Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 3 (2016), pp. 2385-2394

http://www.ripublication.com/gjpam.htm

On Disjoint Restrained Domination in Graphs[1]

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Abstract

Let G be a connected simple graph. A set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. Let D be a minimum restrained dominating set of G. A restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an *inverse restrained dominating set* of G with respect to D. The *inverse restrained domination number* of G denoted by $\gamma_r^{-1}(G)$ is the minimum cardinality of an inverse restrained dominating set of G. An inverse restrained dominating set of cardinality $\gamma_r^{-1}(G)$ is called γ_r^{-1}-set. A disjoint restrained dominating set of G is the set $C = D \cup S \subseteq V(G)$. The *disjoint restrained domination number* of G denoted by $\gamma\gamma_r(G)$ is the minimum cardinality of a disjoint restrained dominating set of G. A disjoint restrained dominating set of cardinality $\gamma\gamma_r(G)$ is called $\gamma\gamma_r$-set. In this paper, we show that every integers k and n with $2 \leq k \leq n$ is realizable as disjoint restrained domination number, and order of G respectively. Further, we give the characterization of the disjoint restrained dominating set and give some important results.

AMS subject classification: 05C69.
Keywords: Dominating set, restrained dominating set, inverse restrained dominating set, disjoint restrained dominating set.

[1]This research is partially funded by the Office of Research University of San Carlos, Cebu City, Philippines.

1. Introduction

The concept of domination in graphs introduced by Claude Berge in 1958 and Oystein Ore in 1962 [7] is currently receiving much attention in literature. Following the article of Ernie Cockayne and Stephen Hedetniemi [1], the domination in graphs became an area of study by many researchers. One type of domination parameter is the restrained domination number in a graph. This was introduced by Telle and Proskurowski [5] indirectly as a vertex partitioning problem. Restrained domination in graphs can be read in the paper of Domke et al. [4]. One practical application of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. To effect security, each prisoner's position is observed by a guard's position. To protect the rights of prisoners, each prisoner's position is seen by at least one other prisoner's position. To be cost effective, it is desirable to place a few guards as possible. The inverse domination in a graph was first found in the paper of Kulli [8] while Hedetmiemi et al. [6] introduced the concept of disjoint dominating sets in graphs. Moreover, for the general concepts not mentioned, readers may refer to [3].

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \varnothing$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v. The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{v \in X} N_G(v)$. The *closed neighborhood* of X in G is the set

$$N_G[X] = \bigcup_{v \in X} N_G[v] = N_G(X) \cup X.$$

When no confusion arises, $N_G[x]$ [resp. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$].

A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \backslash S$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. A set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a subset S of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. Let D be a minimum dominating set in G. The dominating set $S \subseteq V(G) \setminus D$ is called an *inverse dominating set* with respect to D. The minimum cardinality of inverse dominating set is called an *inverse domination number* of G and is denoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is called γ^{-1}-*set* of G. In [6], Hedetniemi et al. defined the disjoint domination as $\gamma\gamma(G) = min\{|S_1| + |S_2| : S_1$ and S_2 are disjoint dominating sets of $G\}$. The two disjoint dominating sets whose union has cardinality $\gamma\gamma(G)$ is a $\gamma\gamma$-*pair* of G.

The paper "Inverse restrained domination in graphs", is currently workout by Punzalan and Enriquez. Accordingly, if D is a minimum restrained dominating set in G, then a restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an *inverse restrained dominating set* of G with respect to D. The *inverse restrained domination number* of G denoted by $\gamma_r^{-1}(G)$ is the minimum cardinality of an inverse restrained dominating set of G. An inverse restrained dominating set of cardinality $\gamma_r^{-1}(G)$ is called γ_r^{-1}-*set*. Motivated by definition of inverse restrained dominating set and disjoint disjoint dominating set, we define the following variant of domination in graphs. A disjoint restrained dominating set of G is the set $C = D \cup S \subseteq V(G)$. The *disjoint restrained domination number* of G denoted by $\gamma\gamma_r(G)$ is the minimum cardinality of a disjoint restrained dominating set of G. A disjoint restrained dominating set of cardinality $\gamma\gamma_r(G)$ is called $\gamma\gamma_r$-*set* of G. Unless otherwise stated, all graphs in this paper are assumed to be simple and connected.

2. Results

One of the classical results in the domination theory which was introduced by Ore in 1962 state the following theorem:

Theorem 2.1. [7] Let G be a graph with no isolated vertex. If $S \subseteq V(G)$ is a γ-*set*, then $V(G) \setminus S$ is also a dominating set in G.

This motivate the introduction of a variant of domination in graphs, the disjoint restrained domination in graphs. Theorem 2.1 guarantees the existence of γ_r^{-1}-*set* and hence of $\gamma\gamma_r$-*set* in some graph G. Since $\gamma\gamma_r(G)$ does not always exists in a connected nontrivial graph G, we denote $\mathcal{RR}(G)$ a family of all graphs with disjoint restrained dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{RR}(\mathcal{G})$. From the definitions, the following result is immediate.

Remark 2.2. Let G be a connected graph of order $n \geq 3$. If D is a γ_r-*set* and S is a γ_r^{-1}-*set* of G, then $D \cap S = \varnothing$ and $C = D \cup S$ is a $\gamma\gamma_r$-*set* of G. Further, the set C need not be a restrained set.

Remark 2.3. Let G be a connected graph of order $n \geq 3$. Then

(i) $2 \leq \gamma\gamma_r(G) \leq n$, and

(ii) $\gamma(G) \leq \gamma_r(G) < \gamma\gamma_r(G)$.

The next result says that the value of the parameter $\gamma\gamma_r$ ranges over all positive integers except 1.

Theorem 2.4. **(Realization Problem)** Given positive integers k and n such that $n \geq 5$ and $2 \leq k \leq n$, there exists a connected nontrivial graph G with $|V(G)| = n$ and $\gamma\gamma_r(G) = k$.

Proof. Consider the following cases:

Case 1. Suppose $k = 2$.
Let $G = K_n$. Then, clearly, $|V(G)| = n$ and $\gamma\gamma_r(G) = 2$.

Case 2. Suppose $3 \leq k \leq n - 2$.
Let $H = K_r$ $(r \geq 3)$ and $P_m = [a_1, a_2, \ldots, a_m]$ $(m \geq 2)$. Consider the graph G obtained from H by adding the edges va_1, va_2, ..., and va_m (see Figure 1).

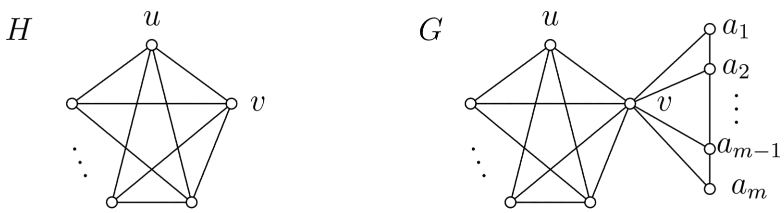

Figure 1: A graph G with $\gamma\gamma_r(G) = k$.

Let $n = m + r$. If $m = 3s - 1$ for some $s \in \mathbb{N}$, then let $k = (m + 7)/3$. The set $D = \{v\}$ is a γ_r-*set* of G and $S = \left\{a_{3j-1} : j = 1, 2, \ldots, \dfrac{m+1}{3}\right\} \cup \{u\}$ is a γ_r^{-1}-*set* of G. Thus, $C = D \cup S$ is a $\gamma\gamma_r$-*set* of G and hence $\gamma\gamma_r(G) = 1 + \left(\dfrac{m+1}{3} + 1\right) = k$. If $m = 3s + 1$ for some $s \in \mathbb{N}$, then let $k = (m + 8)/3$. The set $D = \{v\}$ is a γ_r-*set* of G and $S = \left\{a_{3j-2} : j = 1, 2, \ldots, \dfrac{m+2}{3}\right\} \cup \{u\}$ is a γ_r^{-1}-*set* of G. Thus, $C = D \cup S$ is a $\gamma\gamma_r$-*set* of G and hence $\gamma\gamma_r(G) = 1 + \left(\dfrac{m+2}{3} + 1\right) = k$. If $m = 3s$ for some $s \in \mathbb{N}$, then let $k = (m + 6)/3$. The set $D = \{v\}$ is a γ_r-*set* of G and $S = \left\{a_{3j-1} : j = 1, 2, \ldots, \dfrac{m}{3}\right\} \cup \{u\}$ is a γ_r^{-1}-*set* of G. Thus, $C = D \cup S$ is a $\gamma\gamma_r$-*set* of G and hence $\gamma\gamma_r(G) = 1 + \left(\dfrac{m}{3} + 1\right) = k$. Morover, $|V(G)| = r + m = n$.

Case 3. Suppose $k = n$. Let $G = C_4$ and let $V(G) = \{x_1, x_2, x_3, x_4\}$ with $x_1x_2, x_2x_3, x_3x_4, x_4x_1$. Then $D = \{x_1, x_2\}$ is a minimum restrained dominating set of G and $S = \{x_3, x_4\}$ is a minimum inverse restrained dominating set of G. Thus $C = D \cup S = V(G)$. Hence, $\gamma\gamma_s(G) = n$.

This proves the assertion. ■

Corollary 2.5. The difference $\gamma\gamma_r - \gamma_r$ can be made arbitrarily large.

Proof. Let k be a positive integer. By Theorem 2.4, there exists a connected graph G such that $\gamma\gamma_r(G) = k + 1$ and $\gamma_r(G) = 1$. Thus, $\gamma\gamma_r(G) - \gamma_r(G) = k$. ■

Remark 2.6. $\gamma\gamma_r(G) = n - 1$ if $G = C_3$ and $\gamma\gamma_r(G) = n$ if $G = C_4$.

Theorem 2.7. Let G be a connected graph of order $n \geq 3$. Then $\gamma\gamma_r(G) = 2$ if and only if $G = K_1 + H$ where $\gamma(H) = 1$.

Proof. Suppose that $\gamma\gamma_r(G) = 2$. Let $C = \{a, b\}$ be a $\gamma\gamma_r$-*set* of G. Further, let D and S be γ_r-*set* and γ_r^{-1}-*set* of G respectively. Since D and S are nonempty sets, let $S = \{a\} = V(K_1)$. Then $D = \{b\}$ and $V(H) = V(G) \setminus S$. This implies that $\gamma_r(G) = 1$ and $G = K_1 + H$. Since $D \cap S = \varnothing$ by Remark 2.2, $D \subset V(H)$, that is, $\gamma(H) = 1$. Therefore, $G = K_1 + H$ where $\gamma(H) = 1$.

For the converse, suppose that $G = K_1 + H$ where $\gamma(H) = 1$. Let $D = V(K_1) = \{x\}$ be a γ_r-*set* of G and let $S = \{y\}$ be a dominating set of H. Since D is a dominating set of G and $n \geq 3$, $xz \in E(G)$ for every $z \in V(G) \setminus S$ $(x \neq z)$. Thus, $\langle V(G) \setminus S \rangle$ has no isolated vertices. This implies that S is a restrained dominating set of G. Since $D \cap S = \varnothing$, $S \subseteq (V(G) \setminus D)$, that is, S is a γ_r^{-1}-*set* of G. Hence, $C = \{x, y\}$ is a $\gamma\gamma_r$-*set* of G, that is, $\gamma\gamma_r(G) = 2$. ■

The following result is a direct consequence of Theorem 2.7.

Corollary 2.8. Let G be a connected graph of order $n \geq 3$. Then $\gamma\gamma_r(G) = 2$ if and only if $G = K_2 + H$ for some subgraph H.

Remark 2.9. If G is a complete graph of order $n \geq 3$, then $\gamma\gamma_r(G) = 2$.

Theorem 2.10. Let G be a connected non-complete graph of order $n \geq 4$. Then $\gamma\gamma_r(G) = 4$ if and only if $G \neq K_2 + H$ for any subgraph H and

(1) there exist distinct vertices x and y such that $\{x, y\}$ is a dominating set of G and $\langle V(G) \setminus \{x, y\} \rangle$ has no isolated vertices and satisfies one of the following:

(i) $\gamma(\langle N(x) \setminus \{y\} \rangle) = 1$ and

(a) $\gamma(\langle N(y) \setminus \{x\} \rangle) = 1$ or

(b) $\gamma(\langle (N(y) \setminus \{x\}) \setminus \{c : c \notin N(b) \text{ for some } b, c \in N(y) \setminus \{x\}\} \rangle) = 1$ where $c \in N(a)$ for some $a \in N(x) \setminus \{y\}$.

(ii) $\gamma(\langle (N(x) \setminus \{y\}) \setminus \{d : d \notin N(a) \text{ for some } a, d \in N(x) \setminus \{y\}\} \rangle) = 1$ where $d \in N(b)$ for some $b \in N(y) \setminus \{x\}$ and

(a) $\gamma(\langle N(y) \setminus \{x\} \rangle) = 1$ or

(b) $\gamma(\langle (N(y) \setminus \{x\}) \setminus \{c : c \notin N(b) \text{ for some } b, c \in N(y) \setminus \{x\}\} \rangle) = 1$ where $c \in N(a)$.

(iii) $\gamma(\langle N(x) \rangle) = 1$ and

(a) $\gamma(\langle N(y) \rangle) = 1$ or

(b) $\gamma(\langle N(y) \setminus (N(a) \setminus \{b\})\rangle) = 1$ for some $a \in N(x)$ and $b \in N(y)$ with $ab \in E(G)$.

(iv) $\gamma(\langle N(x) \setminus (N(b) \setminus \{a\})\rangle) = 1$ for some $a \in N(x)$ and $b \in N(y)$ with $ab \in E(G)$ and

(a) $\gamma(\langle N(y)\rangle) = 1$ or

(b) $\gamma(\langle N(y) \setminus (N(a) \setminus \{b\})\rangle) = 1$ with $ab \in E(G)$.

(v) $\gamma(\langle N(x) \setminus (N(b)\rangle) = 1$ for some $b \in N(y)$ and $\gamma(\langle N(y) \setminus N(a)\rangle) = 1$ for some $a \in N(x)$ with $ab \notin E(G)$.

(vi) there exists $a \in N(x)$ and $b \in N(y)$ such that $ab \notin E(G)$ and

(a) $\gamma(\langle N(x) \setminus N(b)\rangle) = 1$ and $\gamma(\langle N(x)\rangle) = 1$ or

(b) $\gamma(\langle N(x)\rangle) = 1$ and $\gamma(\langle N(y) \setminus N(a)\rangle) = 1$.

(2) there exist distinct vertices x, y, a and b such that $\{x, y, b\}$ is a minimum dominating set of $G - a$ with a dominate G and $\langle V(G) \setminus \{x, y, b\}\rangle$ and $\langle V(G) \setminus \{a\}\rangle$ have no isolated vertices.

Proof. Suppose that $\gamma\gamma_r(G) = 4$. Let $C = \{x, y, a, b\}$ be a $\gamma\gamma_r$-*set* of G. Since $\gamma_r(G) \leq \gamma_r^{-1}(G)$, it follows that $\gamma_r^{-1}(G) \neq 1$. Thus $\gamma_r^{-1}(G) = 2$ or $\gamma_r^{-1}(G) = 3$. Consider the following cases.

Case 1. Suppose that $\gamma_r^{-1}(G) = 2$.
Let $S = \{x, y\}$ be a γ_r^{-1}-*set* of G. Then $\{x, y\}$ is a dominating set of G such that $\langle V(G) \setminus \{x, y\}\rangle$ has no isolated vertices. Further, $D = \{a, b\}$ is a γ_r-*set* of G. Suppose that $G = K_2 + H$ for some subgraph H. Then $\gamma_r^{-1}(G) = 1$ by Corollary 2.8 contrary to our assumption. Thus, $G \neq K_2 + H$ for any subgraph H.

Subcase 1. Suppose that $xy \in E(G)$.
Since D is a dominating set in G, let $\{a\}$ be a dominating set of $N(x) \setminus \{y\}$. Then $\gamma(\langle N(x) \setminus \{y\}\rangle) = 1$. If $\{b\}$ is a dominating set of $N(y) \setminus \{x\}$, then $\gamma(\langle N(y) \setminus \{x\}\rangle) = 1$. This proves $(1ia)$. If $\{b\}$ is not a dominating set of $N(y) \setminus \{x\}$, then there exists $c \in N(y) \setminus \{x\}$ such that $c \notin N(b)$ for some $b \in N(y) \setminus \{x\}$. Thus, $\gamma(\langle (N(y) \setminus \{x\}) \setminus \{c : c \notin N(b) \text{ for some } b, c \in N(y) \setminus \{x\}\}\rangle) = 1$. Since D is dominating, $c \in N(a)$. This proves $(1ib)$.

Now, if $\{a\}$ is not a dominating set of $N(x) \setminus \{y\}$, then there exists $d \in N(x) \setminus \{y\}$ such that $d \notin N(a)$ for some $a \in N(x) \setminus \{y\}$. Thus, $\gamma(\langle (N(x) \setminus \{y\}) \setminus \{d : d \notin N(a) \text{ for some } a, d \in N(x)\} \setminus \{y\}\rangle) = 1$. Since D is dominating, $d \in N(b)$. If $\{b\}$ is a dominating set of $N(y) \setminus \{x\}$, then $\gamma(\langle N(y) \setminus \{x\}\rangle) = 1$. This proves $(1iia)$. If $\{b\}$ is not a dominating set of $N(y) \setminus \{x\}$, then $\gamma(\langle (N(y) \setminus \{x\}) \setminus \{c : c \notin N(b) \text{ for some } b, c \in N(y) \setminus \{x\}\}\rangle) = 1$ where $c \in N(a)$ by similar arguments used in $(1iia)$. This proves $(1iib)$.

Subcase 2. Suppose that $xy \notin E(G)$.

Consider $ab \in E(G)$. Let $D_a = \{a\}$ be a dominating set of $\langle N(x)\rangle$. Then $\gamma(\langle N(x)\rangle) = 1$. If $D_b = \{b\}$ is a dominating set of $\langle N(y)\rangle$, then $\gamma(\langle N(y)\rangle) = 1$. This proves $(1iiia)$. Suppose that D_b is not a dominating set of $\langle N(y)\rangle$. Then there exists $c \in N(y)$ such that $c \notin N(b)$. Since $D = \{a, b\}$ is a dominating set of G, it follows that $c \in N(a)$. Thus, D_b is a dominating set of $\langle N(y) \setminus (N(a) \setminus \{b\})\rangle$, that is, $\gamma(\langle N(y) \setminus (N(a) \setminus \{b\})\rangle) = 1$ for some $a \in N(x)$ with $ab \in E(G)$. This proves $(1iiib)$.

Similarly, if D_a is not a dominating set in $\langle N(x)\rangle$, then $\gamma(\langle N(x) \setminus (N(b) \setminus \{a\})\rangle) = 1$ for some vertex $b \in N(y)$ with $ab \in E(G)$. If D_b is a dominating set of $\langle N(y)\rangle$, then $\gamma(\langle N(y)\rangle) = 1$, proving $(1iva)$. If D_b is not a dominating set of $\langle N(y)\rangle$, then $\gamma(\langle N(y) \setminus (N(a) \setminus \{b\})\rangle) = 1$ for some $a \in N(x)$ with $ab \in E(G)$. This proves $(1ivb)$.

Consider $ab \notin E(G)$. Suppose that $D_a = \{a\}$ is not a dominating set of $\langle N(x)\rangle$. Then there exists $c \in N(x)$ such that $c \notin N(a)$. Since $D = \{a, b\}$ is a dominating set of G, $c \in N(b)$. Thus $\gamma(\langle N(x) \setminus N(b)\rangle) = 1$. If $D_b = \{b\}$ is not a dominating set of $\langle N(y)\rangle$, then there exists $d \in N(y)$ such that $d \notin N(b)$. Since $D = \{a, b\}$ is a dominating set of G, $d \in N(a)$. Thus $\gamma(\langle N(y) \setminus N(a)\rangle) = 1$. This shows $(1v)$. If $D_b = \{b\}$ is a dominating set of $\langle N(y)\rangle$, then $\gamma(N(y)) = 1$. This proves $(1via)$. Now, suppose that $D_a = \{a\}$ is a dominating set of $\langle N(x)\rangle$. Then $\gamma(\langle N(x)\rangle) = 1$. If $D_b = \{b\}$ is not a dominating set of $\langle N(y)\rangle$, $\gamma(\langle N(y) \setminus N(a)\rangle) = 1$ by similar arguments used above. This proves $(1vib)$.

Case 2. Suppose that $\gamma_r^{-1}(G) = 3$.

Let $S = \{x, y, b\}$ be a γ_r^{-1}-*set* of G. Then $\{x, y, b\}$ is a dominating set of G such that $\langle V(G) \setminus \{x, y, b\}\rangle$ has no isolated vertices. Further, $D = \{a\}$ is a γ_r-*set* of G. Suppose that $G = K_2 + H$ for some subgraph H. Then $\gamma_r^{-1}(G) = 1$ by Corollary 2.8 contrary to our assumption. Thus, $G \neq K_2 + H$ for any subgraph H. Since S is dominating set of G, for every $u \in V(G) \setminus \{a\}$, there exists $v \in S$ such that $uv \in E(G)$. Thus, $\langle V(G) \setminus \{a\}\rangle$ has no isolated vertices. This proves (2).

For the converse, suppose that $G \neq K_2 + H$ for any subgraph H and there exist distinct vertices x and y such that $\{x, y\}$ is a dominating set of G and $\langle V(G) \setminus \{x, y\}\rangle$ has no isolated vertices and satisfies (i), (ii), (iii), (iv), (v), or (vi).

Suppose first that $(1ia)$ holds. Then $xy \in E(G)$. Let $S = \{x, y\}$ and let $D_a = \{a\}$ be a dominating set of $\langle N(x) \setminus \{y\}\rangle$ and $D_b = \{b\}$ be a dominating set of $\langle N(y) \setminus \{x\}\rangle$. Then, $N[a] = N[x] \setminus \{y\}$ and $N[b] = N[y] \setminus \{x\}$. Thus,

$$\begin{aligned} N[a] \cup N[b] &= (N[x] \setminus \{y\}) \cup (N[y] \setminus \{x\}) \\ &= N[x] \cup N[y] \\ &= V(G). \end{aligned}$$

This implies that $D = \{a, b\}$ is a dominating set of G. Now, let $u, v \in V(G) \setminus D$. If $u = x$ and $v = y$, then $uv \in E(G)$. Suppose that $u = x$ and $v \neq y$. If $v \in N(x) \setminus \{y\}$, then $xv = uv \in E(G)$. If $v \in N(y) \setminus \{x\}$, then $vy, uy \in E(G)$. This implies that u-v is a path in G. Similarly, if $u \neq x$ and $v = y$, then u-v is path in G. Moreover, suppose that $u \neq x$ and $v \neq y$. If $u \in N(x) \setminus \{y\}$ and $v \in N(y) \setminus \{x\}$, then $ux, xy, yv \in E(G)$. Thus, u-v is a path in G. If $u, v \in N(x) \setminus \{y\}$ or $u, v \in N(y) \setminus \{x\}$, then it can be shown u-v is a path in G. In any case, $\langle V(G) \setminus D\rangle$ has no isolated vertices. This implies that D is a restrained dominating set in G. Thus, $\gamma_r(G) \leq |D| = 2$. Suppose that $\gamma_r(G) = 1$. Let $D = \{a\}$ be a γ_r-*set* and $S = \{x, y\}$ be a restrained dominating set of G. Since $a \in V(G) \setminus S$, it follows that $S \subseteq (V(G) \setminus D)$. Thus, S is an inverse restrained dominating set of G, that is, $\gamma_r^{-1}(G) \leq |S| = 2$. Suppose that $\gamma_r^{-1}(G) = 1$. Then there exist a vertex in S, say x, such that x dominate G. Since $x \neq a$, it follows that $\{x\}$ and $\{a\}$ are dominating sets of G. This implies that $G = K_2 + H$ for some subgraph H contrary to our assumption. Thus, $\gamma_r^{-1}(G) = 2$. Suppose that $\gamma_r(G) = 2$. Let $D = \{a, b\}$ be a γ_r-*set* of G and $S = \{x, y\}$ is a restrained dominating set of G. By (ia), $S \cap D = \varnothing$. This implies that $S \subseteq (V(G) \setminus D)$, that is S is an inverse restrained dominating set of G with respect to D. Since $\gamma_r(G) = 2$, it follows that $S = \{x, y\}$ is the minimum inverse restrained dominating set of G with respect to D by Remark 2.3. Hence, $\gamma_r^{-1}(G) = 2$. Accordingly, $C = D \cup S = \{a, b, x, y\}$ is a $\gamma\gamma_r$-*set* of G, that is, $\gamma\gamma_r(G) = 4$.

Suppose that $(1ib)$ holds. Then $xy \in E(G)$. Let $D_a = \{a\}$ be a dominating set of $\langle N(x) \setminus \{y\}\rangle$ and $D_b = \{b\}$ be a dominating set of $\langle (N(y) \setminus \{x\}) \setminus C\rangle$ where $C = \{c : c \notin N(b)$ for some $b, c \in N(y) \setminus \{x\}\}$ and $c \in N(a)$. Then, $N[a] = (N[x] \setminus \{y\}) \cup C$ and $N[b] = (N[y] \setminus \{x\}) \setminus C$. Thus,

$$\begin{aligned} N[a] \cup N[b] &= (N[x] \setminus \{y\}) \cup C \cup (N[y] \setminus \{x\}) \setminus C \\ &= N[x] \cup N[y] \\ &= V(G). \end{aligned}$$

This implies that $D = \{a, b\}$ is a dominating set of G. By following similar arguments in (ia), it follows that $\gamma\gamma_r(G) = 4$.

Suppose that $(1ii)$ holds. By using similar arguments in $(1i)$, it can be shown that $\gamma\gamma_r(G) = 4$. Finally, if any of the conditions $(1iii)$ or $(1iv)$ or $(1v)$ or $(1vi)$ holds, then it is clear that $\gamma\gamma_r(G) = 4$.

Finally, suppose that (2) holds. Let $D = \{a\}$ be a γ_r-*set* and $S = \{x, y, b\}$ be a restrained dominating set of G. Since $a \in V(G) \setminus S$, it follows that $S \subseteq (V(G) \setminus D)$. Thus, S is an inverse restrained dominating set of G, that is, $\gamma_r^{-1}(G) \leq |S| = 3$. Suppose that $\gamma_r^{-1}(G) = 1$. Then there exist a vertex in S, say x, such that x dominate G. Since $x \neq a$, it follows that $\{x\}$ and $\{a\}$ are dominating sets of G. This implies that $G = K_2 + H$ for some subgraph H contrary to our assumption. Suppose that $\gamma_r^{-1}(G) = 2$. Let $S' = \{x, y\}$ be a γ_r^{-1}-*set* of G. Since, S is a minimum dominating set of $G - a$, it follows that S' is not a dominating set of $G - a$ (and hence of G) contrary to our assumption that S' is a γ_r^{-1}-*set* of G. This implies that $\gamma_r^{-1}(G) = 3$. Thus, $C = D \cup S = \{a, x, y, b\}$ is a $\gamma\gamma_r$-*set* of G, that is, $\gamma\gamma_r(G) = 4$. ■

The following result is a direct consequence of Theorem 2.10.

Corollary 2.11. Let $G = K_2$ and H be connected graphs of order $m \geq 2$. Then $\gamma\gamma_r(G \circ H) = 4$ if and only if $\gamma(H) = 1$.

Corollary 2.11, can be generally stated by the following result.

Theorem 2.12. Let G and H be connected graphs of orders n and $m \geq 2$ respectively. Then $\gamma\gamma_r(G \circ H) = 2n$ if and only if $\gamma(H) = 1$.

Proof. Suppose that $\gamma\gamma_r(G \circ H) = 2n$. Since $V(G)$ is a minimum dominating set in $G \circ H$, let $D = V(G)$. Since H is a connected graph or order $m \geq 2$, it follows that $\langle V(G \circ H) \setminus D \rangle$ is a subgraph without isolated vertices. Thus, D is a γ_r*-set* of $G \circ H$ and hence $\gamma_r(G \circ H) = n$, that is, $\gamma_r^{-1}(G \circ H) = n$.

Thus, $|D| = |S|$, where S is a γ_r^{-1}*-set* of $G \circ H$, that is, $S = \bigcup_{i=1}^{n} \{x_i : x_i \in V(H^{v_i}), v_i \in V(G)\}$. Since S is dominating, x_i dominate $V(H^{v_i})$ for each $v_i \in V(G)$ where $i = 1, 2, \ldots, n$. Hence $\gamma(H^{v_i}) = 1$ for each $v_i \in V(G)$ $(i = 1, 2, \ldots, n)$, that is, $\gamma(H) = 1$.

For the converse, suppose that $\gamma(H) = 1$. Let $x \in V(H^v)$ dominate H^v for each $v \in V(G)$. Then $\{x\} \subset V(v + H^v)$ is a minimum dominating set of $v + H^v$ for each $v \in V(G)$. This implies that $D = \bigcup_{i=1}^{n} \{x_i : x_i \in V(H^{v_i}), v_i \in V(G)\}$ is a minimum dominating set of $V(G \circ H)$. Let $v \in V(G)$. Since the order of H is $m \geq 2$, for each $u \in V(H^v) \setminus \{x\}$ where $x \in D$, $uv \in E(v + H^v)$. Thus, $\langle V(v + H^v) \setminus \{x\} \rangle$ has no isolated vertices for each $v \in V(G)$. This implies that $\langle V(G \circ H) \setminus D \rangle$ has no isolated vertices. Hence D is a γ_r*-set* of $G \circ H$. Since $V(G) \subseteq (V(G \circ H) \setminus D)$, it follows that $V(G)$ is an inverse dominating set of $V(G \circ H)$. Since $\langle V(G \circ H) \setminus V(G) \rangle = H$ has no isolated vertices, $V(G)$ is an inverse restrained dominating set of $G \circ H$. Since $|V(G)|$ is a minimum dominating set of $G \circ H$, it follows that $V(G)$ is a γ_r^{-1}*-set* of $G \circ H$. Hence, $C = V(G) \cup D$ is a $\gamma\gamma_r$*-set* of $G \circ H$, that is, $\gamma\gamma_r(G \circ H) = |V(G)| + |D| = n + n = 2n$. ■

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