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Differential equations associated with higher-order Frobenius-Euler numbers

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Abstract

In this paper, we construct a family of linear differential equations from a constant multiple of the generating function of the higher-order Frobenius-Euler numbers. Using those differential equations, we find some new and explicit identities involving higher-order Frobenius-Euler numbers and higher-order Bernoulli numbers.

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1. Introduction

The *Euler polynomials of order r* are defined by generating function

$$\sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^r e^{xt}, \text{ (see [2, 3, 4, 13]).}$$

In the special case, for $x = 0$, $E_n^{(r)}(0) = E_n^{(r)}$ are called the *Euler numbers of order r*. In particular, for $r = 1$, $E_n(x) = E_n^{(1)}(x)$ are called the *ordinary Euler polynomials*.

As is well-known, the *Frobeinus-Euler numbers of order r* are defined by the generating function

$$\left(\frac{1-u}{e^t-u}\right)^r = \sum_{n=0}^{\infty} H_n^{(r)}(u)\frac{t^n}{n!}, \tag{1}$$

where $u(\neq 1) \in \mathbb{C}$. In particular, if we put $u = -1$, then

$$\sum_{n=0}^{\infty} H_n^{(r)}(-1)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^r = \sum_{n=0}^{\infty} E_n^{(r)}\frac{t^n}{n!}, \tag{2}$$

where E_n are the Euler numbers of order r (see [1, 7, 10, 11, 12]). By (2), we know that $E_n^{(r)} = H_n^{(r)}(-1)$, for all $n \in \mathbb{N}$.

In [6, 8], authors attempted a new method for obtaining some new and explicit identities related to Bernoulli numbers of the second kind and Frobenius-Euler numbers arising from nonlinear differential equation. This method turned out to be very for studying special polynomials and numbers and mathematical physics (see [6, 8, 9]).

In this paper, we construct a family of linear differential equations from a constant multiple of the generating function of the higher-order Frobenius-Euler numbers. Using those differential equations, we find some new and explicit identities involving higher-order Frobenius-Euler numbers and higher-order Bernoulli numbers.

2. Some identities of higher-order Frobenius-Euler numbers arising from nonlinear differential equations

Let u be a fixed complex number with $u \neq 1$, $r \in \mathbb{N}$, and let

$$F = F(t; r) = F(t; r, u) = \left(\frac{1}{e^t-u}\right)^r. \tag{3}$$

Throughout this paper, all derivatives will be taken with respect to t. Then, by (3), we get

$$\begin{aligned} F^{(1)}(t;r) &= \frac{d}{dt}F(t;r) = -r(e^t-u)^{-r-1}e^t = -r(e^t-u)^{-r-1}(e^t-u+u) \\ &= -r(F(t;r)+uF(t;r+1)), \end{aligned} \tag{4}$$

$$\begin{aligned} F^{(2)}(t;r) &= -rF^{(1)}(t;r) - ruF^{(1)}(t;r+1) \\ &= -r(-rF(t;r) - ruF(t;r+1)) - ru(-(r+1)F(t;r+1) \\ &\quad -(r+1)uF(t;r+2)) \\ &= r^2F(t;r) + (2r^2+r)uF(t;r+1) + (r^2+r)u^2F(t;r+2). \end{aligned} \tag{5}$$

So, we are led to put

$$F^{(N)}(t;r) = \sum_{i=0}^{N} a_i(N)u^iF(t;r+i), \ N = 0,1,2,\ldots. \tag{6}$$

Taking the derivative with respect to t of (6), we have

$$\begin{aligned} F^{(N+1)}(t;r) &= \left(\frac{d}{dt}\right)\left(F^{(N)}(t;r)\right) = \sum_{i=0}^{N} a_i(N)u^iF^{(1)}(t;r+i) \\ &= \sum_{i=0}^{N} a_i(N)u^i\left\{-(r+i)(F(t;r+i)+uF(t;r+i+1))\right\} \\ &= -\sum_{i=0}^{N}(r+i)a_i(N)u^iF(t;r+i) - \sum_{i=0}^{N}(r+i)a_i(N)u^{i+1}F(t;r+i+1) \\ &= -\sum_{i=0}^{N}(r+i)a_i(N)u^iF(t;r+i) - \sum_{i=1}^{N+1}(r+i-1)a_{i-1}(N)u^iF(t;r+i). \end{aligned} \tag{7}$$

On the other hand, by replacing N by $N+1$ in (6), we get

$$F^{(N+1)}(t;r) = \sum_{i=0}^{N+1} a_i(N+1)u^iF(t;r+i). \tag{8}$$

Comparing the coefficients on both sides of (7) and (8), we have

$$a_0(N+1) = -ra_0(N), \ a_{N+1}(N+1) = -(r+N)a_N(N), \tag{9}$$

and

$$a_i(N+1) = -(r+i-1)a_{i-1}(N) - (r+i)a_i(N), (1 \le i \le N). \tag{10}$$

In addition,

$$F(t;r) = F^{(0)}(t;r) = a_0(n)F(t;r). \tag{11}$$

Thus, by (11), we get

$$a_0(0) = 1. \tag{12}$$

From (4) and (6), we have

$$\begin{aligned} -rF(t;r) - ruF(t;r+1) &= F^{(1)}(t;r) \\ &= a_0(1)F(t;r) + a_1(1)uF(t;r+1). \end{aligned} \tag{13}$$

From (13), we have

$$a_0(1) = -r,\ a_1(1) = -r. \tag{14}$$

By (9), we easily get

$$\begin{aligned} a_0(N+1) &= -ra_0(N) = (-r)^2 a_0(N+1) = \cdots \\ &= (-r)^N a_0(1) = (-r)^{N+1}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} a_{N+1}(N+1) &= -(r+N)a_N(N) = (-1)^2(r+N)(r+N-1)a_{N-1}(N-1) = \cdots \\ &= (-1)^N(r+N)(r+N-1)\cdots(r+1)a_1(1) \\ &= (-1)^{N+1}(r+N)(r+N-1)\cdots(r+1)r \\ &= (-1)^{N+1}(r+N)_{N+1}. \end{aligned} \tag{16}$$

Now, we turn our attention to

$$a_i(N+1) = -(r+i-1)a_{i-1}(N) - (r+i)a_i(N),\ (1 \le i \le N). \tag{17}$$

For $i = 1$ in (10), we have

$$\begin{aligned} a_1(N+1) &= -ra_0(N) - (r+1)a_1(N) \\ &= -ra_0(N) - (r+1)(-ra_0(N-1) - (r+1)a_1(N-1)) \\ &= -r(a_0(N) - (r+1)a_0(N-1)) + (-(r+1))^2 a_1(N-1) \\ &= -r(a_0(N) - (r+1)a_0(N-1)) + (-(r+1))^2 \\ &\quad \times (-ra_0(N-2) - (r+1)a_1(N-2)) \\ &= -r(a_0(N) - (r+1)a_0(N-1) + (-(r+1))^2 a_0(N-2)) \\ &\quad + (-(r+1))^3 a_1(N-2) \\ &= \cdots \\ &= -r\sum_{k=0}^{N-1}(-(r+1))^k a_0(N-k) + (-(r+1))^N a_1(1) \\ &= -r\sum_{k=0}^{N}(-(r+1))^k a_0(N-k). \end{aligned} \tag{18}$$

Similarly to the case $i = 1$, for $i = 2, 3$ in (10), we have

$$a_2(N+1) = -(r+1)\sum_{k=0}^{N-1}(-(r+2))^k a_1(N-k). \tag{19}$$

$$a_3(N+1) = -(r+2)\sum_{k=0}^{N-2}(-(r+3))^k a_2(N-k). \tag{20}$$

Thus, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1) = -(r+i-1)\sum_{k=0}^{N-i+1}(-(r+i))^k a_{i-1}(N-k). \tag{21}$$

Now, we give explicit expressions for $a_i(j)$.
From (15) and (18), we have

$$\begin{aligned} a_1(N+1) &= -r\sum_{k_1=0}^{N}(-(r+1))^{k_1} a_0(N-k_1) \\ &= -r\sum_{k_1=0}^{N}(-(r+1))^{k_1}(-r)^{N-k_1} \\ &= (-1)^{N+1} r \sum_{k_1=0}^{N}(r+1)^{k_1} r^{N-k_1}, \end{aligned} \tag{22}$$

and, by (19) and (22), we get

$$\begin{aligned} a_2(N+1) &= -(r+1)\sum_{k_2=0}^{N-1}(-(r+1))^{k_2} a_1(N-k_2) \\ &= -(r+1)\sum_{k_2=0}^{N}(-(r+2))^{k_2}(-1)^{N-k_2} r \sum_{k_1=0}^{N-1-k_2}(r+1)^{k_1} r^{N-1-k_2-k_1} \\ &= (-1)^{N+1}(r+1)_2 \sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-1-k_2}(r+2)^{k_2}(r+1)^{k_1} r^{N-1-k_2-k_1}. \end{aligned} \tag{23}$$

From (20) and (23), we note that

$$\begin{aligned}
a_3(N+1) &= -(r+2)\sum_{k_3=0}^{N-2}(-(r+3))^{k_3}a_2(N-k_3)\\
&= -(r+2)\sum_{k_3=0}^{N-2}(-(r+3))^{k_3}(-1)^{N-k_3}(r+1)_2\\
&\quad\times\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2}(r+2)^{k_2}(r+1)^{k_1}r^{N-2-k_3-k_2-k_1}\\
&= (-1)^{N+1}(r+2)_3\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2}(r+3)^{k_3}(r+2)^{k_2}(r+1)^{k_1}\\
&\quad\times r^{N-2-k_3-k_2-k_1}.
\end{aligned} \tag{24}$$

Continuing this process, we have

$$\begin{aligned}
a_i(N+1) &= (-1)^{N+1}(r+i-1)_i\\
&\quad\times\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-i+1-k_i}\cdots\sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2} r^{N-i+1-\sum_{l=1}^{i}k_l}\prod_{l=1}^{i}(r+l)^{k_l},
\end{aligned} \tag{25}$$

$(1 \le i \le N)$.

Remark 2.1. The equation (25) holds also for $i = N+1$. Therefore, by (25), we obtain the following theorem.

Theorem 2.2. The following family of differential equations

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N)u^i(e^t-u)^{-i}\right)F \quad (N = 0, 1, 2, \ldots) \tag{26}$$

have a solution

$$F = F(t; r, u) = \left(\frac{1}{e^t-u}\right)^r,$$

where $a_0(N) = (-r)^N$,

$$\begin{aligned}
a_i(N) &= (-1)^N(r+i-1)_i\\
&\quad\times\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-i-k_i}\cdots\sum_{k_1=0}^{N-i-k_i-\cdots-k_2} r^{N-i-\sum_{l=1}^{i}k_l}\prod_{l=1}^{i}(r+l)^{k_l},
\end{aligned}$$

for $1 \le i \le N$.

Recall that the *Frobenius-Euler numbers of order* r, $H_k^{(r)}(u)$, are given by the generating function

$$\left(\frac{1-u}{e^t-u}\right)^r = \sum_{k=0}^{\infty} H_k^{(r)}(u)\frac{t^k}{k!}. \tag{27}$$

From (26), we note that

$$\begin{aligned}&\left(\frac{d}{dt}\right)^N \left(\frac{1-u}{e^t-u}\right)^r \\ &\quad = \sum_{i=0}^{N} a_i(N)u^i(1-u)^{-i}\left(\frac{1-u}{e^t-u}\right)^{r+i}.\end{aligned} \tag{28}$$

Thus, by (28), we get

$$\begin{aligned}&\sum_{k=0}^{\infty} H_{k+N}^{(r)}(u)\frac{t^k}{k!} \\ &\quad = \sum_{i=0}^{N} a_i(N)\left(\frac{u}{1-u}\right)^i \sum_{k=0}^{\infty} H_k^{(r+i)}(u)\frac{t^k}{k!} \\ &\quad = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N} a_i(N)\left(\frac{u}{1-u}\right)^i H_k^{(r+i)}(u)\right)\frac{t^k}{k!}.\end{aligned} \tag{29}$$

Comparing the coefficients on both sides of (29), we obtain the following theorem.

Theorem 2.3. For $k, N = 0, 1, 2, \ldots$,

$$H_{k+N}^{(r)}(u) = \sum_{i=0}^{N} a_i(N)\left(\frac{u}{1-u}\right)^i H_k^{(r+i)}(u),$$

where $a_i(N)'s$ are as in Theorem 2.2.

Corollary 2.4. For $N = 0, 1, 2, \ldots$,

$$H_N^{(r)}(u) = \sum_{i=0}^{N} a_i(N)\left(\frac{u}{1-u}\right)^i.$$

Taking $u = -1$, we have the following corollary.

Recall here that the *Euler numbers of order* r, $E_k^{(r)}$, are given by the generating function

$$\left(\frac{2}{e^t+1}\right)^r = \sum_{k=0}^{\infty} E_k^{(r)}\frac{t^k}{k!} = \sum_{k=0}^{\infty} H_k^{(r)}(-1)\frac{t^k}{k!}. \tag{30}$$

Corollary 2.5. For $k, N = 0, 1, 2, \ldots,$

$$E_{k+N}^{(r)} = \sum_{i=0}^{N} \left(-\frac{1}{2}\right)^{i} a_i(N) E_k^{(r+i)}.$$

Corollary 2.6. For $N = 0, 1, 2, \ldots,$

$$E_N^{(r)} = \sum_{i=0}^{N} \left(-\frac{1}{2}\right)^{i} a_i(N).$$

Recall here that the *Bernoulli numbers of order* r, $B_k^{(r)}$, are defined by the generating function

$$\left(\frac{t}{e^t - 1}\right)^r = \sum_{k=0}^{\infty} B_k^{(r)} \frac{t^k}{k!}. \tag{31}$$

By Theorem 2.2, with $u = 1$, we get

$$\left(\frac{d}{dt}\right)^N \left(\frac{1}{e^t - 1}\right)^r = \sum_{i=0}^{N} a_i(N) \left(\frac{1}{e^t - 1}\right)^{r+i}. \tag{32}$$

Now, we observe that

$$\begin{aligned} \left(\frac{1}{e^t - 1}\right)^r &= \frac{1}{t^r} \left(\frac{t}{e^t - 1}\right)^r \\ &= \sum_{k=0}^{\infty} B_k^{(r)} \frac{t^{k-r}}{k!} \\ &= \sum_{k=0}^{r-1} B_k^{(r)} \frac{t^{k-r}}{k!} + \sum_{k=r}^{\infty} B_k^{(r)} \frac{t^{k-r}}{r!} \\ &= \sum_{k=-r}^{-1} B_{k+r}^{(r)} \frac{t^k}{(k+r)!} + \sum_{k=0}^{\infty} B_{k+r}^{(r)} \frac{t^k}{(k+r)!}. \end{aligned} \tag{33}$$

Thus, by (33), we get

$$\left(\frac{d}{dt}\right)^N \left(\frac{1}{e^t - 1}\right)^r = \sum_{k=-r}^{-1} B_{k+r}^{(r)} (k)_N \frac{t^{k-N}}{(k+r)!} + \sum_{k=N}^{\infty} B_{k+r}^{(r)} (k)_N \frac{t^{k-N}}{(k+r)!}. \tag{34}$$

From (34), we have

$$
\begin{aligned}
& t^{r+N}\left(\frac{d}{dt}\right)^{N}\left(\frac{1}{e^{t}-1}\right)^{r} \\
&= \sum_{k=-r}^{-1} B_{k+r}^{(r)}(k)_N \frac{t^{k+r}}{(k+r)!} + \sum_{k=N}^{\infty} B_{k+r}^{(r)}(k)_N \frac{t^{k+r}}{(k+r)!} \qquad (35) \\
&= \sum_{k=0}^{r-1} B_{k}^{(r)}(k-r)_N \frac{t^{k}}{k!} + \sum_{k=r+N}^{\infty} B_{k}^{(r)}(k-r)_N \frac{t^{k}}{k!}.
\end{aligned}
$$

On the other hand, by (32), we get

$$
\begin{aligned}
& t^{r+N}\sum_{i=0}^{N} a_i(N)\left(\frac{1}{e^{t}-1}\right)^{r+i} \\
&= \sum_{i=0}^{N} a_i(N) t^{N-i}\left(\frac{t}{e^{t}-1}\right)^{r+i} \\
&= \sum_{i=0}^{N} a_i(N) t^{N-i}\sum_{l=0}^{\infty} B_{l}^{(r+i)}\frac{t^{l}}{l!} \\
&= \sum_{i=0}^{N}\sum_{l=0}^{\infty} a_i(N) B_{l}^{(r+i)}\frac{t^{l+N-i}}{l!} \\
&= \sum_{i=0}^{N}\sum_{k=N-i}^{\infty} a_i(N) B_{k+i-N}^{(r+i)}\frac{t^{k}}{(k+i-N)!} \qquad (36) \\
&= \sum_{i=0}^{N}\sum_{k=N-i}^{\infty} a_i(N) B_{k+i-N}^{(r+i)}(k)_{N-i}\frac{t^{k}}{k!} \\
&= \sum_{k=0}^{N-1}\sum_{i=N-k}^{N} a_i(N) B_{k+i-N}^{(r+i)}(k)_{N-i}\frac{t^{k}}{k!} \\
&\quad + \sum_{k=N}^{\infty}\sum_{i=0}^{N} a_i(N) B_{k+i-N}^{(r+i)}(k)_{N-i}\frac{t^{k}}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{i=\max\{N-k,0\}}^{N} a_i(N) B_{k+i-N}^{(r+i)}(k)_{N-i}\frac{t^{k}}{k!}.
\end{aligned}
$$

By comparing the coefficients on the both sides of (35) and (36), we obtain the following theorem.

Theorem 2.7. For $k, N = 0, 1, 2, \ldots$, we have

(a) for $0 \le k \le r-1$ and $k \ge r+N$,

$$B_k^{(r)} = \frac{1}{(k-r)_N} \sum_{i=\max\{N-k,0\}}^{N} a_i(N) B_{k+i-N}^{(r+i)}(k)_{N-i},$$

(b) for $r \le k \le r-1+N$,

$$\sum_{i=\max\{N-k,0\}}^{N} a_i(N) B_{k+i-N}^{(r+i)}(k)_{N-i} = 0,$$

where $a_i(N)'s$ are as in Theorem 2.2.

References

[1] M. Can, M. Cenkci, V. Kurt, and Y. Simsek, *Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions*, Adv. Stud. Contemp. Math., **18** (2009), no. 2, 135–160.

[2] D. Ding and J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials*, Adv. Stud. Contemp. Math., **20** (2010), no. 1, 7–21.

[3] S. Gaboury, R. Tremblay, B.J. Fugère, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials*, Proc. Jangjeon Math. Soc., **17** (2014), 115–123.

[4] L. C. Jang, B. M. Kim, *On identities between sums of Euler numbers and Genocchi numbers of higher order*, J. Comput. anal. appl., **20** (2016), 1240–1247.

[5] D. S. Kim and T. Kim, *Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation*, Bull. Korean Math. Soc., **52** (2015), 2001–2010.

[6] T. Kim, *Identities involving Frobenius-Euler polynomials arising from non-linear differential equations*, J. Number Theory, **132** (2012), no. 12, 2854–2865.

[7] T. Kim and J. Choi, *A note on the product of Frobenius-Euler polynomials arising from the p-adic integral on $\mathbb{Z}_p$*, Adv. Stud. Contemp. Math., **22** (2012), no. 2, 215–223.

[8] T. Kim and D. S. Kim, *A note on non-linear Changhee differential equations*, Russ. J. Math. Phys., **23** (2016), no. 1, 88–92.

[9] T. Kim and D. S. Kim, *Identities involving degenerate Euler numbers and polynomials arising from non-linear differential equations*, J. Nonlinear Sci. Appl., **9** (2016), 2086–2098.

[10] T. Kim, H. I. Kwon and J. J. Seo, *On the degenerate Frobenius-Euler polynomials*, Global J. Pured Appl. Math., **11** (2014), no. 4, 2077–2084.

[11] T. Kim and T. Mansour, *Umbral calculus associated with Frobenius-type Eulerian polynomials,*, Russ. J. Math. Phys., **21** (2014), no. 4, 484–493.

[12] C. S. Ryoo, *A note on the Frobenius-Euler polynomials*, Proc. Jangjeon Math. Soc., **14** (2011), no. 4, 495–501.

[13] E. Şen, *Theorems on Apostol-Euler polynomials of higher order arising from Euler basis*, Adv. Stud. Contemp. Math., **23** (2013), 337–345.