3. (X, d) is a complete cone 2-metric space if every Cauchy sequence in X is convergent in X.

Lemma 2.1: ([14]) Let $P \subset A$. If we have,

1. $a \preccurlyeq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
2. $\theta \preccurlyeq a \ll c$ for each $\theta \ll c$, then $a = \theta$.
3. $x_n \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there is a natural number N such that $x_n \ll c$ for all $n > N$.

Preposition 2.1([11]) Let A be a Banach Algebra with a unit e and $x \epsilon A$. If the spectral radius $r(x) < 1$,

$i.e., r(x) = \lim_{n\to\infty} x^{n^{1}/n} = \inf_{n\geq 1} x^{n^{1}/n} < 1,$

then $e - x$ is invertible and $(e - x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Lemma 2.2: ([14]) For a Banach Algebra A with a unit e,

1. $r(x) \leq x$ for all $x \in A$.
2. If $r(k) < 1$ where $k \in P$, we have $(e - k)^{-1} \in P$.
3. If $r(k) < 1$, then $k^n \to 0$ as $n \to \infty$.

Lemma 2.3: ([14]) Let $x, y \in A$. If x and y commute, then

1. $r(xy) \leq r(x)r(y)$
2. $r(x + y) \leq r(x) + r(y)$
3. $|r(x) - r(y)| \leq r(x - y)$

Lemma 2.4: ([14]) Let A be a Banach Algebra and let k be a vector in A. If $0 \leq r(k) < 1$, then we have $r((e - k)^{-1}) \leq (1 - r(k))^{-1}$.

Lemma 2.5: ([16]) If A is a real Banach space with a solid cone P and if $\|x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, then for any $\theta \ll c$, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$, we have $x_n \ll c$.

Preposition 2.2 Let $P \subset A$. Then

1. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.
2. If $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are c-sequences in P then for all $\alpha, \beta, \gamma \epsilon P$; $\{\alpha u_n + \beta v_n + \gamma w_n\}$ is a c-sequence
3. For any $u, v \epsilon A$ and $k \in P$, we have $ku \preccurlyeq kv$ whenever $u \preccurlyeq v$.

Preposition 2.3:([16]) Let P be a solid cone in the Banach algebra A. Let $x \in A$. If $k \in P$ and $x \ll c$ for any $\theta \ll c$, then $kx \ll c$ for any $\theta \ll c$.

3. MAIN RESULT

Theorem 3.1: ([4])Let (X, d) be a complete metric space and T be a self-mapping of X satisfying the condition for all $x, y \in X$, $d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$

(3. 1)

Where a, b, c, e, f are non-negative and $a + b + c + e + f < 1$. Then T has a unique fixed point in X.

Theorem 3.2: Let (X, d) be a complete cone 2-metric space in a Banach algebra A and P be the underlying cone. Let $\{T_i^{m_i}\}_{i=1}^{\infty}$ be family of self-maps on X satisfying

$$d\left(T_i^{m_i}x, T_j^{m_j}y, a\right) \preccurlyeq k_1 d(x, y, a) + k_2 d(x, T_i^{m_i}x, a) + k_3 d\left(y, T_j^{m_j}y, a\right) + k_4 d\left(x, T_j^{m_j}y, a\right) + k_5 d(y, T_i^{m_i}x, a)$$

(3. 2)

for all $x, y, a \in X$ where $k_i \in P, i = 1, 2, 3, 4, 5$. If k_i commute and $\sum_{i=1}^{5} r(k_i) < 1$. Then the family of maps $\{T_i\}_{i=1}^{\infty}$ have unique common fixed point in X.

Proof: Write $f_i = T_i^{m_i}$ for $i = 1, 2, 3, ...$

Then (3. 2) becomes

$$d\left(f_i x, f_j y, a\right) \preccurlyeq k_1 d(x, y, a) + k_2 d(x, f_i x, a) + k_3 d\left(y, f_j y, a\right) + k_4 d\left(x, f_j y, a\right) + k_5 d(y, f_i x, a)$$

(3. 3)

Choose $x_0 \in X$ arbitrary and define the sequence $\{x_n\}$ by $x_n = f_n(x_{n-1})$ for $n = 1, 2, 3, ...$

Now we show that $\{x_n\}$ is a Cauchy sequence in X.

$d(x_{n+1}, x_n, a) = d(f_{n+1}x_n, f_n x_{n-1}, a)$

$\preccurlyeq k_1 d(x_n, x_{n-1}, a) + k_2 d(x_n, x_{n+1}, a) + k_3 d(x_{n-1}, x_n, a) + k_4 d(x_n, x_n, a) + k_5 d(x_{n-1}, x_{n+1}, a)$

$\preccurlyeq k_1 d(x_n, x_{n-1}, a) + k_2 d(x_n, x_{n+1}, a) + k_3 d(x_{n-1}, x_n, a)$

$+ k_5[d(x_{n-1}, x_{n+1}, x_n) + d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a)]$

$i.e., (e - k_2 - k_5) d(x_{n+1}, x_n, a) \preccurlyeq (k_1 + k_3 + k_5) d(x_n, x_{n-1}, a)$

$i.e., d(x_{n+1}, x_n, a) \preccurlyeq \alpha d(x_n, x_{n-1}, a)$ where $\alpha = (e - k_2 - k_5)^{-1}(k_1 + k_3 + k_5) \in P$.

(3. 4)

Because of symmetry of the cone 2-metric d, we have

$d(x_{n+1}, x_n, a) = d(x_n, x_{n+1}, a) = d(f_n x_{n-1}, f_{n+1} x_n, a)$

$\preccurlyeq k_1 d(x_{n-1}, x_n, a) + k_2 d(x_{n-1}, x_n, a) + k_3 d(x_n, x_{n+1}, a) + k_4 d(x_{n-1}, x_{n+1}, a) + k_5 d(x_n, x_n, a)$

$\preccurlyeq k_1 d(x_{n-1}, x_n, a) + k_2 d(x_{n-1}, x_n, a) + k_3 d(x_n, x_{n+1}, a)$

$+ k_4[d(x_{n-1}, x_{n+1}, x_n) + d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a)]$

$i.e., (e - k_3 - k_4) d(x_{n+1}, x_n, a) \preccurlyeq (k_1 + k_2 + k_4) d(x_n, x_{n-1}, a)$

Or $d(x_{n+1}, x_n, a) \preccurlyeq \beta d(x_n, x_{n-1}, a)$ where $\beta = (e - k_3 - k_4)^{-1}(k_1 + k_2 + k_4) \in P$.

(3. 5)

Then we claim that, either $r(\alpha) < 1$ or$r(\beta) < 1$.

Suppose if $r(\alpha) > 1$ and $r(\beta) > 1$. Then we get,

$$1 < r(\alpha) \leq \frac{r(k_1) + r(k_3) + r(k_5)}{1 - r(k_2) + r(k_5)}$$

which implies $r(k_1) + r(k_2) + r(k_3) + 2r(k_5) > 1$

(3. 6)

$$1 < r(\alpha) \leq \frac{r(k_1) + r(k_3) + r(k_5)}{1 - r(k_2) + r(k_5)}$$

which implies $r(k_1) + r(k_2) + r(k_3) + 2r(k_4) > 1$

(3. 7)

Adding (3.6) and (3.7), we get$r(k_1) + r(k_2) + r(k_3) + r(k_4) + r(k_5) > 1$, a contradiction.

Thus, our claim is true.

Take $k = \begin{cases} \alpha \; if \; r(\alpha) < 1 \; or \\ \beta \; if \; r(\beta) < 1 \; or \\ \alpha \; or \; \beta \; if \; r(\alpha) < 1 \; and \; r(\beta) < 1 \end{cases}$

Then clearly $k \in P$ and $r(k) < 1$.

(3. 8)

Thus $d(x_{n+1}, x_n, a) \preccurlyeq kd(x_n, x_{n-1}, a)$ for all $n \geq 1$ where $r(k) < 1$.

(3. 9)

Suppose $l < n$. Then

$d(x_{n+1}, x_n, x_l) \preccurlyeq kd(x_n, x_{n-1}, x_l)$

……..

$\preccurlyeq k^{n-l} d(x_{l+1}, x_l, x_l) = \theta$

Thus, for all $l < n$, we have $d(x_{n+1}, x_n, x_l) = \theta$.

(3. 10)

For $n > m$, using (3. 10) in the below steps, we have

$d(x_n, x_m, a) \preccurlyeq d(x_n, x_m, x_{n-1}) + d(x_n, x_{n-1}, a) + d(x_{n-1}, x_m, a)$

$\preccurlyeq k^{n-1} d(x_1, x_0, a) + d\big(x_{n-1}, x_m, x_{n-2)}\big) + d(x_{n-1}, x_{n-2}, a) + d(x_{n-2}, x_m, a)$

$\preccurlyeq (k^{n-1} + k^{n-2})\, d(x_1, x_0, a) + d(x_{n-2}, x_m, a)$

$\preccurlyeq (k^{n-1} + k^{n-2} + \cdots \ldots \ldots \ldots \ldots .. + k^{m+1}) d(x_1, x_0, a) + d(x_{m+1}, x_m, a)$

$\preccurlyeq (x^{n-1} + k^{n-2} + \cdots \ldots \ldots \ldots \ldots \ldots + k^{m+1} + k^m) d(x_1, x_0, a)$

$= (e + k + \cdots \ldots \ldots \ldots .. + k^{n-m+1}) k^m d(x_1, x_0, a)$

$\preccurlyeq \left(\sum_{i=0}^{\infty} k^i \right) k^m d(x_1, x_0, a)$

$= (e - k)^{-1} k^m d(x_1, x_0, a)$

(3. 11)

Hence from Lemma 2. 5 and the fact that $\|(e - k)^{-1} k^m d(x_1, x_0, a)\| \longrightarrow 0 \; as \; n \longrightarrow \infty$, we have for any $c \in A$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, we have

$d(x_n, x_m, a) \preccurlyeq (e - k)^{-1} k^m d(x_1, x_0, a) \ll c$, showing that $\{x_n\}$is a Cauchy sequence in X. Since X is complete, there exists $x \in X$ such that $x_n \longrightarrow x \; as \; n \longrightarrow \infty$.

Now we show that this x is the unique common fixed point of the family of maps $\{T_i\}_{i=1}^{\infty}$.

$d(f_n x, x, a) \preccurlyeq d(f_n x, x, x_{m+1}) + d(f_n x, x_{m+1}, a) + d(x_{m+1}, x, a)$

$= d(x_{m+1}, x, a) + d(f_n x, f_{m+1} x_m, x) + d(f_n x, f_{m+1} x_m, a)$

$\preccurlyeq d(x_{m+1}, x, a)$

$+[k_1 d(x,x_m,x)+k_2 d(x,f_n x,x)+k_3 d(x_m,x_{m+1},x)+k_4 d(x,x_{m+1},x)+k_5 d(x_m,f_n x,x)]$
$+[k_1 d(x,x_m,a)+k_2 d(x,f_n x,a)+k_3 d(x_m,x_{m+1},a)+k_4 d(x,x_{m+1},a)+k_5 d(x_m,f_n x,a)]$
$\preccurlyeq d(x_{m+1},x,a)+k_3 d(x_m,x_{m+1},x)+k_5 d(x_m,f_n x,x)$
$+k_1 d(x,x_m,a)+k_2 d(f_n x,x,a)+k_3 d(x_m,x_{m+1},a)+k_4 d(x,x_{m+1},a)$
$+k_5[d(x_m,f_n x,x)+d(x_m,x,a)+d(f_n x,x,a)]$
$i.e., (e-k_2-k_5)d(f_n x,x,a)\preccurlyeq d(x_{m+1},x,a)+k_3 d(x_m,x_{m+1},x)+k_1 d(x,x_m,a)$
$+k_3 d(x_m,x_{m+1},a)+k_4 d(x,x_{m+1},a)+2k_5 d(x_m,f_n x,x)+k_5 d(x_m,x,a)$

(3. 12)

Keeping n fixed and using the Preposition 2. 2(2), the R. H. S of the above inequality (3. 12) is a c-sequence in P, say $\{g_m\}$.
Therefore, for any $c\in A$ with $\theta \ll c$, we have $(e-k_2-k_5)d(f_n x,x,a)\preccurlyeq g_m \ll c$ for all $a\in X$.
Since $r(k_2+k_5)<1$, by Lemma 2. 2(2), we have $(e-k_2-k_5)$ is invertible and hence $d(f_n x,x,a)\ll c$. This is true for any $n\in\mathbb{N}$ and for any c with $\theta\ll c$. Thus we have, $d(f_n x,x,a)=\theta$ for all $n\in\mathbb{N}$ and $a\in X$.
Therefore, $f_n x=x$ for all $n\geq 1$ which implies x is common fixed point of
$f_n, n\geq 1$.

(3. 13)

Suppose y be any other common fixed point of $f_n, n\geq 1$.
$i.e., f_n y=y, n\geq 1$. Then using (3. 2)
$d(x,y,a)=d(f_n x,f_n y,a)\preccurlyeq k_1 d(x,y,a)+k_2 d(x,x,a)+k_3 d(y,y,a)+k_4 d(x,y,a)+k_5 d(y,x,a)$
$i.e., (e-k_1-k_4-k_5)d(x,y,a)\preccurlyeq\theta$ which implies $d(x,y,a)=\theta$ for all $a\in X$.
Therefore $x=y$ is the unique common fixed point of the family of maps $\{f_n\}_{n=1}^{\infty}$.
Thus we have, $x=f_n x=T_n^{m_n}x$

(3. 14)

Also, $T_n(x)=T_n\left(T_n^{m_n+1}x\right)=T_n^{m_n}(T_n x)=f_n(T_n x)$
$i.e., T_n(x)=f_n(T_n x)$ which implies $T_n(x)$ is also fixed point of f_n.
But x is the unique fixed point of $f_n$by (3. 14), which means $T_n x=x$.
Suppose z is any other fixed point of $T_n i.e., T_n z=z$
Then $T_n^{m_n}z=z$ and hence $f_n z=T_n^{m_n}z=z$
Since fixed point of f_n is unique, which is x we get $x=z$, proving the uniqueness.
The following list of corollaries can be obtained from our main theorem.

Corollary 3.1: Let (X, d) be a complete cone 2-metric space in a Banach algebra A and P be the underlying cone. Let $\{T_i^{m_i}\}_{i=1}^{\infty}$ be family of self-maps on X satisfying
$d\left(T_i^{m_i}x,T_j^{m_j}y,a\right)\preccurlyeq k_1 d(x,y,a)+k_2 d(x,T_i^{m_i}x,a)+k_3 d\left(y,T_j^{m_j}y,a\right)$

(3. 15)

for all $x,y,a\in X$ where $k_i\in P, i=1,2,3$. If k_i commute and $\sum_{i=1}^{3} r(k_i)<1$. Then the family of maps $\{T_i\}_{i=1}^{\infty}$ have unique common fixed point in X.
The above fixed point theorem was proved by Wang et. al in [16].

Corollary 3.2: Let (X, d) be a complete cone 2-metric space in a Banach algebra A and P be the underlying cone. Let $\{T_i^{m_i}\}_{i=1}^{\infty}$ be family of self-maps on X satisfying

$$d(T_i^{m_i}x, T_j^{m_j}y, a) \preccurlyeq k[d(x, T_i^{m_i}x, a) + d(y, T_j^{m_j}y, a)] \tag{3. 16}$$

for all $x, y, a \in X$ where $k \in P$ and $r(k) < \frac{1}{2}$. Then the family of maps $\{T_i\}_{i=1}^{\infty}$ have unique common fixed point in X.

Corollary 3.3: Let (X, d) be a complete cone 2-metric space in a Banach algebra A and P be the underlying cone. Let $\{T_i^{m_i}\}_{i=1}^{\infty}$ be family of self-maps on X satisfying

$$d(T_i^{m_i}x, T_j^{m_j}y, a) \preccurlyeq k[d(x, T_i^{m_i}y, a) + d(y, T_j^{m_j}x, a)] \tag{3. 17}$$

for all $x, y, a \in X$ where $k \in P$ and $r(k) < \frac{1}{2}$. Then the family of maps $\{T_i\}_{i=1}^{\infty}$ have unique common fixed point in X.

Corollary 3.4: Let (X, d) be a complete cone 2-metric space in a Banach algebra A and P be the underlying cone. Let $\{T_i^{m_i}\}_{i=1}^{\infty}$ be family of self-maps on X satisfying

$$d(T_i^{m_i}x, T_j^{m_j}y, a) \preccurlyeq kd(x, y, a) \tag{3. 18}$$

for all $x, y, a \in X$ where $k \in P$ and $r(k) < 1$. Then the family of maps $\{T_i\}_{i=1}^{\infty}$ have unique common fixed point in X.

EXAMPLE 3.1: Let $A = R^2$ for each $(x_1, x_2) \in A, \|(x_1, x_2)\| = |x_1| + |x_2|$. The multiplication is defined by $xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$. Then A is a Banach algebra with unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in R^2 x_1, x_2 \geq 0\}$ then P is a cone in A.

Let $X = \{(x, 0) \in R^2 | 0 \leq x \leq 1\} \cup \{(0, x) \in R^2 | 0 \leq x \leq 1\}$.

The metric is defined by $d(\alpha_1, \alpha_2, \alpha_3) = d(\beta_1, \beta_2)$ where $\alpha_1, \alpha_2, \alpha_3 \in X$, $\beta_1, \beta_2 \in \{\alpha_1, \alpha_2, \alpha_3\}$ are such that $\|\beta_1 - \beta_2\| = \min\{\|\alpha_1 - \alpha_2\|, \|\alpha_2 - \alpha_3\|, \|\alpha_3 - \alpha_2\|\}$ and

$$d_1((x, 0), (y, 0)) = \left(\frac{5}{4}|x - y|, |x - y|\right)$$

$$d_1((0, x), (0, y)) = \left(|x - y|, \frac{3}{4}|x - y|\right)$$

$$d_1((x, 0), (0, y)) = d_1((0, y), (x, 0)) = \left(\frac{5}{4}x + y, x + \frac{3}{4}y\right)$$

Then (X, d) is a complete cone 2-metric space over the Banach Algebra A.

Now define $T_i : X \to X$ $(i \geq 1)$ by

$$T_i((x, 0)) = \left(0, (3)^{\frac{i-2}{2i-1}}(2)^{\frac{-i+1}{2i-1}}x\right)$$

And $T_i((0, x)) = \left((3)^{\frac{-i+1}{2i-1}}(2)^{\frac{i-2}{2i-1}}x, 0\right)$

Then we have $T_i^{2i-1}(x, 0) = \left(0, \frac{1}{12}x\right)$ and $T_i^{2i-1}(0, x) = \left(\frac{1}{18}x, 0\right)$

Also T_i satisfies the contractive condition of the main Theorem 3. 2 with $m_i = 2i - 1, k_1 = \left(\frac{1}{4}, 0\right), k_2 = k_3 = k_4 = k_5 = (\frac{1}{6}, 0)$. Also $r(k_1) = \frac{1}{4}$ & $r(k_2) = r(k_3) =$

$r(k_4) = r(k_5) = \frac{1}{6}$. Hence by our main Theorem 3. 2, T_i has a unique fixed point which is $(0, 0)$ for all $i \geq 1$.

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