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Revisit nonlinear differential equations arising from degenerate Euler numbers

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Abstract

Recently, Kim-Kim have studied nonlinear differential equations arising from degenerate Euler numbers. In this paper, we revisit nonlinear differential equations which are derived from the generating functions of degenerate Euler numbers for doing obtain some explicit formulas and identities of degenerate Euler numbers.

AMS subject classification:
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1. Introduction

As is well know, the Euler numbers are defined by the generating function to be

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \text{(see [1–7])}. \tag{1.1}$$

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In [1, 2], L Carlitz considered degenerate Euler numbers which are given by the generating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}=\sum_{n=0}^{\infty}\xi_{n,\lambda}\frac{t^n}{n!}. \tag{1.2}$$

Thus, we note that $\lim_{\lambda\to 0}\xi_{n,\lambda}=E_n$, $(n\geq 0)$. For $r\in\mathbb{N}$, the higher-order degenerate Euler numbers are also defined by the generating function to be

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r=\sum_{n=0}^{\infty}\xi_n^{(r)}\frac{t^n}{n!},\ \text{(see [1, 2])}. \tag{1.3}$$

In [5], Kim-Kim gave some explicit identities involving degenerate Euler numbers and polynomials arising form nonlinear differential equations. In this paper, we revisit nonlinear differential equations which are derived from the generating functions of degenerate Euler numbers for doing obtain some explicit formulas and identities of degenerate Euler numbers.

2. Nonlinear differential equations arising from the generating functions of degenerate Euler numbers

Let

$$F=F(t)=\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}. \tag{2.4}$$

Then, by (2.1), we get

$$F^{(1)}=\frac{d}{dt}F(t)=\frac{(-1)}{1+\lambda t}(F-F^2). \tag{2.5}$$

Thus, by (2.2), we have

$$F^2=(1+\lambda t)F^{(1)}+F. \tag{2.6}$$

Let us take derivative in (2.3) with respect to t. Then we note that

$$2FF^{(1)}=(\lambda+1)F^{(1)}+(1+\lambda t)F^{(2)}. \tag{2.7}$$

From (2.2) and (2.4), we can derive the following equation

$$2!F^3=2F+(1+\lambda t)(3+\lambda)F^{(1)}+(1+\lambda t)^2F^{(2)}. \tag{2.8}$$

Continuing this process, we can write

$$N!F^{N+1}=\sum_{i=0}^{N}(1+\lambda t)^i a_i(N,\lambda)F^{(i)},\ (N=0,1,2,\ldots), \tag{2.9}$$

where

$$F^{(i)} = \left(\frac{d}{dt}\right)^{i} F(t).$$

Taking derivative with respect to t in (2.6), we have

$$\begin{aligned}(N+1)!\frac{F^{N+2}-F^{N+1}}{1+\lambda t} &= (N+1)!F^{N}F^{(1)} \\ &= \sum_{i=0}^{N}\lambda i(1+\lambda t)^{i-1}a_i(N,\lambda)F^{(i)} + \sum_{i=0}^{N}(1+\lambda t)^{i}a_i(N,\lambda)F^{(i+1)}.\end{aligned} \tag{2.10}$$

By (2.6) and (2.7), we get

$$\begin{aligned}&(N+1)!F^{N+2} \\ =&(N+1)!F^{N+1} + \sum_{i=0}^{N}\lambda i(1+\lambda t)^{i}a_i(N,\lambda)F^{(i)} + \sum_{i=0}^{N}(1+\lambda t)^{i+1}a_i(N,\lambda)F^{(i+1)} \\ =&(N+1)\sum_{i=0}^{N}a_i(N,\lambda)(1+\lambda t)^{i}F^{(i)} + \sum_{i=0}^{N}\lambda i a_i(N,\lambda)(1+\lambda t)^{i}F^{(i)} \\ &+ \sum_{i=1}^{N+1}(1+\lambda t)^{i}a_{i-1}(N,\lambda)F^{(i)} \\ =&\sum_{i=0}^{N}\{(N+1)a_i(N,\lambda)+\lambda i a_i(N,\lambda)\}(1+\lambda t)^{i}F^{(i)} + \sum_{i=1}^{N+1}a_{i-1}(N,\lambda)(1+\lambda t)^{i}F^{(i)}.\end{aligned} \tag{2.11}$$

On the other hand, by replacing N by $N+1$ in (2.6), we get

$$(N+1)!F^{N+2} = \sum_{i=0}^{N+1}a_i(N+1,\lambda)(1+\lambda t)^{i}F^{(i)}. \tag{2.12}$$

By (2.8) and (2.9), we note that

$$a_0(N+1,\lambda) = (N+1)a_0(N,\lambda), \quad a_{N+1}(N+1,\lambda) = a_N(N,\lambda), \tag{2.13}$$

and

$$a_i(N+1,\lambda) = (N+1+\lambda i)a_i(N,\lambda) + a_{i-1}(N,\lambda), \ (1 \le i \le N). \tag{2.14}$$

From $F = a_0(0,\lambda)F$, we have $a_0(0,\lambda) = 1$. By (2.10), we easily get

$$a_0(N+1,\lambda) = (N+1)a_0(N,\lambda) = (N+1)Na_0(N-1,\lambda) = \cdots = (N+1)!, \tag{2.15}$$

$$a_{N+1}(N+1,\lambda)=a_N(N,\lambda)=\cdots=a_1(1,\lambda)=a_0(0,\lambda)=1. \tag{2.16}$$

For $i=1,2,3$ in (2.11), we have

$$a_1(N+1,\lambda)=\sum_{k=0}^{N}(N+\lambda+1)_k a_0(N-k,\lambda), \tag{2.17}$$

$$a_2(N+1,\lambda)=\sum_{k=0}^{N-1}(N+2\lambda+1)_k a_1(N-k,\lambda), \tag{2.18}$$

and

$$a_3(N+1,\lambda)=\sum_{k=0}^{N-2}(N+3\lambda+1)_k a_2(N-k,\lambda), \tag{2.19}$$

where $(x)_n=x(x-1)\cdots(x-n+1),\ (n\geq 1),\ (x)_0=1$.
Continuing this process, we get

$$a_i(N+1,\lambda)=\sum_{k=0}^{N-i+1}(N+i\lambda+1)_k a_{i-1}(N-k,\lambda),\ (1\leq i\leq N). \tag{2.20}$$

Here, we observe that the matrix $(a_i(j,\lambda))_{0\leq i,j\leq N}$ is given by

$$\begin{array}{c} \\ 0\\ 1\\ 2\\ 3\\ \vdots\\ N \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \dots & N \end{array}\\ \begin{pmatrix} 1 & 1! & 2! & 3! & \dots & N!\\ 0 & 1 & \cdot & \cdot & \dots & \cdot\\ 0 & 0 & 1 & \cdot & \dots & \cdot\\ 0 & 0 & 0 & 1 & \dots & \cdot\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}\end{array}$$

From (2.12), we easily note that

$$a_1(N+1,\lambda)=\sum_{k_1=0}^{N}(N+\lambda+1)_{k_1}(N-k_1)!, \tag{2.21}$$

$$a_2(N+1,\lambda)=\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-1-k_2}(N+2\lambda+1)_{k_2}(N-k_2+\lambda)_{k_1}(N-k_2-k_1-1)!, \tag{2.22}$$

and

$$\begin{aligned} a_3(N+1,\lambda)=&\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2}(N+3\lambda+1)_{k_3}(N-k_3+2\lambda)_{k_2}\\ &\times(N-k_3-k_2-1+\lambda)_{k_1}(N-k_3-k_2-k_1-2)!, \end{aligned} \tag{2.23}$$

Continuing this process, we obtain

$$\begin{aligned}
a_i(N+1,\lambda) = &\sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-i+1-k_i} \cdots \sum_{k_1=0}^{N-i+1-k_i\cdots-k_2} (N+\lambda i+1)_{k_i} \\
&\times (N+\lambda(i-1)-k_i)_{k_{i-1}} (N+(i-2)\lambda-k_i-k_{i-1}-1)_{k_{i-2}} \\
&\times (N+(i-3)\lambda-k_i-k_{i-1}-k_{i-2}-2)_{k_{i-3}} \times \cdots \\
&\times (N+\lambda-k_i-k_{i-1}-\cdots-k_2-i+2)_{k_1} (N-k_i-k_{i-1}-\cdots-k_1-i+1)!, \\
&\text{where } \quad \le i \le N.
\end{aligned} \tag{2.24}$$

Therefore, we obtain the following theorem.

Theorem 2.1. For $N \in \mathbb{N}$, the nonlinear differential equations

$$N!F^{N+1} = \sum_{i=1}^{N} a_i(N,\lambda)(1+\lambda t)^i F^{(i)}$$

$$\textit{have a solution } F = F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}.$$

From (1.3) and (1.5), we have

$$N!2^{N+1}F^{N+1} = N!\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{N+1} = N!\sum_{n=0}^{\infty} \xi_{n,\lambda}^{(N+1)} \frac{t^n}{n!}. \tag{2.25}$$

Now, we observe that

$$2F^{(i)} = \left(\frac{d}{dt}\right)^i \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right) = \sum_{l=0}^{\infty} \xi_{l+i,\lambda} \frac{t^l}{l!}. \tag{2.26}$$

By Theorem 2.1, (2.22) and (2.23), we get

$$\begin{aligned}
N!\sum_{n=0}^{\infty} \xi_{n,\lambda}^{(N+1)} \frac{t^n}{n!} &= N!2^{N+1}F^{N+1} = 2^N \sum_{i=1}^{N} a_i(N,\lambda)(1+\lambda t)^i 2F^{(i)} \\
&= 2^N \sum_{i=1}^{N} a_i(N,\lambda) \sum_{k=0}^{\infty} (i)_k \lambda^k \frac{t^k}{k!} \sum_{l=0}^{\infty} \xi_{l+i,\lambda} \frac{t^l}{l!} \\
&= 2^N \sum_{i=1}^{N} a_i(N,\lambda) \sum_{n=0}^{\infty} \sum_{k=0}^{n} (i)_k \lambda^k \xi_{n-k+i,\lambda} \binom{n}{k} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(2^N \sum_{i=1}^{N} \sum_{k=0}^{n} \binom{n}{k} a_i(N,\lambda)(i)_k \lambda^k \xi_{n-k+i,\lambda}\right) \frac{t^n}{n!}.
\end{aligned} \tag{2.27}$$

By comparing the coefficients on the both sides of (2.24), we obtain the following theorem.

Theorem 2.2. For $n, N \in \mathbb{N} \cup \{0\}$, we have

$$\xi_{n,\lambda}^{(N+1)} = \frac{2^N}{N!} \sum_{i=1}^{N} \sum_{k=0}^{n} \binom{n}{k} a_i(N, \lambda)(i)_k \lambda^k \xi_{n-k+i,\lambda}.$$

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