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Differential equations associated with higher-order Bernoulli numbers of the second kind

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Abstract

The purpose of this paper is to derive differential equations associated with the generating function of higher-order Bernoulli numbers of the second kind. In addition, we find some new and interesting identities involving those numbers arising from our differential equations.

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1. Introduction

For a given positive integer r, the *Bernoulli polynomials of order* r are given by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \text{ (see [1, 2, 8, 10]).}$$

In the special case, for $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the *Bernoulli numbers of order* r.

For a given positive integer r, the *Bernoulli polynomials of the second kind of order* r are defined by the generating function

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!}.$$

When $x = 0$, $b_n^{(r)} = b_n^{(r)}(0)$ are called the *Bernoulli numbers of the second kind of order* r (see [3, 9, 11]).

In [4, 5], authors introduced new methods of using diffferential equations in order to obtain some interesting identities related to Bernoulli numbers of the second kind and Frobenius-Euler numbers of higher order. This idea of using differential equations turned out to be very useful tools for studying special polynomials and mathematical physics (see [3, 4, 5, 6, 7]).

In this paper, we derive differential equations associated with the generating function of higher-order Bernoulli numbers of the second kind. In addition, we find some new and interesting identities involving those numbers arising from our differential equations.

2. Some identities of higher-order Bernoulli numbers of second kind arising from nonlinear differential equations

Throughout this paper, all derivatives will be taken with respect to t.

For $r \in \mathbb{Z}_{>0}$, we let

$$F = F(t,r) = \left(\frac{1}{\log(1+t)}\right)^r = (\log(1+t))^{-r}. \tag{2.1}$$

Then, by (2.1), we get

$$F^{(1)}(t,r) = -r(1+t)^{-1}F(t,r+1), \tag{2.2}$$

$$F^{(2)}(t,r) = r(1+t)^{-2}F(t,r+1) + <r>_2 (1+t)^{-2}F(t,r+2), \tag{2.3}$$

$$\begin{aligned} F^{(3)}(t,r) = -2r(1+t)^{-3}F(t,r+1) - 3 <r>_2 (1+t)^{-3}F(t,r+2) \\ - <r>_3 (1+t)^{-3}F(t,r+3). \end{aligned} \tag{2.4}$$

where $< x >_n = x(x+1)\cdots(x+n-1)$, $(n \geq 1)$, and $< x >_0 = 1$.

So, we are led to put

$$F^{(N)}(t,r) = (-1)^N (1+t)^{-N} \sum_{i=1}^{N} a_i(N) < r >_i F(t, r+i), \tag{2.5}$$

where $N = 1, 2, \ldots$.

Taking the derivative with respect to t of (2.5), we have

$$\begin{aligned} F^{(N+1)}(t,r) &= (-1)^N(-N)(1+t)^{-N-1}\sum_{i=1}^{N} a_i(N) < r >_i F(t,r+i) \\ &\quad + (-1)^N (1+t)^{-N} \sum_{i=1}^{N} a_i(N) < r >_i F^{(1)}(t,r+i) \\ &= N(-1)^{N+1}(1+t)^{-N-1}\sum_{i=1}^{N} a_i(N) < r >_i F(t,r+i) \\ &\quad + (-1)^N (1+t)^{-N} \sum_{i=1}^{N} a_i(N) < r >_i \\ &\quad \times (-(r+i))(1+t)^{-1} F(t,r+i+1) \\ &= N(-1)^{N+1}(1+t)^{-N-1}\sum_{i=1}^{N} a_i(N) < r >_i F(t,r+i) \\ &\quad + (-1)^{N+1}(1+t)^{-N-1}\sum_{i=1}^{N} a_i(N) < r >_{i+1} F(t,r+i+1) \\ &= (-1)^{N+1}(1+t)^{-N-1}\sum_{i=1}^{N} N a_i(N) < r >_i F(t,r+i) \\ &\quad + (-1)^{N+1}(1+t)^{-N-1}\sum_{i=2}^{N+1} a_{i-1}(N) < r >_i F(t,r+i). \end{aligned} \tag{2.6}$$

As initial conditions, we have

$$\begin{aligned} F^{(1)}(t,r) &= -(1+t)^{-1} r a_1(1) F(t,r+1) \\ &= -r(1+t)^{-1} F(t,r+1). \end{aligned} \tag{2.7}$$

Thus, by (2.7), we get $a_1(1) = 1$.

$$\begin{aligned} F^{(2)}(t,r) &= (1+t)^{-2}\left\{a_1(2) r F(t,r+1) + a_2(2) < r >_2 F(t,r+2)\right\} \\ &= (1+t)^{-2}\left\{r F(t,r+1) + < r >_2 F(t,r+2)\right\}. \end{aligned} \tag{2.8}$$

Thus, by (2.8), we get

$$a_1(2) = a_2(2) = 1. \tag{2.9}$$

Also, comparing the above, we obtain

$$a_1(N+1) = Na_1(N),\ a_{N+1}(N+1) = a_N(N), \tag{2.10}$$

and

$$a_i(N+1) = a_{i-1}(N) + Na_i(N),\ (2 \leq i \leq N). \tag{2.11}$$

From (2.10), we have

$$\begin{aligned} a_1(N+1) &= Na_1(N) = N(N-1)a_1(N-1) = \cdots \\ &= N(N-1)\cdots 2a_1(2) \\ &= N! \end{aligned} \tag{2.12}$$

and

$$a_{N+1}(N+1) = a_N(N) = \cdots = a_1(1) = 1. \tag{2.13}$$

For $i = 2$ in (2.11), we have

$$\begin{aligned} a_2(N+1) &= a_1(N) + Na_2(N) \\ &= a_1(N) + N\left(a_1(N-1) + (N-1)a_2(N-1)\right) \\ &= a_1(N) + Na_1(N-1) + (N)_2 a_2(N-1) \\ &= a_1(N) + Na_1(N-1) + (N)_2\left(a_1(N-2) + (N-2)a_2(N-2)\right) \\ &= a_1(N) + Na_1(N-1) + (N)_2 a_1(N-2) + (N)_3 a_2(N-2) \\ &= \cdots \\ &= \sum_{k=0}^{N-2} (N)_k a_1(N-k) + (N)_{k-1} a_2(2) \\ &= \sum_{k=0}^{N-1} (N)_k a_1(N-k), \end{aligned} \tag{2.14}$$

where $(x)_r = x(x-1)\cdots(x-r+1)$, $(r \geq 1)$, and $(x)_0 = 1$.

Similarly to the $i = 2$ case, for $i = 3, 4$ in (2.11), we have

$$a_3(N+1) = \sum_{k=0}^{N-2} (N)_k a_2(N-k). \tag{2.15}$$

$$a_4(N+1) = \sum_{k=0}^{N-3} (N)_k a_3(N-k). \tag{2.16}$$

Thus, we can deduce that, for $2 \le i \le N$,

$$a_i(N+1) = \sum_{k=0}^{N-i+1} (N)_k a_{i-1}(N-k). \tag{2.17}$$

Now, we give explicit expressions for $a_i(j)$.

From (2.12) and (2.14), we have

$$\begin{aligned} a_2(N+1) &= \sum_{k=0}^{N-1} (N)_k a_1(N-k) \\ &= \sum_{k=0}^{N-1} (N)_k (n-k-1)! \\ &= N! \sum_{k=0}^{N-1} \frac{1}{N-k} \\ &= N!\left(\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}\right) \\ &= N! H_N, \end{aligned} \tag{2.18}$$

$$\begin{aligned} a_3(N+1) &= \sum_{k=0}^{N-2} (N)_k a_2(N-k) \\ &= \sum_{k=0}^{N-2} (N)_k (n-k-1)! H_{N-k-1} \\ &= N! \sum_{k=0}^{N-2} \frac{H_{N-k-1}}{N-k} \\ &= N!\left(\frac{H_{N-1,1}}{N} + \frac{H_{N-2,1}}{N-1} + \cdots + \frac{H_{1,1}}{2}\right) \\ &= N! H_{N,2}, \end{aligned} \tag{2.19}$$

where

$$H_{N,j} = \frac{H_{N-1,j-1}}{N} + \cdots + \frac{H_{j-1,j-1}}{j}, \ (2 \le j \le N),$$

and

$$H_{N,1} = H_N = \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}.$$

Thus, we deduce that for $2 \le i \le N$,

$$a_i(N+1) = N! H_{N,i-1}. \tag{2.20}$$

Note that

$$H_{N,N} = \frac{1}{N!}.$$

So, (2.20) also valid for $i = N + 1$. We also define $H_{N,0} = 1$ for all N. Now, we have the following theorem.

Theorem 2.1. The following family of differential equations

$$\begin{aligned} F^{(N)} &= (-1)^N (1+t)^{-N} (N-1)! \\ &\quad \times \sum_{i=1}^{N} H_{N-1,i-1} < r >_i (\log(1+t))^{-i} F, \ (N = 1, 2, \ldots) \end{aligned} \tag{2.21}$$

have a solution

$$F = F(t, r) = \left(\frac{1}{\log(1+t)}\right)^r,$$

where $H_{N,0} = 1$, for all N,

$$\begin{aligned} H_{N,1} = H_N &= \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}, \\ H_{N,j} &= \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{j-1,j-1}}{j}, \ (2 \le j \le N). \end{aligned}$$

3. Applications

Recall that the *Bernoulli numbers of the second kind* $b_k^{(r)}$ *of order* r are defined by the generating function

$$\left(\frac{t}{\log(1+t)}\right)^r = \sum_{k=0}^{\infty} b_k^{(r)} \frac{t^k}{k!}. \tag{3.1}$$

From Theorem 2.1, we note that

$$\begin{aligned} &\left(\frac{d}{dt}\right)^N \left(\frac{1}{\log(1+t)}\right)^r \\ &= (-1)^N (1+t)^{-N} (N-1)! \sum_{i=1}^{N} H_{N-1,i-1} < r >_i \left(\frac{1}{\log(1+t)}\right)^{r+i}. \end{aligned} \tag{3.2}$$

Now, we observe that

$$\begin{aligned} \left(\frac{1}{\log(1+t)}\right)^r &= \frac{1}{t^r} \sum_{k=0}^{\infty} b_k^{(r)} \frac{t^k}{k!} = \sum_{k=0}^{r-1} b_k^{(r)} \frac{t^{k-r}}{k!} + \sum_{k=r}^{\infty} b_k^{(r)} \frac{t^{k-r}}{k!} \\ &= \sum_{k=-r}^{-1} b_{k+r}^{(r)} \frac{t^k}{(k+r)!} + \sum_{k=0}^{\infty} b_{k+r}^{(r)} \frac{t^k}{(k+r)!}. \end{aligned} \tag{3.3}$$

Thus, by (3.3), we get

$$\left(\frac{d}{dt}\right)^N\left(\frac{1}{\log(1+t)}\right)^r=\sum_{k=-r}^{-1}b_{k+r}^{(r)}(k)_N\frac{t^{k-N}}{(k+r)!}+\sum_{k=N}^{\infty}b_{k+r}^{(r)}(k)_N\frac{t^{k-N}}{(k+r)!}. \quad (3.4)$$

From (3.4), we note that

$$\begin{aligned}&t^{r+N}\left(\frac{d}{dt}\right)^N\left(\frac{1}{\log(1+t)}\right)^r\\&=\sum_{k=-r}^{-1}b_{k+r}^{(r)}(k)_N\frac{t^{k+r}}{(k+r)!}+\sum_{k=N}^{\infty}b_{k+r}^{(r)}(k)_N\frac{t^{k+r}}{(k+r)!}\\&=\sum_{k=0}^{r-1}b_k^{(r)}(k-r)_N\frac{t^k}{k!}+\sum_{k=N+r}^{\infty}b_k^{(r)}(k-r)_N\frac{t^k}{k!}.\end{aligned} \quad (3.5)$$

By (3.2), we get

$$\begin{aligned}&t^{r+N}\left(\frac{d}{dt}\right)^N\left(\frac{1}{\log(1+t)}\right)^r\\&=(-1)^N(1+t)^{-N}(N-1)!\sum_{i=1}^{N}H_{N-1,i-1}<r>_i\,t^{N-i}\left(\frac{t}{\log(1+t)}\right)^{r+i}.\end{aligned} \quad (3.6)$$

Now,

$$\begin{aligned}&(-1)^N(1+t)^{-N}(N-1)!\sum_{i=1}^{N}H_{N-1,i-1}<r>_i\,t^{N-i}\left(\frac{t}{\log(1+t)}\right)^{r+i}\\&=(-1)^N(N-1)!\sum_{i=1}^{N}H_{N-1,i-1}<r>_i\,t^{N-i}\sum_{l=0}^{\infty}(-1)^l(N+l-1)_l\frac{t^l}{l!}\\&\quad\times\sum_{m=0}^{\infty}b_m^{(r+i)}\frac{t^m}{m!}\\&=(-1)^N(N-1)!\sum_{i=1}^{N}H_{N-1,i-1}<r>_i\,t^{N-i}\sum_{s=0}^{\infty}\sum_{l=0}^{s}\binom{s}{l}(-1)^l(N+l-1)_l b_{s-l}^{(r+i)}\frac{1}{s!}t^s\\&=(-1)^N(N-1)!\sum_{i=1}^{N}H_{N-1,i-1}<r>_i\\&\quad\times\sum_{k=N-i}^{\infty}\sum_{l=0}^{k+i-N}\binom{k+i-N}{l}(-1)^l(N+l-1)_l b_{k+i-N-l}^{(r+i)}\frac{t^k}{(k+i-N)!}\end{aligned}$$

$$
\begin{aligned}
&= (-1)^N (N-1)! \sum_{i=1}^{N} \sum_{k=N-i}^{\infty} \sum_{l=0}^{k+i-N} H_{N-1,i-1} < r >_i \\
&\quad \times \binom{k+i-N}{l} (-1)^l (N+l-1)_l b_{k+i-N-l}^{(r+i)} \frac{t^k}{(k+i-N)!} \\
&= (-1)^N (N-1)! \sum_{k=0}^{\infty} \sum_{i=\max\{N-k,1\}}^{N} \sum_{l=0}^{k+i-N} H_{N-1,i-1} < r >_i \\
&\quad \times \binom{k+i-N}{l} (-1)^l (N+l-1)_l b_{k+i-N-l}^{(r+i)} \frac{k!}{(k+i-N)!} \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} (-1)^N (N-1)! k! \sum_{i=\max\{N-k,1\}}^{N} \sum_{l=0}^{k+i-N} (-1)^l < r >_i \\
&\quad \times \binom{N+l-1}{l} \frac{1}{(k+i-N-l)!} H_{N-1,i-1} b_{k+i-N-l}^{(r+i)} \frac{t^k}{k!}.
\end{aligned} \tag{3.7}
$$

Therefore, by (3.5), (3.6) and (3.7), we obtain the following theorem.

Theorem 3.1. For $k = 0, 1, 2, \ldots$ and $N = 1, 2, 3, \ldots$, we have the following:

(a) for k with $0 \le k \le r-1$ or $N+r \le k$,

$$
\begin{aligned}
b_k^{(r)} =& \frac{(-1)^N (N-1)! k!}{(k-r)_N} \sum_{i=\max\{N-k,1\}}^{N} \sum_{l=0}^{k+i-N} (-1)^l < r >_i \binom{N+l-1}{l} \\
& \times \frac{1}{(k+i-N-l)!} H_{N-1,i-1} b_{k+i-N-l}^{(r+i)}.
\end{aligned}
$$

(b) for $r \le k \le N+r-1$, we have

$$
\begin{aligned}
& \sum_{i=\max\{N-k,1\}}^{N} \sum_{l=0}^{k+i-N} (-1)^l < r >_i \binom{N+l-1}{l} \\
& \times \frac{1}{(k+i-N-l)!} H_{N-1,i-1} b_{k+i-N-l}^{(r+i)} = 0.
\end{aligned}
$$

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