Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 3 (2016), pp. 2349–2354

http://www.ripublication.com/gjpam.htm

A note on Daehee numbers arising from differential equations

Hyuck In Kwon

Department of Mathematics,
Kwangwoon University,
Seoul 139-701, Republic of Korea.

Taekyun Kim

Department of Mathematics,
Kwangwoon University,
Seoul 139-701, Republic of Korea.

Jong Jin Seo[1]

Department of Applied Mathematics,
Pukyong National University,
Busan, Republic of Korea.

Abstract

In this paper, we study some differential equations arising from the generating function of Daehee numbers and we investigate some new explicit identities of Daehee numbers.

AMS subject classification: 11B68, 11S80.
Keywords: Daehee numbers Differential equations.

[1]Corresponding author.

1. Introduction

As is know, the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{(see [1–10])}. \tag{1.1}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

In [4], Daehee polynomials are defined by the generating function to be

$$\frac{log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad \text{(see [2–6])}. \tag{1.2}$$

When $x = 0$, $D_n = D_n(0)$ are called Daehee numbers.

For $r \in \mathbb{N}$, the higher-order Daehee numbers are given by the generating funcion to be

$$\left(\frac{log(1+t)}{t}\right)^r = \sum_{n=0}^{\infty} D_n^{(r)} \frac{t^n}{n!}, \quad \text{(see [4, 5])}. \tag{1.3}$$

By (1.1), we easily get

$$\begin{aligned} \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} &= \sum_{m=0}^{\infty} B_m(x) \frac{1}{m!} \Big(log(1+t)\Big)^m \\ &= \sum_{m=0}^{\infty} B_m(x) \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} B_m(x) S_1(n,m) \right) \frac{t^n}{n!}, \end{aligned} \tag{1.4}$$

where $S_1(n, m)$ is the Stirling number of the first kind.

From (1.4), we note that

$$D_n(x) = \sum_{m=0}^{n} B_m(x) S_1(n,m) \quad (n \geq 0).$$

Recently, Kim and Kim introduced nonlinear Changhee differential equations which are derived from the generating function of Changhee numbers and they gave some interesting identities between Changhee numbers and Euler numbers. In this paper, we study differential equations which are derived from the generating function of Daehee numbers and we give some new explicit identities of Daehee numbers.

2. Daehee numbers associated with differential equations

Let

$$F = F(t) = log(1 + t) \tag{2.5}$$

Then, by (2.5), we get

$$F^{(1)} = \frac{dF(t)}{dt} = \frac{1}{1+t} = e^{-log(1+t)} = e^{-F}, \tag{2.6}$$

$$F^{(2)} = (-1)e^{-F}F^{(1)} = (-1)e^{-2F} \tag{2.7}$$

$$F^{(3)} = (-1)^2 2!e^{-2F}F^{(1)} = (-1)^2 2!e^{-3F} \tag{2.8}$$

and

$$F^{(4)} = (-1)^3 3!e^{-4F}. \tag{2.9}$$

Continuing this process, we have

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t) = (-1)^{N-1}(N-1)!e^{-NF} \\ &= (-1)^{N-1}(N-1)!\sum_{n=0}^{\infty}(-1)^n N^n \frac{1}{n!}F^n \end{aligned} \tag{2.10}$$

.

Now, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty}(-1)^n N^n \frac{1}{n!}F^n &= \sum_{n=0}^{\infty}(-1)^n N^n \sum_{m=n}^{\infty} S_1(m,n)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}(-1)^n N^n S_1(m,n)\Big)\frac{t^m}{m!}, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N \Big(log(1+t)\Big) = \left(\frac{d}{dt}\right)^N \Big(\frac{log(1+t)}{t}\cdot t\Big) \\ &= \left(\frac{d}{dt}\right)^N \Big(\sum_{m=0}^{\infty} D_m \frac{t^{m+1}}{m!}\Big) \\ &= \left(\frac{d}{dt}\right)^N \Big(\sum_{m=0}^{\infty} mD_{m-1} \frac{t^m}{m!}\Big) \\ &= \sum_{m=0}^{\infty}(m+N)D_{m+N-1}\frac{t^m}{m!}. \end{aligned} \tag{2.12}$$

Therefore, by (2.10),(2.11) and (2.12), we obtain the following theorem.

Theorem 1. For $N \in \mathbb{N}$, $m \geq 0$, we have

$$D_{m+N-1} = \frac{(-1)^{N-1}(N-1)!}{m+N} \sum_{n=0}^{m} (-1)^n N^n S_1(m,n).$$

From (1.3), we note that

$$\begin{aligned} F^n &= \big(log(1+t)\big)^n = \Big(\frac{log(1+t)}{t}\Big)^n t^n \\ &= \Big(\sum_{l=0}^{\infty} D_l^{(n)} \frac{t^l}{l!}\Big) t^n = \sum_{l=n}^{\infty} D_{l-n}^{(n)} \frac{t^{l-n}}{(l-n)!} t^n \\ &= \sum_{l=n}^{\infty} D_{l-n}^{(n)} (l)_n \frac{t^l}{l!} = \sum_{l=0}^{\infty} D_l^{(l)} \big(l+n\big)_n \frac{t^{l+n}}{(l+n)!}. \end{aligned} \tag{2.13}$$

Thus, by (2.13), we get

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n N^n \frac{1}{n!} F^n &= \sum_{n=0}^{\infty} (-1)^n N^n \frac{1}{n!} \sum_{l=0}^{\infty} D_l^{(l)} \big(l+n\big)_n \frac{t^{l+n}}{(l+n)!} \\ &= \sum_{m=0}^{\infty} \Big(\sum_{n=0}^{m} (-1)^n N^n \binom{m}{n} D_{m-n}^{(n)} \Big) \frac{t^m}{m!}. \end{aligned} \tag{2.14}$$

Therefore, by (2.10),(2.12) (2.14), we obtain the following theorem.

Theorem 2.1. For $N \in \mathbb{N}$, $m \geq 0$, we have

$$D_{m+N-1} = \frac{(-1)^N (N-1)!}{m+N} \sum_{n=0}^{m} \binom{m}{n} (-1)^n N^n D_{m-n}^{(n)}.$$

From (2.14), we have

$$\begin{aligned} e^{-Nt} &= \sum_{m=0}^{\infty} \Big(\sum_{l=0}^{m} (-1)^l N^l \binom{m}{l} D_{m-l}^{(l)} \Big) \frac{\big(e^t-1\big)^m}{m!} \\ &= \sum_{m=0}^{\infty} \Big(\sum_{l=0}^{m} (-1)^l N^l \binom{m}{l} D_{m-l}^{(l)} \Big) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{n} \sum_{l=0}^{m} (-1)^l N^l \binom{m}{l} D_{m-l}^{(l)} S_2(n,m) \Big) \frac{t^n}{n!} \end{aligned} \tag{2.15}$$

and

$$e^{-Nt}=\sum_{n=0}^{\infty}(-1)^n N^n \frac{t^n}{n!}. \tag{2.16}$$

From (2.15), and (2.16), we obtain the following theorem.

Theorem 2.2. For $N \in \mathbb{N}$, $m \geq 0$, we have

$$N^n=\sum_{m=0}^{n}\sum_{l=0}^{m}(-1)^{n-l}N^l\binom{m}{l}D_{m-l}^{(l)}S_2(n,m)$$

As is well know, the higher-order Daehee numbers are given by the generating function to be

$$\left(\frac{log(1+t)}{t}\right)^x=\sum_{n=0}^{\infty}D_n^{(x)}\frac{t^n}{n!}, \quad (x \in \mathbb{R}). \tag{2.17}$$

Thus, by (2.17), we get

$$\begin{aligned}\left(\frac{log(1+t)}{t}\right)^x &= \left(\frac{log(1+t)}{t}\right)^r\left(\frac{log(1+t)}{t}\right)^{x-r}\\ &= \left(\sum_{l=0}^{\infty}D_l^{(r)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}D_m^{(x-r)}\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}D_{n-l}^{(x-r)}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.18}$$

Thus, by (2.18), we obtain the following convolved theory of Daehee numnbers.

Theorem 2.3. For $r \in \mathbb{N}$, we have

$$D_n^{(x)}=\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}D_{n-l}^{(x-r)}$$

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