Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 3 (2016), pp. 2313–2321

http://www.ripublication.com/gjpam.htm

On Classes of Multivalent Functions Defined in Terms of Katugampola Fractional Derivatives

Amjad S. Barham

Department of Applied Mathematics,
College of Applied Sciences,
Palestine Polytechnic University,
Hebron, Palestine.

Abstract

In the present paper, some subclasses of multivalent functions defined in terms of Katugampola fractional derivative are studied. Coefficient inequalities and other interesting results are obtained.

AMS subject classification:
Keywords:

1. Introduction

Let $A(p)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p}^{Z^{k+p}} p \in N = 1, 2, 3, \ldots \tag{1}$$

which are analytic and multivalent in the open unit half-disk

$$U = \{z : z \in C, Rez > 0, |z| < 1\}$$

Motivating by the definition of fractional derivative given by Katugampola [5] (see, e.g. Anderson and Ulness [1]), for real valued functions, we introduce the fractional derivative for complex valued functions as follows:

Definition 1.1. Let the function $f(z)$ be analytic in a simply connected region (of the z-plane) containing the origin and RE $z > 0$. Then the fractional derivative of order λ is defined by

$$D^{\lambda} f(z) = f^{(\lambda)}(z) = \begin{cases} \lim\limits_{\epsilon \to 0} f(ze^{\epsilon z^{-\lambda}}) - f(z), & \lambda \in (0,1) \\ f(z), & \lambda = 0 \end{cases} \tag{2}$$

For $\lambda \in [0, 1\}$ and Re $z > 0$, the above definition yields the following results:

$$D^{\lambda}[af + bg](z) = aD^{\lambda} f(z) + bD^{\lambda} g(z) \tag{3}$$

$$D^{\lambda}[fg](z) = f(z)D^{\lambda} g(z) + g(z)D^{\lambda} f(z) \tag{4}$$

$$D^{\lambda} z^{p} = pz^{p-\lambda},\ p \in R \tag{5}$$

$$D^{\lambda} f(z) = z^{1-\lambda} \frac{\mathrm{df}}{\mathrm{d}z} \tag{6}$$

Definition 1.2. (Extended Fractional derivative) Under the hypothesis of Definition 1.1, the fractional derivative of order $\lambda + n$ is defined for $\lambda \in [0, 1)$; $n \in N_0 = N \cup \{0\}$ by

$$D^{n+\lambda} f(z) = f^{(n+\lambda)}(z) = \lim_{\epsilon \to 0} \frac{f^{(n)}(ze^{\epsilon z^{n-\lambda}}) - f^{(n)}(z)}{\epsilon} \tag{7}$$

It follows easily that

$$D^{n+\lambda} f(z) = z^{1-\lambda} f^{(n+1)}(z) \tag{8}$$

$$D^{n+\lambda} z^{p} = \frac{\tau(p+1)}{\tau(p-n)} z^{p-\lambda-n},\ p \in R \tag{9}$$

where $\tau(.)$ is Euler's Gamma function. Making use of (9) to both sides of (1), we get

$$D^{q+\lambda} f(z) = \psi(p, q) z^{p-q-\lambda} + \sum_{k=1}^{\infty} \psi(k+p, q) a_{k+p} z^{k+p-q-\lambda} \tag{10}$$

where $p \in R, q \in N_0, \lambda \in [0, 1)$, and $\psi(p, q) = \dfrac{\tau(1+p)}{\tau(p-q)}$.

Definition 1.3. A function $f(z) \in A(p)$ is said to be p-valent λ-starlike function of order α, if it satisfies the inequality

$$Re \left\{ \frac{zD^{\lambda+1} f(z)}{D^{\lambda} f(z)} \right\} > \alpha,\ (z \in U) \tag{11}$$

where $0 \leq \alpha < p, \lambda \in [0, 1)$ and $p \in R$ We denote by $S^{\alpha}(p, \lambda)$ the class of p-valent λ -starlike functions of order λ.

Definition 1.4. A function $f(z) \in A(p)$ is said to be p-valent λ-convex if it satisfies the inequality

$$Re\left\{1+\frac{zD^{\lambda+2}f(z)}{D^{\lambda+1}f(z)}\right\} > \alpha, \ (z \in U) \tag{12}$$

where $0 \leq \alpha < p, \lambda \in [0, 1)$ and $p \in R$ We denote by $C^{\alpha}(p, \lambda)$ the class of p-valent λ-convex functions of order λ.

Note that when $\lambda = 0$, then $S^{\alpha}(p) \equiv S^{\alpha}(p, 0)$ and $C^{\alpha}(p) \equiv C^{\alpha}(p, 0)$ are the well known classes of p-valent starlike and convex functions of order α; respectively (see, e.g. [2,6]).

Now, let us define the following subclass of multivalent analytic functions as follows:

Definition 1.5. The class $T^{\alpha}(p, q, \lambda)$ consists of functions $f(z) \in A(p)$ satisfying the inequality

$$\left|\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} - p + q\right| < p - \alpha \tag{13}$$

where $0 \leq \alpha < p, \lambda \in [0, 1)$ and $p \in R$.

Note that on setting $q = 0$ and $q = 1$ in Definition 1.5, we easily get the classes $S^{\alpha}(p, \lambda)$ and $C^{\alpha}(p, \lambda)$respectively. In the following sections we focus on the functions $f(z)$ belonging to the class $T^{\alpha}(p, q, \lambda)$ defined by (13) to obtain coefficients inequalities and some other interesting results. It is useful here to recall Jack's Lemma [4], which is needed in the present work.

Lemma 1.6. Let $\omega(z)$ be a non constant analytic function in U with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then $z_0 w'(z_0) = cw(z_0)$, where c is a real number satisfying $c \geq 1$.

2. Results for the class $T^{\alpha}(p, q, \lambda)$

We begin by proving the following theorem in which coefficient inequality for functions $f(z) \in A(p)$ belonging to the class $T^{\alpha}(p, q, \lambda)$ is established.

Theorem 2.1. Let $f(z) \in A(p)$ satisfy

$$\sum_{k=1}^{\infty} \frac{(1+p)_k}{(p-q)_k}(k+p-\alpha-\lambda)\left|a_{k+p}\right| \leq p-\alpha+\lambda \tag{14}$$

where $p \in R, q \in N_0, \lambda \in [0, 1\}$ and $0 \leq \alpha < p$. Then $f(z) \in T^{\alpha}(p, q, \lambda)$.

Proof. Making use of (1) and (10), we get

$$\begin{aligned}
&\left|\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)}-p+q\right| \\
&=\left|\frac{zD^{q+\lambda+1}f(z)-(p-q)D^{q+\lambda}}{D^{q+\lambda}f(z)}\right| \\
&=\left|\frac{[\psi(p,q+1)-(p-q)\psi(p,q)]z^{p-q-\lambda}+\sum_{k=1}^{\infty}[\psi(k+p,q+1)-(p-q)\psi(k+p,q)]a_{k+p}z^{k+p-q-\lambda}}{\psi(p,q)z^{p-q-\lambda+\sum_{k=1}^{\infty}[\psi(k+p,q)a_{k+p}z^{k+p-q-\lambda}]}}\right| \\
&\le\left|\frac{[\psi(p,q+1)-(p-q)\psi(p,q)]|z|^{p-q-\lambda}+\sum_{k=1}^{\infty}[\psi(k+p,q+1)-(p-q)\psi(k+p,q)]|a_{k+p}||z|^{k+p-q-\lambda}}{\psi(p,q)|z|^{p-q-\lambda-\sum_{k=1}^{\infty}[\psi(k+p,q)|a_{k+p}||z|^{k+p-q-\lambda}]}}\right| \\
&<\frac{|\psi(p,q+1)-(p-q)\psi(p,q)|+\sum_{k=1}^{\infty}|\psi(k+p,q+1)-(p-q)\psi(k+p,q)|a_{k+p}|}{\psi(p,q)-\sum_{k=1}^{\infty}\psi(k+p,q)|a_{k+p}|}
\end{aligned}$$

This expression is bounded by $p-\alpha$ if

$$\begin{aligned}
&\sum_{k=1}^{\infty}[\psi(k+p,q+1)-(p-\alpha)\psi(k+p,q)]|a_{k+p}| \\
&\quad\le(p-\alpha)\psi(p,q)-|\psi(p,q+1)-(p-\alpha)\psi(p,q)| \\
&\quad\le(2p-q-\alpha)\psi(p,q)-\psi(p,q+1)
\end{aligned}$$

Using the Pochhammer symbol $\dfrac{\tau(\alpha+n)}{\tau(\alpha)}=(\alpha)_n$ and simple calculations, yields the required result. ■

Theorem 2.2. If $f(z)\in A(p)$ satisfies the inequality.

$$\left|\frac{1+\frac{zD^{q+\lambda+2}f(z)}{D^{q+\lambda+1}f(z)}-p+q+\lambda}{\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda+1}f(z)}-p+q+\lambda}-1\right|<\frac{1}{2p-q-\alpha-2\lambda} \tag{15}$$

where $p\in R, q\in N_0, \lambda\in[0,1)$ and $0\le\alpha<p-\lambda$. Then $f(z)\in T^{\alpha}(p,q,\lambda)$.

Proof. Define the function $\omega(z)$ by

$$\omega(z)=\frac{1}{p-\alpha-\lambda}\left[\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda+1}f(z)}-p+q+\lambda\right] \tag{16}$$

It can be easily verified that $\omega(z)$ satisfies the hypothesis of lemma 1.6. By logarithmic differentiation of (16), we obtain

$$1+\frac{zD^{q+\lambda+2}f(z)}{D^{q+\lambda+1}f(z)}-p+q+\lambda$$

$$= (p-\alpha-\lambda)\left[1+\frac{z\omega'(z)}{\omega(z)}.\frac{1}{(p-q-\lambda)+(p-\alpha-\lambda)\omega(z)}\right] \tag{17}$$

Now, let

$$G(z) = \frac{1+\frac{zD^{q+\lambda+2}f(z)}{D^{q+\lambda+1}f(z)}-p+q+\lambda}{\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)}-p+q+\lambda}-1$$

$$= \frac{z\omega'(z)}{\omega(z)}.\frac{1}{(p-q-\lambda)+(p-\alpha-\lambda)\omega(z)} \tag{18}$$

Therefore, from lemma 1.6 and (18), we get

$$|G(z_0)| = \left|\frac{z\omega'(z_0)}{\omega(z_0)}.\frac{1}{(p-q-\lambda)+(p-\alpha-\lambda)\omega(z_0)}\right| \geq \frac{c}{2p-q-\alpha-2\lambda}, c \geq 1$$

which contradicts the inequality (15). Hence, we must have $|\omega(z)| < 1$ for $z \in U$ So, we have

$$\frac{1}{p-\alpha-\lambda}\left\{\left|\frac{zD^{q+\lambda}f(z)}{D^{q+\lambda+1}f(z)}-p-q\right|-\lambda\right\} < |\omega(z)| < 1$$

which directly yields that $f(z) \in T^{p,q,\lambda}$. ■

Corollary 2.3. If $f(z) \in A(p)$ satisfies

$$\left|\frac{1+\frac{zD^{\lambda+2}f(z)}{D^{\lambda+1}f(z)}-p+\lambda}{\frac{zD^{\lambda+1}f(z)}{D^{\lambda}f(z)}-p+\lambda}-1\right| < \frac{1}{2p-1-\alpha-2\lambda} \tag{19}$$

where $p \in R, q \in N_0, \lambda \in [0, 1)$ and $0 \leq \alpha < p$. Then $f(z) \in S^{\alpha}(p, \lambda)$.

Corollary 2.4. If $f(z) \in A(p)$ satisfies

$$\left|\frac{2+\frac{zD^{\lambda+2}f(z)}{D^{\lambda+1}f(z)}-p+\lambda}{\frac{zD^{\lambda+1}f(z)}{D^{\lambda}f(z)}-p+\lambda}-1\right| < \frac{1}{2p-1-\alpha-2\lambda} \tag{20}$$

where $p \in R, q \in N_0, \lambda \in [0, 1)$ and $0 \leq \alpha < p$. Then $f(z) \in C^{\alpha}(p, \lambda)$. Moreover, let $\lambda = 0$ in Corollaries 2.3 and 2.4, we get analogues results for the classical well known classes $S^{\alpha}(po)$ and $C^{\alpha}(p)$, respectively.

3. More Results

In their paper, Irmak and Cho [3] studied the multivalent analytic functions in the open unit disk by using integer order differential operator. In this section it is intended to extend these results by using Katugampola fractional derivative defined by (2) and (7).

Theorem 3.1. If $f(z) \in A(p)$, then

$$Re\left[\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)}\right] < p-q-\mu \Rightarrow \left|D^{q+\lambda}f(z)\right| < \psi(p,q)|z|^{p-q-\lambda-1} \qquad (21)$$

where $p \in R, q \in N_0, \lambda \in [0, 1)$.

Proof. Let $f(z) \in A(p)$ and $\omega(z)$ is defined as

$$D^{q+\lambda}f(z) = \psi(p,q)z^{p-q-\lambda-1}\left[z + \sum_{k=1}^{\infty}\frac{\psi(k+p,q)}{\psi(p,q)}a_{k+p}z^{k+p}\right]$$

$$= \psi(p,q)z^{p-q-\lambda-1}\omega(z) \qquad (22)$$

Differentiating (22) yields

$$zD^{q+\lambda+1}f(z) = \psi(p,q)\omega(z)\left[p-q-\lambda-1+\frac{z\omega'(z)}{\omega(z)}\right]z^{p-q-\lambda-1} \qquad (23)$$

So, in view of (22) and (23), we have

$$\frac{zD^{q+\lambda+1}f(z)}{zD^{q+\lambda}f(z)} = p-q-\lambda-1+\frac{z\omega'(z)}{\omega(z)}, \ (\omega(z) \neq 0) \qquad (24)$$

It is clear that $\omega(z)$ satisfies the hypothesis of Lemma 1.6. We claim that $|\omega(z)| < 1$. Indeed, if not, there exists $z_0 \in U$ such that $\max\limits_{|z|\leq|z_0|} |\omega(z)| = |\omega(z_0)| = 1$. Since, we have $z_0\omega(z_0) = c\omega(z_0)$, $c \geq 1$ from Lemma 1.6. Thus with $z = z_0$,we have from (24) that

$$Re\left[\frac{z_0D^{q+\lambda+1}f(z_0)}{D^{q+\lambda}f(z_0)}\right] = p-q-\alpha-1+Re\left[\frac{z_0\omega'(z_0)}{\omega(z_0)}\right] \geq p-q-\lambda$$

which contradicts the condition in (21). Therefore (22) yields

$$\left|\frac{D^{q+\lambda}f(z)}{z^{p-q-\lambda-1}}\right| = \psi(p,q)|\omega(z)| \leq \psi(p,q) \qquad (25)$$

which directly implies the result (21). ■

Theorem 3.2. Let $f(z) \in A(n)$ and $g(z) \in A(m)$ with $p = n - m$, $(p, n, m \in N)$ and suppose that

$$Re\left[\frac{D^{q+\lambda}g(z)}{zD^{q+\lambda+1}g(z)}\right] > \beta, \ (z \in U, q \in N_0, \lambda \in (0, 1], \beta \geq 0) \qquad (26)$$

If the inequality

$$\left|\frac{D^{q+\lambda}g(z)}{D^{q+\lambda}f(z)}\left[\frac{D^{q+\lambda+1}f(z)}{D^{q+\lambda+1}g(z)} - \frac{\psi(n,q)}{\psi(m,q)}\right]z^p\right| < \beta\left[p+\frac{1}{2}\right]-\frac{1}{2} \qquad (27)$$

holds, Then

$$\left|\frac{\psi(m,q)}{\psi(n,q)}.\frac{D^{q+\lambda}f(z)}{D^{q+\lambda}g(z)} - z^p\right| < |z|^p \tag{28}$$

where $p \in R, q \in N_0, \lambda \in [0,1)$.

Proof. In view of (1) and (10), it is easily seen that

$$\begin{aligned}\frac{\psi(m,q)}{\psi(n,q)}.\frac{D^{q+\lambda}f(z)}{D^{q+\lambda}g(z)} &= \frac{\psi(m,q)\left[\psi(n,q)z^{n-q-\lambda} + \sum_{k=1}^{\infty}\psi(k+n,q)a_{k+p}z^{k+n-q-\lambda}\right]}{\psi(n,q)\left[\psi(m,q)z^{m-q-\lambda} + \sum_{k=1}^{\infty}\psi(k+m,q)a_{k+p}z^{k+m-q-\lambda}\right]}\\ &= z^p.\frac{1+\sum_{k=1}^{\infty}\frac{\psi(k+n,q)}{\psi(n,q)}a_{k+p}z^k}{1+\sum_{k=1}^{\infty}\frac{\psi(k+m,q)}{\psi(m,q)}a_{k+p}z^k}\\ &= z^p\left[1+\sum_{k=1}^{\infty}c_{k+p}z^k\right] \in A(p) = A(n-m)\end{aligned}$$

Now, define the function $\omega(z)$ by

$$\frac{\psi(m,q)}{\psi(n,q)}.\frac{D^{q+\lambda}f(z)}{D^{q+\lambda}g(z)} = z^p[1+\omega(z)] \tag{29}$$

Differentiating (29), then we get

$$\frac{\psi(m,q)}{\psi(n,q)}.\frac{D^{q+\lambda+1}f(z)}{z^pD^{q+\lambda+1}g(z)} - 1 = \omega(z) + [z\omega'(z) + p(1+\omega(z))].\frac{D^{q+\lambda}g(z)}{zD^{q+\lambda+1}f(z)}$$

Now, define the function $F(z)$ by

$$F(z) = \frac{\left[\frac{D^{q+\lambda+1}f(z)}{z^pD^{q+\lambda+1}g(z)} - \frac{\psi(n,q)}{\psi(m,q)}\right]}{\left[\frac{D^{q+\lambda}f(z)}{D^{q+\lambda}g(z)}\right]} \tag{30}$$

Then in view of (29), we have

$$F(z) = \frac{\omega(z)}{1+\omega(z)} + \left[p + \frac{z\omega'(z)}{\omega(z)}\right]\frac{D^{q+\lambda}g(z)}{zD^{q+\lambda+1}g(z)} \tag{31}$$

Now, as in the proof of Theorem 3.1, we claim that $|\omega(z)| < 1$ otherwise,

$$\begin{aligned}|F(z_0)| &= \left|\frac{\omega(z_0)}{1+\omega(z_0)} + \left[p + \frac{z\omega'(z_0)}{\omega(z_0)}\right]\frac{D^{q+\lambda}g(z_0)}{z_0D^{q+\lambda+1}g(z_0)}\right|\\ &\geq \left|\left[p + \frac{z\omega'(z_0)}{\omega(z_0)}\right]\frac{D^{q+\lambda}g(z_0)}{z_0D^{q+\lambda+1}g(z_0)}\right| - \left|\frac{\omega(z_0)}{1+\omega(z_0)}\right|\\ &\geq Re\left[p + \frac{z_0\omega'(z_0)}{\omega(z_0)}\right]Re\left[\frac{D^{q+\lambda}g(z_0)}{z_0D^{q+\lambda+1}g(z_0)}\right| - Re\left[\frac{\omega(z_0)}{1+\omega(z_0)}\right]\\ &\geq \beta\left[p+\frac{1}{2}\right] - \frac{1}{2}\end{aligned}$$

which contradicts (27). Hence $|\omega(z)| < 1$ for all $z \in U$ and (29) evidently yields the inequality (28). ■

Theorem 3.3. If $f(z) \in A(p)$, then

$$\left| D^{q+\lambda} f(z) \left[\frac{z D^{q+\lambda+1} f(z)}{D^{q+\lambda} f(z)} - p + q + \lambda \right] \right| < |z|^{p-q-\lambda}$$

$$\Rightarrow |D^{q+\lambda} f(z) - \psi(p,q) z^{p-q-\lambda}| < |z|^{p-q-\lambda} \tag{32}$$

where $p \in R, q \in N_0, \lambda \in [0, 1)$.

Proof. Making use of (10) for $f(z) \in A(p)$ we get

$$\frac{D^{q+\lambda} f(z)}{z^{p-q-\lambda}} - \psi(p,q) = \sum_{k=1}^{\infty} \psi(k+p,q) a_{k+p} z^k = \omega(z) \tag{33}$$

Differentiating (32) implies

$$\frac{D^{q+\lambda} f(z)}{z^{p-q-\lambda}} = (p - q - \lambda)[\psi(p,q) + \omega(z)] + z\omega'(z)$$

So, we have

$$z\omega'(z) = \frac{D^{q+\lambda} f(z)}{z^{p-q-\lambda}} \left[\frac{z D^{q+\lambda+1} f(z)}{D^{q+\lambda} f(z)} - p + q + \lambda \right]$$

Now, applying lemma 1.6, if $z = z_0$, we obtain $|z_0\omega'(z_0)| = c|\omega'(z_0)| = c \geq 1$, which contradicts the condition (33). So, we have $|\omega(z)| < 1$ for all $z \in U$, which completes the proof of the theorem. ■

Note that by setting $\lambda = 0$ in the above theorems, we get the results obtained by Irmak and Cho [3] as special cases.

References

[1] Anderson D.R. and Ulness D.J., Properties of Katugampola fractional derivative with potential application in quantum mechanics, J. Math. Phys. 56, 063502 (2015).

[2] Duren P.L., Univalent Functions, Springer-Verlag. New York, 1983.

[3] Irmak, H. and Cho N.E., A differential operator and its applications to certain multivalent analytic functions, Hacettepe J. Math. Sat. 36(1) (2007), 1–6.

[4] Jack I.S., Functions starlike and convex of order, J. London Math. Soc. 3(1971), 469–474.

[5] Katugampola U.N., A new fractional derivative with classical properties. E-print, arXiv: 14106563.

[6] Srivastava H.M., and Owa S., Current Topics in Analytic Function Theory, World Scientific, Singapore, 1992.