Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 3 (2016), pp. 2513–2522

http://www.ripublication.com/gjpam.htm

Reliability Analysis of Mukherjee–Islam Distribution under three Different Prior Distributions

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Abstract

In this paper Mukherjee–Islam distribution has been taken for Bayes' estimators of its parameter under three different prior distributions. The prior distributions considered here are, uniform distribution, inverted gamma distributions and distribution by Siddiqui et al. (1992).

AMS subject classification:
Keywords: Bayes' estimators, Prior distribution.

1. Introduction

Theory of Bayesian inference has priority over the classical theory of inference in many situations. It is preferred strongly where the prior information on parameter are available or can be made available. A number of research workers have worked on Bayesian estimation like Bhattacharya (1967), Siddiqui et al. (1991, 1995, 1997 and 2014) and many more.

1.1. Basics of Mukherjee-Islam distribution

Mukherjee-Islam (1983) is a well established probability distribution and has been used by many research workers as a failure distribution for the purpose of reliability and Bayesian analysis. The main features of the distribution are: Pdf,

$$f(x; \theta, p) = (p/\theta^p)x^{(p-1)}; \qquad p, \theta > 0; \;\; x \geq 0 \tag{1.1}$$

Where p and θ are the shape and scale parameters of the distribution. Cdf,

$$F(x) = [\, x/\theta \,]^p \tag{1.2}$$

The reliability function of the distribution is given as;

$$R(t) = P[T > t] = 1 - [\, t/\theta \,]^p \tag{1.3}$$

Hazard rate function,

$$h(t) = \frac{f(t)}{R(t)}$$

or,

$$h(t) = \frac{(\, p/\theta^p)\, x^{(p-1)}}{1 - [t/\theta]^p}$$

or,

$$h(t) = \frac{p \;\; t^{(p-1)}}{\theta^p - t^p} \tag{1.4}$$

the m.l.e of p is finally obtained as;

$$\widehat{p} = \frac{n}{log\, \theta - \Sigma log\, x_i} \tag{1.5}$$

the m.l.e for the parameter θ, i.e.

$$\widehat{\theta} = x_(n) = Max(x_1, x_2, \ldots, x_n) \tag{1.6}$$

2. Bayesian Analysis of the Distribution

The three prior distributions are considered for the purpose of the Bayesian analysis they are, Uniform distribution, Inverted Gamma distribution and the one proposed by Siddiqui et al. (1992).

2.1. Bayes' estimates when Uniform distribution is taken as prior distribution

$$g(\theta) = \frac{(\alpha - 1)(\alpha\beta)^{\alpha-1}}{\beta^{\alpha-1} - \alpha^{\alpha-1}} . \frac{1}{\theta^\alpha}; \;\; 0 < \alpha \;\leq\; \theta \;\leq\; \beta < \infty \tag{2.7}$$

Now, suppose $t_1, t_2, \ldots, t_n$ denotes n observation then the likelihood conditional of θ is given by

$$\begin{aligned} L(\theta/t_1, t_2, \ldots, t_n) &= \prod_i^n f(t_i/\theta) \\ &= \prod_i^n (\frac{p}{\theta^p}) t_i^{p-1} \\ &= p^n \theta^{-np} \prod_i^n \; t_i^{p-1} \end{aligned}$$

$$S = p^n \theta^{-np} \tag{2.8}$$

where

$$S = \prod_{i}^{n} t_i^{p-1}$$

The likelihood conditional under the assumption of prior density (7) provides the following posterior distribution

$$G(\theta/t_1, t_2, \ldots, t_n) = \frac{S.p^n \theta^{-np} \frac{(\alpha-1)(\alpha\beta)^{\alpha-1}}{\beta^{\alpha-1}-\alpha^{\alpha-1}} \cdot \frac{1}{\theta^\alpha}}{\int_\alpha^\beta S.p^n \theta^{-np} \frac{(\alpha-1)(\alpha\beta)^{\alpha-1}}{\beta^{\alpha-1}-\alpha^{\alpha-1}} \cdot \frac{1}{\theta^\alpha}} d\theta$$
$$= \frac{\theta^{-np-a}}{\int_\alpha^\beta \theta^{-np-a}} d\theta$$

Which comes out to be in the following form

$$G(\theta/t1, t2, \ldots, tn) = \frac{\theta^{-np-a}}{\beta^{-np-a+1} - \alpha^{-np-a+1}}(1 - a - np)$$
$$= \frac{N_1}{\theta^{np+a}}$$

Where

$$N_1 = \frac{(np + a - 1)}{(1/\alpha)^{np+a-1} - (1/\beta)^{np+a-1}} \tag{2.9}$$

The Bayes estimate of the parameter θ is;

$$\theta* = \int_\alpha^\beta \theta \frac{N_1}{\theta^{np+a}} d\theta$$
$$= \frac{N_1[(1/\alpha)^{np+a-2} - (1/\beta)^{np+a-2}]}{(np + a - 2)}$$
$$= \frac{(np + a - 1)}{np + a - 2)} \cdot \frac{[\alpha\beta^{np+a-1} - \beta\alpha^{np+a-1}]}{\beta^{np+a-1} - \alpha^{np+a-1}} \tag{2.10}$$

The Bayes estimate of the variance of θ can be obtained through the following expression;

$$E(\theta^2) = \int_\alpha^\beta \theta^2 \frac{N_1}{\theta^{np+a}} d\theta$$
$$= \frac{N_1[(1/\alpha)^{np+a-3} - (1/\beta)^{np+a-3}]}{(np + a - 3)}$$
$$= \frac{(np + a - 1)}{(np + a - 3)} \cdot \frac{[\alpha^2\beta^{np+a-1} - \beta^2\alpha^{np+a-1}]}{\beta^{np+a-1} - \alpha^{np+a-1}} \tag{2.11}$$

Using these values in the following expression, variance comes out as;

$$
\begin{aligned}
V^*(\theta) &= E(\theta^2) - (\theta^*)^2 \\
&= \frac{(np+a-1)}{(np+a-3)} . \frac{[\alpha^2\beta^{np+a-1} - \beta^2\alpha^{np+a-1}]}{\beta^{np+a-1} - \alpha^{np+a-1}} \\
&\quad - \left[\frac{(np+a-1)}{(np+a-2)} . \frac{[\alpha\beta^{np+a-1} - \beta\alpha^{np+a-1}]}{\beta^{np+a-1} - \alpha^{np+a-1}} \right]^2
\end{aligned} \tag{2.12}
$$

The Bayes estimate of the reliability function R(t);

$$
\begin{aligned}
R^*(t) &= \int_\alpha^\beta \left[1 - \left(\frac{t}{\theta} \right)^p \right] \frac{N_1}{\theta^{np+a}} \, d\theta \\
&= 1 - t^p N_1 \int_\alpha^\beta \theta^{-np-a-p} d\theta \\
&= 1 - t^p N_1 \frac{[\beta^{-p(n+1)-a+1} - \alpha^{-p(n+1)-a+1}]}{-p(n+1) - a + 1}
\end{aligned}
$$

Which, has the following form.

$$
R^*(t) = 1 - \frac{(np+a-1)}{(n-1)p+a-1} . \frac{[\beta^{p(n+1)+a-1} - \alpha^{p(n+1)+a-1}]}{\alpha^p \, \beta^{p(n+1)+a-1} - \beta^p \, \alpha^{p(n+1)+a-1}} . t^p \tag{2.13}
$$

The hazard rate function of the distribution is

$$
h(t) = \frac{pt^{p-1}}{\theta^p - t^p}
$$

The Bayes estimate of hazard rate function $h(t)$ will be

$$
\begin{aligned}
h^*(t) &= \int_\alpha^\beta \frac{pt^{p-1}}{\theta^p - t^p} . \frac{N_1}{\theta^{np+a}} \\
&= pt^{p-1} N_1 \int_\alpha^\beta \theta^{-np-a-p} [1 - (t/\theta)^p]^{-1} d\theta \\
&= pt^{p-1} N_1 \int_\alpha^\beta \theta^{-(n+1)p-a-p} \Sigma_{k=0}^\infty (t/\theta)^{pk} d\theta \\
&= pt^{p-1} N_1 \, \Sigma_{k=0}^\infty \, t^{pk} \, \frac{[\beta^{p(n+k+1)+a-1} - \alpha^{p(n+k+1)+a-1}]}{p(n+k+1) + a - 1} \\
&= pt^{p-1} N_1 \, \Sigma_{k=0}^\infty \, t^{pk} \, \frac{(np+a-1)}{(n+k+1)\, p + a - 1} \\
&\quad . \frac{[\beta^{p(n+k+1)+a-1} - \alpha^{p(n+k+1)+a-1}]}{[\alpha^{(k+1)p} \, \beta^{p(n+k+1)+a-1} - \beta^{(k+1)p} \, \alpha^{p(n+k+1)+a-1}]}
\end{aligned} \tag{2.14}
$$

2.2. Bayes' Estimators when Siddiqui et al. (1992) distribution is taken as a prior distribution

In this section a continuous distribution is being used as prior density developed by Siddiqui et al. (1992). The probability density function of the distribution is,

$$g(\theta) = \frac{h\theta^{h-1}}{q^h - h^h}; \quad h \leq \theta \leq q; \quad h, q > 0 \tag{2.15}$$

The likelihood conditional under the assumption of prior density provides the following posterior distribution

$$G(\theta/t_1, t_2, \ldots, t_n) = \frac{S.p^n\theta^{-np}.\frac{h\theta^{h-1}}{q^h-h^h}}{\int_h^q S.p^n\theta^{-np}\ \frac{h\theta^{h-1}}{q^h-h^h}d\theta}$$

which comes out in the following form :

$$\begin{aligned} G(\theta/t_1, t_2, \ldots, t_n) &= \frac{\theta^{-np+h-1}}{\int_h^q \theta^{-np+h-1}d\theta} \\ &= \frac{N_2}{\theta^{np+h+1}} \end{aligned} \tag{2.16}$$

where

$$N_2 = \frac{h - np}{q^{h-np} - h^{h-np}}$$

The Bayes estimate of the parameter θ can be obtained as

$$\begin{aligned} \theta^* &= \int_h^q \theta.\frac{N_2}{\theta^{np+h+1}} \\ &= \frac{h - np}{h - np + 1}.\frac{q^{h-np+1} - h^{h-np+1}}{q^{h-np} - h^{h-np}} \end{aligned} \tag{2.17}$$

The Bayes estimate of the variance of θ can be obtained after obtaining the following expression

$$\begin{aligned} E(\theta^2) &= \int_h^q \theta^2\frac{N_2}{\theta^{np+h+1}}d\theta \\ &= \frac{h - np}{h - np + 2}.\frac{q^{h-np+2} - h^{h-np+2}}{q^{h-np} - h^{h-np}} \end{aligned}$$

Using these values in the following expression, variance can be calculated.

$$\begin{aligned} V^*(\theta) &= E(\theta^2) - (\theta^*)^2 \\ &= \frac{h - np}{h - np + 2}.\frac{q^{h-np+2} - h^{h-np+2}}{q^{h-np} - h^{h-np}} - \left[\frac{h - np}{h - np + 1}.\frac{q^{h-np+1} - h^{h-np+1}}{q^{h-np} - h^{h-np}}\right]^2 \end{aligned} \tag{2.18}$$

The Bayes estimate of the reliability function R(t);

$$\begin{aligned}R^*(t) &= \int_h^q [1-(t/\theta)^p].\frac{N_2}{\theta^{np+h+1}}d\theta \\ &= 1 - t^p N_2 \int_h^q \theta^{-(n+1)p-h-1} d\theta \\ &= 1 - t^p N_2 \frac{q^{h-(n+1)p} - h^{h-(n+1)p}}{h-(n+1)p}\end{aligned}$$

That has the following form:

$$R*(t) = 1 - t^p \frac{h-np}{h-(n+1)p} \frac{q^{h-(n+1)p} - h^{h-(n+1)p}}{q^{h-np} - h^{h-np}} \tag{2.19}$$

The Bayes estimate of the variance of R(t) can be expressed as

$$V^*[R(t)] = E[R^2(t)] - [R^*(t)]^2$$

That needs,

$$\begin{aligned}E[R^2(t)] &= \int_h^q [1-(t/\theta)^p]^2 \frac{N_2}{\theta^{np+h+1}} d\theta \\ &= 1 - t^{2p} N_2 \int_h^q \theta^{-(n+2)p-h-1}\end{aligned}$$

That has the following form:

$$\begin{aligned}E[R^2(t)] = 1 + t^{2p} &\frac{h-np}{h-(n+2)p} \frac{q^{h-(n+2)p} - h^{h-(n+2)p}}{q^{h-np} - h^{h-np}} \\ - 2t^p &\frac{h-np}{h-(n+1)p} \frac{q^{h-(n+1)p} - h^{h-(n+1)p}}{q^{h-np} - h^{h-np}}\end{aligned} \tag{2.20}$$

This will help in giving variance, $V*[R(t)]$. The Bayes estimate of hazard rate function; *(In this case hazard rate function is being denoted by w(t))*

$$\begin{aligned}w*(t) &= \int_h^q \frac{p\,t^{p-1}}{\theta^p - t^p} \frac{N_2}{\theta^{np+h+1}} \\ &= pt^{p-1} N_2 \int_h^q \theta^{-np-h-p-1}[1-(t/\theta)^p]^{-1} d\theta \\ &= pt^{p-1} N_2 \int_h^q \theta^{-np-h-p-1} \Sigma_{k=0}^{\infty} (t/\theta)^{pk} d\theta\end{aligned}$$

Finally;

$$= pt^{p-1}\, \Sigma_{k=0}^{\infty}\, t^{pk} \frac{h-np}{h-(n+k+1)p} \frac{q^{h-(n+k+1)p} - h^{h-(n+k+1)p}}{q^{h-np} - h^{h-np}} \tag{2.21}$$

Now, Bayes estimate of the variance of the same function comes through,

$$
\begin{aligned}
E[w^2(t)] &= \int_h^q [\frac{p\,t^{p-1}}{\theta^p - t^p}]^2 \frac{N_2}{\theta^{np+h+1}} d\theta \\
&= p^2 t^{2(p-1)} N_2 \int_h^q \theta^{-np-h-2p-1}[1-(t/\theta)^p]^{-2} d\theta \\
&= p^2 t^{2(p-1)} N_2 \int_h^q \theta^{-p(n+2)-h-1} \Sigma_{k=0}^{\infty} (k+1)\,(t/\theta)^{pk} d\theta
\end{aligned}
$$

Finally the above expression gives,

$$
E[w^2(t)] = p^2 t^{2(p-1)}\ \Sigma_{k=0}^{\infty} (k+1) t^{pk} \frac{h-np}{h-(n+k+2)p} \frac{q^{h-(n+k+2)p} - h^{h-(n+k+2)p}}{q^{h-np} - h^{h-np}} \tag{2.22}
$$

These equations provide estimate of variance of hazard rate function;

$$
V^*(w(t)) = E(w^2(t)] - [w^*(t)]^2
$$

2.3. Inverted Gamma Prior Density

The probability density function of Inverted gamma distribution is;

$$
g(\theta) = (v/\theta)^{d+1} \frac{e^{-(v/\theta)}}{\Gamma(d+1)}; \quad v, d, \theta \geq 0 \tag{2.23}
$$

The applicability of this model in the Bayesian analysis is already being discussed by Folks and Chhikara (1978). The likelihood conditional under the assumption of prior density (2.44) provides the following posterior distribution

$$
G(\theta/t_1, t_2, \ldots, t_n) = \frac{S.\ p^n\ \theta^{-np}\ (v/\theta)^{d+1} \frac{e^{-(v/\theta)}}{\Gamma(d+1)}}{\int_0^\infty S.\ p^n\ \theta^{-np}\ (v/\theta)^{d+1} \frac{e^{-(v/\theta)}}{\Gamma(d+1)} d\theta}
$$

That gives the following form:

$$
G(\theta/t_1, t_2, \ldots, t_n) = \frac{\theta^{-np-d-1} e^{-(v/\theta)}}{\int_0^\infty \theta^{-np-d-1} e^{-(v/\theta)} d\theta}
$$

$$
G(\theta/t_1, t_2, \ldots, t_n) = \theta^{-np-d-1} e^{-(v/\theta)} N_3 \tag{2.24}
$$

where

$$
N_3 = \frac{v^{np+d}}{\Gamma(np+d)}
$$

The Bayes estimate of the parameter θ can be obtained as;

$$\theta^* = \int_0^\infty \theta.\, \theta^{-np-d-1}\, e^{-(v/\theta)}\, N_3\, d\theta$$
$$= N_3 \int_0^\infty \theta^{-np-d}\, e^{-(v/\theta)} d\,\theta$$

That gives;

$$\theta^* = \frac{v}{(np+d-1)} \tag{2.25}$$

The Bayes estimate of the variance of θ is obtained after obtaining the following expression

$$E(\theta^2) = \int_0^\infty \theta^2.\, \theta^{-np-d-1}\, e^{-(v/\theta)}\, N_3\, d\theta$$
$$= N_3 \int_0^\infty \theta^{-np-d+1}\, e^{-(v/\theta)}\, d\theta$$

Finally;

$$E(\theta^2) = \frac{v^2}{(np+d-2)(np+d-1)} \tag{2.26}$$

Putting these values in the expression the estimate of variance can be calculated as,

$$V^*(\theta) = \frac{v^2}{(np+d-2)(np+d-1)} - \left[\frac{v}{(np+d-1)}\right]^2$$

$$V^*(\theta) = \frac{v^2}{(np+d-2)(np+d-1)^2} \tag{2.27}$$

The Bayes estimate of reliability function,

$$R^*(t) = \int_0^\infty [1-(t/\theta)^p]\, \theta^{-np-d-1}\, e^{-(v/\theta)}\, N_3\, d\theta$$
$$= 1 - t^p\, N_3 \int_0^\infty \theta^{-(n+1)p-d-1}\, e^{-(v/\theta)}\, d\theta$$
$$= 1 - t^p\, N_3\, \frac{\Gamma[(n+1)p+d]}{v^{(n+1)p+d}}$$

That has the following form:

$$R*(t) = 1 - t^p v^p \frac{\Gamma[(n+1)p+d]}{\Gamma(np+d)} \tag{2.28}$$

The Bayes estimate of hazard rate function,

$$\begin{aligned} h^*(t) &= \int_0^\infty \frac{pt^{p-1}}{\theta^p - t^p}\, \theta^{-np-d-1}\, e^{-(v/\theta)}\, N_3\, d\theta \\ &= pt^{p-1} N_3 \int_0^\infty \theta^{-(n+1)p-d-1}\, e^{-(v/\theta)}\, \Sigma_{k=0}^{\infty}\, (t/\theta)^{pk}\, d\theta \\ &= pt^{p-1} N_3\, \Sigma_{k=0}^{\infty}\, t^{pk} \int_0^\infty \theta^{-(n+k+1)p-d-1}\, e^{-(v/\theta)} d\theta \end{aligned}$$

Finally;

$$h * (t) = pt^{p-1} \Sigma_{k=0}^{\infty} t^{pk} v^{(k+1)p} . \frac{\Gamma[(n+k+1)p+d]}{\Gamma(np+d)} \tag{2.29}$$

and also,

$$\begin{aligned} E * [h^2(t)] &= \int_0^\infty \left[\frac{pt^{p-1}}{\theta^p - t^p}\right]^2 \theta^{-np-d-1}\, e^{-(v/\theta)} N_3\, d\theta \\ &= p^2 t^{2(p-1)}\, N_3 \int_0^\infty \theta^{-(n+2)p-d-1}\, e^{-(v/\theta)} \Sigma_{k=0}^{\infty}\, (k+1)\, (t/\theta)^{pk}\, d\theta \end{aligned}$$

Solving the above expression, we get;

$$E^*[h^2(t)] = p^2 t^{2(p-1)}\, \Sigma_{k=0}^{\infty}\, (k+1)\, t^{pk}\, v^{(k+2)p} \frac{\Gamma[(n+k+2)p+d]}{\Gamma(np+d)} \tag{2.30}$$

These equations will help in giving Bayes estimate of the variance of hard rate function on using the following expression

$$V^*[h(t)] = E(h^2(t)] - [h^*(t)]^2$$

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