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Common Fixed Points of Two Maps in Cone Pentagonal Metric Spaces

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Abstract

In this paper, we prove existence of common fixed points for a pair of self mappings in non-normal cone pentagonal metric spaces. Our results extend and improve the recent results of Azam et al. [Banach contraction principle on cone rectangular metric spaces, Applicable Analysis and Discrete Mathematics, 3(2), 236–241, 2009], Rashwan and Saleh [Some Fixed Point Theorems in Cone Rectangular Metric Spaces, Mathematica Aeterna, 2(6): 573–587, 2012], Garg and Agarwal, [Banach Contraction Principle on Cone Pentagonal Metric Space, Journal of Advanced Studies in Topology, 3(1), 12–18, 2012], and others.

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1. Introduction

Banach [5] introduced the concept of classical Banach contraction principle. Due to wide applications of this concept, the study of existence and uniqueness of fixed points of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the Banach contraction principle in various generalized metric spaces (e.g., see [4, 6, 7, 9]).

Long-Guang and Xian [9] introduced the concept of a cone metric space, they replaced the set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have (for e.g., [1, 8, 11]) proved some fixed point theorems for different contractive types conditions in cone metric spaces.

Azam et al. [4] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a normal cone rectangular metric space setting. Rashwan and Saleh [10] extended and improved the result of Azam et al. [4] by omitting the assumption of normality condition.

Recently, Garg and Agarwal [6] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting.

Very recently, Auwalu [2] studied common fixed point of a self mapping in cone pentagonal metric space and proved Banach fixed point theory in a cone pentagonal metric space setting by removing the normality condition of the paper [6].

Motivated and inspired by the results of [2, 6, 10], it is our purpose in this paper to continue the study of common fixed points of a pair of self mappings in non-normal cone pentagonal metric space setting. Our results extend and improve the results of [2, 4, 6, 10], and others.

2. Preliminaries

The following definitions and Lemmas are needed in the sequel.

Definition 2.1. [9] Let E be a real Banach space and P subset of E. P is called a cone if and only if:

(1) P is closed, nonempty, and $P \neq \{0\}$;

(2) $a, b \in \mathbb{R}, \;\; a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$;

(3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

A cone P is called normal if there is a number $k \geq 1$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Longrightarrow \|x\| \leq k\|y\|. \tag{1}$$

The least positive number k satisfying (1) is called the normal constant of P.

In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.2. [9] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, \infty)$ (e.g., see [9]).

Definition 2.3. [4] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$ for all $x, y, z \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [rectangular property].

Then ρ is called a cone rectangular metric on X, and (X, ρ) is called a cone rectangular metric space.

Remark 2.4. Every cone metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [4]).

Definition 2.5. [6] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u, \in X - \{x, y\}$ [pentagonal property].

Then d is called a cone pentagonal metric on X, and (X, d) is called a cone pentagonal metric space.

Remark 2.6. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [6]).

Let (X, d) be a cone pentagonal metric space. Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ and that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone pentagonal metric space.

Definition 2.7. [10] Let P be a cone defined as above and let Φ be the set of non decreasing continuous functions $\varphi : P \to P$ satisfying:

1. $0 < \varphi(t) < t$ for all $t \in P \setminus \{0\}$,
2. the series $\sum_{n\geq 0} \varphi^n(t)$ converge for all $t \in P \setminus \{0\}$.

From (1), we have $\varphi(0) = 0$, and from (2), we have $\lim_{n\to 0} \varphi^n(t) = 0$ for all $t \in P \setminus \{0\}$.

Let T and S be self maps of a nonempty set X. If $w = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S. Also, T and S are said to be weakly compatible if they commute at their coincidence points, that is, $Tx = Sx$ implies that $TSx = STx$.

Lemma 2.8. [1] Let T and S be weakly compatible self mappings of nonempty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.

Lemma 2.9. [3, 12] Let (X, d) be a complete cone pentagonal metric space. Let $\{x_n\}$ be a Cauchy sequence in X and suppose that there is natural number N such that:

1. $x_n \neq x_m$ for all $n, m > N$;
2. x_n, x are distinct points in X for all $n > N$;
3. x_n, y are distinct points in X for all $n > N$;
4. $x_n \to x$ and $x_n \to y$ as $n \to \infty$.

Then $x = y$.

3. Main Results

In this section, we derive the main results of our work, which is an extension of Banach contraction principle in cone pentagonal metric space to a pair of two self mappings. We give an example to illustrate the result.

Theorem 3.1. Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:

$$d(Sx, Sy) \leq \varphi\big(d(fx, fy)\big), \tag{2}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \subseteq f(X)$ and $f(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and f have a unique point of coincidence in X. Moreover, if S and f are weakly compatible then S and f have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $S(X) \subseteq f(X)$, we can choose $x_1 \in X$ such that $Sx_0 = fx_1$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} such that

$$Sx_n = fx_{n+1}, \text{ for all } n = 0, 1, 2, \ldots.$$

We assume that $Sx_n \neq Sx_{n-1}$ for all $n \in \mathbb{N}$. Then from (2), it follows that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq \varphi\big(d(fx_n, fx_{n+1})\big) = \varphi\big(d(Sx_{n-1}, Sx_n)\big) \\ &\leq \varphi^2\big(d(fx_{n-1}, fx_n)\big) \\ &\vdots \\ &\leq \varphi^n\big(d(Sx_0, Sx_1)\big). \end{aligned} \tag{3}$$

In similar way, it again follows that

$$d(Sx_n, Sx_{n+2}) \leq \varphi^n\big(d(Sx_0, Sx_2)\big), \tag{4}$$

$$d(Sx_n, Sx_{n+3}) \leq \varphi^n\big(d(Sx_0, Sx_3)\big). \tag{5}$$

Similarly for $k = 1, 2, 3, \ldots$, it further follows that

$$d(Sx_n, Sx_{n+3k+1}) \leq \varphi^n\big(d(Sx_0, Sx_{3k+1})\big), \tag{6}$$

$$d(Sx_n, Sx_{n+3k+2}) \leq \varphi^n\big(d(Sx_0, Sx_{3k+2})\big), \tag{7}$$

$$d(Sx_n, Sx_{n+3k+3}) \leq \varphi^n\big(d(Sx_0, Sx_{3k+3})\big). \tag{8}$$

By pentagonal property and (3), we have

$$\begin{aligned} d(Sx_0, Sx_4) &\leq d(Sx_0, Sx_1) + d(Sx_1, Sx_2) + d(Sx_2, Sx_3) + d(Sx_3, Sx_4) \\ &\leq d(Sx_0, Sx_1) + \varphi\big(d(Sx_0, Sx_1)\big) + \varphi^2\big(d(Sx_0, Sx_1)\big) + \varphi^3\big(d(Sx_0, Sx_1)\big) \\ &\leq \sum_{i=0}^{3} \varphi^i\big(d(Sx_0, Sx_1)\big), \end{aligned}$$

and

$$\begin{aligned} d(Sx_0, Sx_7) &\leq d(Sx_0, Sx_1) + d(Sx_1, Sx_2) + d(Sx_2, Sx_3) + d(Sx_3, Sx_4) \\ &\quad + d(Sx_4, Sx_5) + d(Sx_5, Sx_6) + d(Sx_6, Sx_7) \\ &\leq \sum_{i=0}^{6} \varphi^i\big(d(Sx_0, Sx_1)\big). \end{aligned}$$

By induction, we have for each $k = 1, 2, 3, \ldots$

$$d(Sx_0, Sx_{3k+1}) \leq \sum_{i=0}^{3k} \varphi^i\big(d(Sx_0, Sx_1)\big). \tag{9}$$

Also using (3), (4) and pentagonal property, we have that

$$d(Sx_0, Sx_5) \leq \sum_{i=0}^{2} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^3\big(d(Sx_0, Sx_2)\big),$$

and

$$d(Sx_0, Sx_8) \leq \sum_{i=0}^{5} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^6\big(d(Sx_0, Sx_2)\big).$$

By induction, we have for each $k = 1, 2, 3, \ldots$

$$d(Sx_0, Sx_{3k+2}) \leq \sum_{i=0}^{3k-1} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^{3k}\big(d(Sx_0, Sx_2)\big). \tag{10}$$

Again using (3), (5) and pentagonal property, we have that

$$d(Sx_0, Sx_6) \leq \sum_{i=0}^{2} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^3\big(d(Sx_0, Sx_3)\big),$$

and

$$d(Sx_0, Sx_9) \leq \sum_{i=0}^{5} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^6\big(d(Sx_0, Sx_3)\big).$$

By induction, we have for each $k = 1, 2, 3, \ldots$

$$d(Sx_0, Sx_{3k+3}) \leq \sum_{i=0}^{3k-1} \varphi^i\big(d(Sx_0, Sx_1)\big) + \varphi^{3k}\big(d(Sx_0, Sx_3)\big). \tag{11}$$

Using (6) and (9), for $k = 1, 2, 3, \ldots,$ we have

$$\begin{aligned} d(Sx_n, Sx_{n+3k+1}) &\le \varphi^n \sum_{i=0}^{3k} \varphi^i \big(d(Sx_0, Sx_1)\big) \\ &\le \varphi^n \Big[\sum_{i=0}^{3k} \varphi^i \big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big)\Big] \\ &\le \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i \big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big)\Big]. \end{aligned} \tag{12}$$

Similarly for $k = 1, 2, 3, \ldots,$ (7) and (10) implies that

$$\begin{aligned} d(Sx_n, Sx_{n+3k+2}) &\le \varphi^n \Big[\sum_{i=0}^{3k-1} \varphi^i \big(d(Sx_0, Sx_1)\big) + \varphi^{3k}\big(d(Sx_0, Sx_2)\big)\Big] \\ &\le \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i \big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big)\Big]. \end{aligned} \tag{13}$$

Again, for $k = 1, 2, 3, \ldots,$ (8) and (11) implies that

$$d(Sx_n, Sx_{n+3k+3}) \le \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i \big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big)\Big]. \tag{14}$$

Thus, by (12), (13) and (14), for each m, we have

$$d(Sx_n, Sx_{n+m}) \le \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i \big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big)\Big]. \tag{15}$$

Since $\sum_{i=0}^{\infty} \varphi^i \big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big)$ converges (by definition 2.7), where $d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3) \in P \setminus \{0\}$, and P is closed, then $\sum_{i=0}^{\infty} \varphi^i \big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big) \in P \setminus \{0\}$. Hence

$$\lim_{n\to\infty} \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i \big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big)\Big] = 0.$$

Then for given $c \gg 0$, there is a natural number N_1 such that

$$\varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i \big(d(Sx_0, Sx_1) + d(Sx_0, Sx_2) + d(Sx_0, Sx_3)\big)\Big] \ll c, \quad \forall n \ge N_1. \tag{16}$$

Thus, from (15) and (16), we have

$$d(Sx_n, Sx_{n+m}) \ll c, \text{ for all } n \geq N_1.$$

Therefore $\{Sx_n\}$ is a Cauchy sequence in X. Suppose $S(X)$ is a complete subspace of X, then there exists a point $z \in S(X)$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_{n+1} = z$. Also, we can find a point $y \in X$ such that $fy = z$.

We show that $Sy = z$. Given $c \gg 0$, we choose a natural number N_2, N_3 such that $d(z, fx_n) \ll \frac{c}{4}, \ \forall n \geq N_2$, and $d(Sx_n, Sx_{n-1}) \ll \frac{c}{4}, \ \forall n \geq N_3$. Since $x_n \neq x_m$ for $n \neq m$, by pentagonal property we have that

$$\begin{aligned} d(Sy, z) &\leq d(Sy, Sx_n) + d(Sx_n, fx_n) + d(fx_n, fx_{n-1}) + d(fx_{n-1}, z) \\ &\leq \varphi\big(d(fy, fx_n)\big) + d(Sx_n, Sx_{n-1}) + d(Sx_{n-1}, Sx_{n-2}) + d(fx_{n-1}, z) \\ &< d(z, fx_n) + d(Sx_n, Sx_{n-1}) + d(Sx_{n-1}, Sx_{n-2}) + d(fx_{n-1}, z) \\ &\ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c, \text{ for all } n \geq N, \end{aligned}$$

where $N := \max\{N_2, N_3\}$. Since c is arbitrary, we have $d(Sy, z) \ll \frac{c}{m}, \ \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(Sy, z) \to -d(Sy, z)$ as $m \to \infty$. Since P is closed, $-d(Sy, z) \in P$. Hence $d(Sy, z) \in P \cap -P$. By definition of cone we get that $d(Sy, z) = 0$, and so $Sy = fy = z$. Hence, z is a point of coincidence of S and f.

Next, we show that z is unique. For suppose z' be another point of coincidence of S and f, that is $Sx = fx = z'$, for some $x \in X$, then

$$d(z, z') = d(Sy, Sx) \leq \varphi\big(d(fy, fx)\big) = \varphi\big(d(z, z')\big) < d(z, z').$$

Hence $z = z'$. Since S and f are weakly compatible, by Lemma 2.16, z is the unique common fixed point of S and f. This completes the proof of the theorem. ■

Corollary 3.2. Let (X, d) be a cone pentagonal metric space and P be a normal cone with normal constant k. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:

$$d(Sx, Sy) \leq \lambda d(fx, fy),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Suppose that $S(X) \subseteq f(X)$ and $f(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and f have a unique point of coincidence in X. Moreover, if S and f are weakly compatible then S and f have a unique common fixed point in X.

Proof. Define $\varphi : P \to P$ by $\varphi(t) = \lambda t$. Then it is clear that φ satisfies the conditions in definition 2.7. Hence the results follows from Theorem 3.1. ■

Corollary 3.3. (see [10]) Let (X, d) be a cone rectangular metric space. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:

$$d(Sx, Sy) \leq \varphi\big(d(fx, fy)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \subseteq f(X)$ and $f(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and f have a unique point of coincidence in X. Moreover, if S and f are weakly compatible then S and f have a unique common fixed point in X.

Proof. This follows from the Remark 2.6 and Theorem 3.1. ■

Corollary 3.4. (see [2]) Let (X, d) be a cone pentagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:

$$d(Sx, Sy) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. Putting $f = I$ in Theorem 3.1, where I is the identity mapping. This completes the proof. ■

Corollary 3.5. (see [10]) Let (X, d) be a cone rectangular metric space. Suppose the mapping $S : X \to X$ satisfy the following:

$$d(Sx, Sy) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.6 and Corollary 3.4. ■

Corollary 3.6. (see [6]) Let (X, d) be a cone pentagonal metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfies the contractive condition:

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. Putting $f = I$ in Corollary 3.2, where I is the identity mapping. This completes the proof. ■

Corollary 3.7. (see [4])Let (X, d) be a cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfies:

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.6 and Corollary 3.6. ■

Theorem 3.8. Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, f : X \to X$ satisfy the contractive condition:

$$d(Sfx, Sfy) \leq \varphi\big(d(Sx, Sy)\big), \tag{17}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that S is one to one, $S(X)$ is a complete subspace of X, then the mapping f have a unique fixed point in X. Moreover, if S and f are commuting at the fixed point of f, then S and f have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n, \text{ for all } n = 0, 1, 2, \ldots.$$

We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Then, from (17), it follows that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &= d(Sfx_{n-1}, Sfx_n) \\ &\leq \varphi\big(d(Sx_{n-1}, Sx_n)\big) \\ &\leq \varphi^2\big(d(Sx_{n-2}, Sx_{n-1})\big) \\ &\;\;\vdots \\ &\leq \varphi^n\big(d(Sx_0, Sx_1)\big). \end{aligned} \tag{18}$$

In similar way, it again follows that

$$d(Sx_n, Sx_{n+2}) \leq \varphi^n\big(d(Sx_0, Sx_2)\big), \tag{19}$$

and

$$d(Sx_n, Sx_{n+3}) \leq \varphi^n\big(d(Sx_0, Sx_3)\big). \tag{20}$$

Similarly for $k = 1, 2, 3, \ldots,$ It further follows that

$$d(Sx_n, Sx_{n+3k+1}) \leq \varphi^n\big(d(Sx_0, Sx_{3k+1})\big), \tag{21}$$

$$d(Sx_n, Sx_{n+3k+2}) \leq \varphi^n\big(d(Sx_0, Sx_{3k+2})\big), \tag{22}$$

$$d(Sx_n, Sx_{n+3k+3}) \leq \varphi^n\big(d(Sx_0, Sx_{3k+3})\big). \tag{23}$$

Using the same argument in the proof of Theorem 3.1, we can show that $\{Sx_n\}$ is a Cauchy sequence in X.

Since $S(X)$ is a complete subspace of X, then there exists a point $z \in S(X)$ such that $\lim_{n\to\infty} Sx_{n+1} = \lim_{n\to\infty} Sfx_n = z$. Also, we can find a point $y \in X$ such that $Sy = z$.

We show that $Sfy = Sy$. Given $c \gg 0$, we choose a natural number M_1, M_2 such that $d(z, Sx_n) \ll \frac{c}{4}, \;\; \forall n \geq M_1$ and $d(Sx_n, Sx_{n+1}) \ll \frac{c}{4}, \;\; \forall n \geq M_2$. Since $x_n \neq x_m$ for $n \neq m$, by pentagonal property, we have that

$$\begin{aligned} d(Sy, Sfy) &\leq d(Sy, Sx_n) + d(Sx_n, Sfx_n) + d(Sfx_n, Sfx_{n+1}) + d(Sfx_{n+1}, Sfy) \\ &\leq d(z, Sx_n) + d(Sx_n, Sx_{n+1}) + \varphi\big(d(Sx_n, Sx_{n+1})\big) + \varphi\big(d(Sx_{n+1}, Sy)\big) \\ &< d(z, Sx_n) + d(Sx_n, Sx_{n+1}) + d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, z) \\ &\ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c, \text{ for all } n \geq M, \end{aligned}$$

where $M := \max\{M_1, M_2\}$. Since c is arbitrary, we have $d(Sy, Sfy) = 0$. Therefore, $Sy = Sfy = z$. Since S is one to one, $y = fy$. Hence, y is a fixed point of f.

Next, we show that y is unique. For suppose y' be another fixed point of f, that is $fy' = y'$, then

$$d(Sy, Sy') = d(Sfy, Sfy') \leq \varphi\big(d(Sy, Sy')\big) < d(Sy, Sy').$$

Hence $Sy = Sy'$. Since S is one to one, we conclude that $y = y'$.

Since S and f are commuting at the fixed point of f, $Sfy = fSy = Sy$. Therefore Sy is a fixed point of f. Since f has a unique fixed point, we have $Sy = y$. Hence $Sy = fy = y$. This completes the proof of the theorem. ■

Corollary 3.9. (see [2]) Let (X, d) be a cone pentagonal metric space. Suppose the mapping $f : X \to X$ satisfy the following:

$$d(fx, fy) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then f has a unique fixed point in X.

Proof. Putting $S = I$ in Theorem 3.8, where I is the identity mapping. This completes the proof. ■

Corollary 3.10. (see [10]) Let (X, d) be a cone rectangular metric space. Suppose the mappings $S, f : X \to X$ satisfies the contractive condition:

$$d(Sfx, Sfy) \leq \varphi\big(d(Sx, Sy)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that S is one to one, $S(X)$ is a complete subspace of X, then the mapping f have a unique fixed point in X. Moreover, if S and f are commuting at the fixed point of f, then S and f have a unique common fixed point in X.

Proof. This follows from the Remark 2.6 and Theorem 3.8. ■

Example 3.11. Let $X = \{r, s, t, u, v\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a cone in E. Define $d : X \times X \to E$ as follows:

$$\begin{aligned}
d(x, x) &= 0, \forall x \in X;\\
d(r, s) &= d(s, r) = (4, 8);\\
d(r, t) &= d(t, r) = d(t, u) = d(u, t) = d(s, t)\\
&= d(t, s) = d(s, u) = d(u, s) = d(r, u) = d(u, r) = (1, 2);\\
d(r, v) &= d(v, r) = d(s, v) = d(v, s) = d(t, v)\\
&= d(v, t) = d(u, v) = d(v, u) = (3, 6).
\end{aligned}$$

Then (X, d) is a complete cone pentagonal metric space, but (X, d) is not a complete cone rectangular metric space because it lacks the rectangular property:

$$\begin{aligned}(4, 8) = d(r, s) &> d(r, t) + d(t, u) + d(u, s)\\ &= (1, 2) + (1, 2) + (1, 2)\\ &= (3, 6), \text{ as } (4, 8) - (3, 6) = (1, 2) \in P.\end{aligned}$$

Now, we define a mapping $S, f : X \to X$ as follows

$$S(x) = \begin{cases} u, & \text{if } x \neq v;\\ s, & \text{if } x = v.\end{cases}$$

$$f(x) = \begin{cases} t, & \text{if } x = r;\\ r, & \text{if } x = s;\\ s, & \text{if } x = t;\\ u, & \text{if } x = u;\\ v, & \text{if } x = v.\end{cases}$$

Clearly $S(X) \subseteq f(X)$, $f(X)$ is a complete subspace of X and the pairs (S, f) is weakly compatible. The inequality (2) holds for all $x, y \in X$, where $\varphi(t) = \frac{1}{3}t$ and $u \in X$ is the unique common fixed point of the mappings S and f.

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