \geq 0)$.

Hence, the matrix $\left(b_i^{(\alpha)}(j,x)\right)$ is given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ N \end{array}\begin{array}{c}\begin{array}{ccccc} 0 & 1 & 2 & & N \end{array} \\ \left[\begin{array}{ccccc} 1 & \alpha+1 & (\alpha+2)_2 & \cdots & (\alpha+N)_N \\ & (-x) & & & \\ & & (-x)^2 & & \\ & & & \ddots & \\ & 0 & & & (-x)^N \end{array}\right]\end{array}$$

By (2.16), we get

$$\begin{aligned} & b_1^{(\alpha)}(N+1,x) \\ &= -xb_0^{(\alpha)}(N,x) + (\alpha+N+2)\, b_1^{(\alpha)}(N,x) \\ &= -x\left\{b_0^{(\alpha)}(N,x) + (\alpha+N+2)\, b_0^{(\alpha)}(N-1,x)\right\} \\ &\quad + (\alpha+N+2)(\alpha+N+1)\, b_1^{(\alpha)}(N-1,x) \\ &\vdots \\ &= -x\sum_{i=0}^{N-1} (\alpha+N+2)_i\, b_0^{(\alpha)}(N-i,x) \\ &\quad + (\alpha+N+2)(\alpha+N+1)\cdots(\alpha+3)\, b_1^{(\alpha)}(1,x) \\ &= -x\sum_{i=0}^{N} (\alpha+N+2)_i\, b_0^{(\alpha)}(N-i,x), \end{aligned} \tag{2.23}$$

$$\begin{aligned} & b_2^{(\alpha)}(N+1,x) \\ &= -xb_1^{(\alpha)}(N,x) + (\alpha+N+3)\, b_2^{(\alpha)}(N,x) \\ &= -x\left\{b_1^{(\alpha)}(N,x) + (\alpha+N+3)\, b_1^{(\alpha)}(N-1,x)\right\} \\ &\quad + (\alpha+N+3)(\alpha+N+2)\, b_2^{(\alpha)}(N-1,x) \\ &\vdots \\ &= -x\sum_{i=0}^{N-2} (\alpha+N+3)_i\, b_1^{(\alpha)}(N-i,x) \\ &\quad + (\alpha+N+3)(\alpha+N+2)\cdots(\alpha+5)\, b_2^{(\alpha)}(2,x) \\ &= -x\sum_{i=0}^{N-1} (\alpha+N+3)_i\, b_1^{(\alpha)}(N-i,x), \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} b_3^{(\alpha)}(N+1,x) &= -xb_2^{(\alpha)}(N,x) + (\alpha+N+4)\, b_3^{(\alpha)}(N,x) \\ &= -x\left\{b_2^{(\alpha)}(N,x) + (\alpha+N+4)\, b_2^{(\alpha)}(N-1,x)\right\} \\ &\quad + (\alpha+N+4)(\alpha+N+3)\, b_3^{(\alpha)}(N-1,x) \\ &\ \vdots \\ &= -x\sum_{i=0}^{N-3}(\alpha+N+4)_i\, b_2^{(\alpha)}(N-i,x) \\ &\quad + (\alpha+N+4)(\alpha+N+3)\cdots(\alpha+7)\, b_3^{(\alpha)}(3,x) \\ &= -x\sum_{i=0}^{N-2}(\alpha+N+4)_i\, b_2^{(\alpha)}(N-i,x)\,. \end{aligned} \tag{2.25}$$

Hence, by continuing this process, we get

$$b_j^{(\alpha)}(N+1,x) = -x\sum_{i=0}^{N-j+1}(\alpha+N+j+1)_i\, b_{j-1}^{(\alpha)}(N-i,x)\,. \tag{2.26}$$

Now, we give explicit expressions for $b_j^{(\alpha)}(N+1,x)$. From (2.21), we can derive the following equations:

$$\begin{aligned} &b_1^{(\alpha)}(N+1,x) \\ &= -x\sum_{i_1=0}^{N}(\alpha+N+2)_{i_1}\, b_0^{(\alpha)}(N-i_1,x) \\ &= -x\sum_{i_1=0}^{N}(\alpha+N+2)_{i_1}\,(\alpha+N-i_1)_{N-i_1}\,, \end{aligned} \tag{2.27}$$

$$\begin{aligned} &b_2^{(\alpha)}(N+1,x) \\ &= -x\sum_{i_2=0}^{N-1}(\alpha+N+3)_{i_2}\, b_1^{(\alpha)}(N-i_2,x) \\ &= (-x)^2\sum_{i_2=0}^{N-1}\sum_{i_1=0}^{N-i_2-1}(\alpha+N+3)_{i_2} \\ &\quad\times(\alpha+N-i_2+1)_{i_1}\,(\alpha+N-i_2-i_1-1)_{N-i_2-i_1-1} \end{aligned} \tag{2.28}$$

and

$$
\begin{aligned}
& b_3^{(\alpha)}(N+1,x) \\
& = -x \sum_{i_3=0}^{N-2} (\alpha+N+4)_{i_3}\, b_2^{(\alpha)}(N-i_3,x) \\
& = (-x)^3 \sum_{i_3=0}^{N-2} \sum_{i_2=0}^{N-i_3-2} \sum_{i_1=0}^{N-i_3-i_2-2} (\alpha+N+4)_{i_3} \\
& \quad \times (\alpha+N-i_3+2)_{i_2} (\alpha+N-i_3-i_2)_{i_1} \\
& \quad \times (\alpha+N-i_3-i_2-i_1-2)_{N-i_3-i_2-i_1-2}\,.
\end{aligned}
$$

Continuing this process, we get

$$
\begin{aligned}
& b_j^{(\alpha)}(N+1,x) \\
& = (-x)^j \sum_{i_j=0}^{N-j+1} \sum_{i_{j-1}=0}^{N-j+1-i_j} \cdots \sum_{i_1=0}^{N-j+1-i_j-\cdots-i_2} \prod_{k=1}^{j} \left(\alpha+N-j+1-\sum_{l=k+1}^{j} i_l + 2k\right)_{i_k} \\
& \quad \times \left(\alpha+N-j+1-\sum_{l=1}^{j} i_l\right)_{N-j+1-\sum_{l=1}^{j} i_l}.
\end{aligned}
\tag{2.29}
$$

Therefore, we obtain the following theorem.

Theorem 2.1. The linear differential equations

$$
\begin{aligned}
F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t \mid x,\alpha) \\
&= \left(\sum_{i=N}^{2N} b_{i-N}^{(\alpha)}(N,x)\,(1-t)^{-i}\right) F, \quad (N \in \mathbb{N}\cup\{0\}),
\end{aligned}
$$

has a solution

$$
F = F(t \mid x,\alpha) = \frac{1}{(1-t)^{\alpha+1}} e^{-\frac{xt}{1-t}},
$$

where $b_0^{(\alpha)}(N,x) = (\alpha+N)_N$, $b_N^{(\alpha)}(N,x) = (-x)^N$,

$$
\begin{aligned}
b_j^{(\alpha)}(N,x) &= (-x)^j \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} \prod_{k=1}^{j} \left(\alpha+N-j-\sum_{l=k+1}^{j} i_l + 2k\right)_{i_k} \\
& \quad \times \left(\alpha+N-j-\sum_{l=1}^{j} i_l\right)_{N-j-\sum_{l=1}^{j} i_l}, \quad (j=1,\ldots,N-1).
\end{aligned}
$$

From (1.2), we note that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F\left(t \mid x, \alpha\right) \tag{2.30}$$
$$= \sum_{n=N}^{\infty} (n)_N \, L_n^{(\alpha)}(x)\, t^{n-N}$$
$$= \sum_{n=0}^{\infty} L_{n+N}^{(\alpha)}(x)\,(n+N)_N\, t^n.$$

On the other hand, by Theorem 2.1, we get

$$F^{(N)} = \left(\sum_{i=N}^{2N} b_{i-N}^{(\alpha)}(N,x)\,(1-t)^{-i}\right) F \tag{2.31}$$
$$= \sum_{i=N}^{2N} b_{i-N}^{(\alpha)}(N,x) \sum_{l=0}^{\infty} \binom{i+l-1}{l} t^l \sum_{k=0}^{\infty} L_k^{(\alpha)}(x)\, t^k$$
$$= \sum_{i=N}^{2N} b_{i-N}^{(\alpha)}(N,x) \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{i+l-1}{l} L_{n-l}^{(\alpha)}(x)\right) t^n$$
$$= \sum_{n=0}^{\infty} \left\{\sum_{i=N}^{2N} b_{i-N}^{(\alpha)}(N,x) \sum_{l=0}^{n} \binom{i+l-1}{l} L_{n-l}^{(\alpha)}(x)\right\} t^n.$$

Therefore, by (2.30) and (2.31), we obtain the following theorem.

Theorem 2.2. For $n, N \in \mathbb{N} \cup \{0\}$, we have

$$L_{n+N}^{(\alpha)}(x) = \frac{1}{(n+N)_N} \sum_{i=N}^{2N} b_{i-N}^{(\alpha)}(N,x) \sum_{l=0}^{n} \binom{i+l-1}{l} L_{n-l}^{(\alpha)}(x),$$

where

$$b_0^{(\alpha)}(N,x) = (\alpha+N)_N, \quad b_N^{(\alpha)}(N,x) = (-x)^N,$$

and

$$b_j^{(\alpha)}(N,x) = (-x)^j \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} \prod_{k=1}^{j} \left(\alpha+N-j-\sum_{l=k+1}^{j} i_l + 2k\right)_{i_k}$$
$$\times \left(\alpha+N-j-\sum_{l=1}^{j} i_l\right)_{N-j-\sum_{l=1}^{j} i_l}$$

$(j = 1, 2, \ldots, N-1)$.

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