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The Generalized Triple Difference of Γ^3 Sequence Spaces

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Abstract

In this paper we define some new sequence spaces and give some topological properties of the sequence spaces $\Gamma^3\left(\Delta_v^m, s, p\right)$ and $\Lambda^3\left(\Delta_v^m, s, p\right)$ and investigate some inclusion relations.

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1. Introduction

Throughout w, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial work on double series is found in *Apostol* [1] and double sequence spaces is found in *Hardy* [2], *Subramanian et al.* [3–9], and many others. Later on investigated by some initial work on triple sequence spaces is found in *Sahiner et al.* [10], *Esi et al.* [11–15], *Subramanian et al.* [16–25] and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} (m, n, k = 1, 2, 3, \ldots).$$

A sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. The space Λ^3 and Γ^3 is a metric space with the metric

$$d(x, y) = sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, \ldots \right\}, \tag{1}$$

forall $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ $in\, \Gamma^3$. Let $\phi = \{\text{finite sequences}\}$.

Consider a triple sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \delta_{ijq}$ for all $m, n, k \in \mathbb{N}$,

$$\delta_{mnk} = \begin{bmatrix} 0 & 0 & \ldots 0 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \\ . \\ . \\ . \\ 0 & 0 & \ldots 1 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \end{bmatrix}$$

with 1 in the $(m, n, k)^{th}$ position and zero otherwise.

Consider a triple sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\Im_{ijq}$ for all $m, n, k \in \mathbb{N}$; where $\Im_{ijq}$ denotes the triple sequence whose only non zero term is a $\frac{1}{(i+j+k)!}$ in the $(i, j, k)^{th}$ place for each $i, j, k \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mnk})$ is a Schauder basis for X, or equivalently $x^{[m,n,k]} \to x$.

An FDK-space is a triple sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings are continuous.

If X is a sequence space, we give the following definitions:

(i) $X^{'}$ is continuous dual of X;

(ii) $X^{\alpha} = \left\{ a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} |a_{mnk}x_{mnk}| < \infty, \; for \;\; each \;\; x \in X \right\}$;

(iii) $X^{\beta} = \left\{ a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} a_{mnk}x_{mnk} \;\; is \;\; convergent, \;\; for \;\; each \;\; x \in X \right\}$;

(iv) $X^{\gamma} = \left\{ a = (a_{mn}) : sup_{m,n \geq 1} \left| \sum_{m,n,k=1}^{M,N,K} a_{mnk}x_{mnk} \right| < \infty, \; for \;\; each \;\; x \in X \right\}$;

(v) Let X be an FK-space $\supset \phi$; then $X^{f} = \left\{ f(\Im_{mnk}) : f \in X^{'} \right\}$;

(vi) $X^{\delta} = \{a = (a_{mnk}) : sup_{m,n,k} |a_{mnk}x_{mnk}|^{1/m+n+k} < \infty$, for each $x \in X\}$; $X^{\alpha}.X^{\beta}$, X^{γ} are called α-(or Köthe-Toeplitz) dual of X, β-(or generalized-Köthe-Toeplitz) dual of X, γ-dual of X, δ-dual of X respectively. X^{α} is defined by Gupta and Kamptan [26]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [27] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and

$$\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$$

for all $m, n \in \mathbb{N}$.

Let

$$w^3, \Gamma^3 (\Delta_{mnk}), \Lambda^3 (\Delta_{mnk})$$

be denote the spaces of all, triple gai difference sequence space and triple analytic difference sequence space respectively and is defined as

$$\begin{aligned}\Delta^m x_{mn} &= \Delta\Delta^{m-1} x_{mn} \\ &= \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{mn+2} - \Delta^{m-1} x_{m+1n} \\ &\quad - \Delta^{m-1} x_{m+1n+1} - \Delta^{m-1} x_{m+1n+2} - \Delta^{m-1} x_{m+2n} \\ &\quad - \Delta^{m-1} x_{m+2n+1} - \Delta^{m-1} x_{m+2n+2}\end{aligned}$$

2. Definitions and Preliminaries

A sequence $x = (x_{mnk})$ is said to be triple analytic if $sup_{mnk} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty$. The vector space of all triple analytic sequences is usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if $|x_{mnk}|^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$. The vector space of triple entire sequences is usually denoted by Γ^3.

Throughout the article $w^3, \Gamma^3 (\Delta), \Lambda^3 (\Delta)$ denote the spaces of all, triple entire difference sequence spaces and triple analytic difference sequence spaces respectively.

For a triple sequence $x \in w^3$, we define the sets

$$\Gamma^3 (\Delta) = \left\{x \in w^3 : |\Delta x_{mnk}|^{1/m+n+k} \to 0 \text{ as } m, n, k \to \infty\right\}$$

$$\Lambda^3 (\Delta) = \left\{x \in w^3 : sup_{m,n,k} |\Delta x_{mnk}|^{1/m+n+k} < \infty\right\}.$$

The space $\Lambda^3 (\Delta)$ is a metric space with the metric

$$d(x, y) = sup_{m,n,k} \left\{|\Delta x_{mnk} - \Delta y_{mnk}|^{1/m+n} : m, n, k = 1, 2, \ldots\right\}$$

for all $x = (x_{mnk})$ and $y = (y_{mnk})$ in $\Lambda^3 (\Delta)$.

Now we define the following sequence spaces: Let $s \geq 0$ be real number and $v = (v_{mnk})$ be non-zero real number sequence, then

$$\begin{aligned}\Gamma^3 \left(\Delta_v^m, s, p\right) &= \Big\{x = (x_{mnk}) \in w^3 : (mnk)^{-s} \left(\left|\Delta_v^m x_{mnk}\right|^{1/m+n+k}\right)^{p_{mnk}} \\ &\qquad \to 0 \,(m, n, k \to \infty), s \geq 0\Big\} \\ \Lambda^3 \left(\Delta_v^m, s, p\right) &= \Big\{x = (x_{mnk}) \in w^3 : sup_{m,n,k} (mnk)^{-s} \left(\left|\Delta_v^m x_{mnk}\right|^{1/m+n+k}\right)^{p_{mnk}} \\ &\qquad < \infty, s \geq 0\Big\}\end{aligned}$$

$$\begin{aligned}
\Delta_v^0 x_{mnk} &= (v_{mnk}x_{mnk})\,,\ \Delta_v x_{mnk} \\
&= v_{mn}x_{mn} - v_{mn+1}x_{mn+1} - v_{mn+2}x_{mn+2} \\
&\quad - v_{m+1n}x_{m+1n} - v_{m+1n+1}x_{m+1n+1} - v_{m+1n+2}x_{m+1n+2} \\
&\quad - v_{m+2n}x_{m+2n} - v_{m+2n+1}x_{m+2n+1} - v_{m+2n+2}x_{m+2n+2} \\
\Delta_v^m x_{mn} &= \Delta\Delta_v^{m-1}x_{mn} \\
&= \Delta_v^{m-1}x_{mn} - \Delta_v^{m-1}x_{mn+1} - \Delta_v^{m-1}x_{mn+2} - \Delta_v^{m-1}x_{m+1n} \\
&\quad - \Delta_v^{m-1}x_{m+1n+1} - \Delta_v^{m-1}x_{m+1n+2} - \Delta_v^{m-1}x_{m+2n} \\
&\quad - \Delta_v^{m-1}x_{m+2n+1} - \Delta_v^{m-1}x_{m+2n+2}
\end{aligned}$$

we get the following sequence spaces from the above sequence spaces by choosing some special p, m, s and v. If $s = 0, m = 1$ and

$$v = \begin{bmatrix} 1 & 1 & ...1 & 1 & 0... \\ 1 & 1 & ...1 & 1 & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1 & 1 & ...1 & 1 & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix}$$

with 1 upto $(m, n, k)^{th}$ position and zero otherwise and $p_{mnk} = 1$ for all m, n, k. We have

$$\Gamma^3(\Delta) = \{x = (x_{mnk}) : \Delta x \in \Gamma^3\},$$

$$\Lambda^3(\Delta) = \left\{x = (x_{mnk}) : \Delta x \in \Lambda^3\right\}.$$

If $s = 0$ and $p_{mnk} = 1$ for all m, n, k we have the following sequence spaces

$$\Gamma^3\left(\Delta_v^m\right) = \left\{x = (x_{mnk}) \in w^3 : \Delta_v^m x \in \Gamma^3\right\},$$

$$\Lambda^3\left(\Delta_v^m\right) = \left\{x = (x_{mnk}) \in w^3 : \Delta_v^m x \in \Lambda^3\right\}.$$

If $s = 0, m = 0$ and

$$v = \begin{bmatrix} 1 & 1 & ...1 & 1 & 0... \\ 1 & 1 & ...1 & 1 & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1 & 1 & ...1 & 1 & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix}$$

with 1 upto $(m,n,k)^{th}$ position and zero otherwise. We have the following sequence spaces

$$\Gamma^3(p) = \{x = (x_{mnk}) \in w^3 : |x_{mnk}|^{p_{mnk}/m+n+k} \to 0, (m,n,k \to \infty)\}$$
$$\Lambda^3(p) = \{x = (x_{mnk}) \in w^3 : sup_{m,n,k} |x_{mnk}|^{p_{mnk}/m+n+k} < \infty\}$$

If $m = 0$ and

$$v = \begin{bmatrix} 1 & 1 & ...1 & 1 & 0... \\ 1 & 1 & ...1 & 1 & 0... \\ . \\ . \\ . \\ 1 & 1 & ...1 & 1 & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix}$$

with 1 upto $(m,n,k)^{th}$ position and zero otherwise. We have the following sequence spaces

$$\Gamma^3(p,s) = \{x = (x_{mnk}) \in w^3 : (mnk)^{-s} |x_{mnk}|^{p_{mnk}/m+n+k} \to 0, (m,n,k \to \infty), s \geq 0\},$$
$$\Lambda^3(p,s) = \left\{x = (x_{mnk}) \in w^3 : sup_{m,n,k} (mnk)^{-s} |x_{mnk}|^{p_{mnk}/m+n+k} < \infty, s \geq 0\right\},$$

If $s = 0$, $m = 0$ and $p_{mnk} = 1$

$$v = \begin{bmatrix} 1 & 1 & ...1 & 1 & 0... \\ 1 & 1 & ...1 & 1 & 0... \\ . \\ . \\ . \\ 1 & 1 & ...1 & 1 & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix}$$

for all m, n, k with 1 upto $(m,n,k)^{th}$ position and zero otherwise. We have Γ^3 and Λ^3.

If $s = 0$ we have $\Gamma^3\left(\Delta_v^m, p\right)$ and $\Lambda^3\left(\Delta_v^m, p\right)$.

For a subspace ψ of a linear space is said to be sequence algebra if $x, y \in \psi$ implies that $x \cdot y = (x_{mnk} y_{mnk}) \in \psi$, see Kamptan and Gupta [10].

A sequence E is said to be solid (or normal) if $(\lambda_{mnk} x_{mnk}) \in E$, whenever $(x_{mnk}) \in E$ for all sequences of scalars $(\lambda_{mnk} = k)$ with $|\lambda_{mnk}| \leq 1$.

If X is a linear space over the field $\mathbb{C}$, then a paranorm on X is a function $g : g(\theta) = 0$ where

$$\theta = (0,0,0,\ldots), g(-x) = g(x), g(x+y) \leq g(x) + g(y)$$

and

$$|\lambda - \lambda_0| \to 0, g(x - x_0)$$

imply $g(\lambda x - \lambda_0 x_0) \to 0,$ where $\lambda, \lambda_0 \in C$ and $x, x_0 \in X$. A paranormed space is a linear space X with a paranorm g and is written (X, g).

In this paper, we define some new sequence spaces and give some topological properties of the sequence spaces

$$\Gamma^3\left(\Delta_v^m, s, p\right)$$

and

$$\Lambda^3\left(\Delta_v^m, s, p\right)$$

and investigate some inclusion relations.

3. Main Results

3.1. Theorem

The following statements are hold

(i) $\Gamma^3\left(\Delta_v^m, s\right) \subset \Lambda^3\left(\Delta_v^m, s\right)$ and the inclusion is strict.

(ii) $X\left(\Delta_v^m, s, p\right) \subset X\left(\Delta_v^{m+1}, s, p\right)$ does not hold in general for any $X = \Gamma^3$ and Λ^3.

Proof. (i) If we choose $s = 0$,

$$x = \begin{bmatrix} 1 & 0 & ...1 & 0 & 0... \\ 1 & 0 & ...1 & 0 & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1 & 0 & ...1 & 0 & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix}$$

and

$$v = \begin{bmatrix} 1 & 1 & ...1 & 1 & 0... \\ 1 & 1 & ...1 & 1 & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1 & 1 & ...1 & 1 & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix}$$

Hence $x \in \Lambda^3\left(\Delta_v^m, s\right)$, but $x \notin \Gamma^3\left(\Delta_v^m, s\right)$.

(ii) Let

$$v = \begin{bmatrix} 1 & 1 & ...1 & 1 & 0... \\ 1 & 1 & ...1 & 1 & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1 & 1 & ...1 & 1 & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix},$$

$p = (p_{mnk})$ and $x = (x_{mnk})$ given by

$$p_{mnk} = 1, |x_{mnk}|^{1/m+n+k} = m^2 n^2 k^2$$

if m, n, k is odd

$$p_{mnk} = 2, |x_{mnk}|^{1/m+n+k} = mnk$$

if m, n, k if even 0 otherwise.
Since for

$$\begin{aligned} m, n, k \geq 1, \left|\Delta_v^0 x_{mnk}\right|^{p_{mnk}/m+n+k} &= |x_{mnk}|^{p_{mnk}/m+n+k} = m^2 n^2 k^2 \\ m^{-3} n^{-3} k^{-3} \left|\Delta_v^0 x_{mnk}\right|^{p_{mnk}/m+n+k} &= m^{-3} n^{-3} k^{-3} m^2 n^2 k^2 \\ &= m^{-1} n^{-1} k^{-1} \to 0 \, (m, n, k \to \infty) \text{ and for } j \geq 1 \\ \left|\Delta_v x_{2j,2j,2j}\right|^{p_{2j,2j,2j}/6j} &= \left(6j^3 + 6j^2 + 1\right)^2, (6j)^{-3} \left|\Delta_v x_{2j,2j,2j}\right|^{p_{2j,2j,2j}/6j} \\ &\geq 6j \to \infty \, (j \to \infty). \end{aligned}$$

Now, we can see that $x \in \Gamma^3 \left(\Delta_v^0, 3, p\right)$ and $x \notin \Lambda^3 \left(\Delta_v^0, 3, p\right)$, which imply that $X \left(\Delta_v^m, s, p\right)$ is not a subset of $X \left(\Delta_v^{m+1}, s, p\right)$. This completes the proof. ∎

3.2. Theorem

$\Gamma^3 \left(\Delta_v^m, s, p\right)$ and $\Lambda^3 \left(\Delta_v^m, s, p\right)$ are linear spaces over the complex field $\mathbb{C}$.

Proof. suppose that $M = max\left(1, sup_{m,n,k \geq \mathbb{N}} p_{mnk}\right)$. Since $p_{mnk}/M \leq 1$, we have for all m, n, k

$$\left|\Delta_v^m \left(x_{mnk} + y_{mnk}\right)\right|^{p_{mnk}/M} \leq \left(\left|\Delta_v^m x_{mnk}\right|^{p_{mnk}/M} + \left|\Delta_v^m y_{mnk}\right|^{p_{mnk}/M}\right) \tag{2}$$

and for all $\lambda \in \mathbb{C}$

$$|\lambda|^{p_{mnk}/M} \leq Max\,(1, |\lambda|) \tag{3}$$

Now the linearity follows from (2) and (3). This completes the proof. ∎

3.3. Theorem

Let

$$N_1 = \min\left\{n_0 : sup_{m,n,k \geq n_0} (mnk)^{-s} \left(\left|\Delta_v^m x_{mnk}\right|^{1/m+n+k}\right)^{p_{mnk}} < \infty\right\},$$
$$N_2 = \min\left\{n_0 : sup_{m,n,k \geq n_0} p_{mnk} < \infty\right\},$$
$$N_3 = \min\left\{n_0 : sup_{m,n,k \geq n_0} < \infty\right\} \text{ and}$$
$$N = \max\{N_1, N_2, N_3\}$$

$\Gamma^3\left(\Delta_v^m, s, p\right)$ is a paranormed space with

$$g(x) = \sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{r} |x_{mnk}| + lim_{N\to\infty} sup_{m,n,k\geq N} (mnk)^{-S/M} \left|\Delta_v^m x_{mnk}\right|^{p_{mnk}/M} \quad (4)$$

if and only if $\mu > 0$, where $\mu = lim_{N\to\infty} inf_{m,n,k\geq N}\, p_{mnk}$ and
$M = max\left(1, sup_{m,n,k\geq N}\, p_{mnk}\right)$.

Proof. (i) **Necessity:** Let $\Gamma^3\left(\Delta_v^m, s, p\right)$ be a paranormed space with (4) and suppose that $\mu = 0$. Then $\alpha = inf_{m,n,k\geq N}\, p_{mnk} = 0$ for all $N \in \mathbb{N}$ and hence we obtain

$$g(\lambda x) = \sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{r} |x_{mnk}| + lim_{N\to\infty} sup_{m,n,k\geq N} (mnk)^{-s} |\lambda|^{p_{mnk}/M} = 1$$

for all $\lambda \in (0, 1]$, where $x = \alpha \in \Gamma^3\left(\Delta_v^m, s, p\right)$. whence $\lambda \to 0$ does not imply $\lambda x \to \theta$, when x is fixed. But this contradicts to (4) to be a paranorm.

Sufficiency: Let $\mu > 0$. It is trivial that

$$g(\theta) = 0, g(-x) = g(x)$$

and

$$g(x + y) \leq g(x) + g(y).$$

Since $\mu > 0$ there exists a positive number β such that $p_{mnk} > \alpha, \beta$ for sufficiently large positive integer m, n, k. Hence for any $\lambda \in \mathbb{C}$, we may write

$$|\lambda|^{p_{mnk}} \leq \max\left(|\lambda|^M, |\lambda|^\alpha, |\lambda|^\beta\right)$$

for sufficiently large positive integers $m, n, k \geq \mathbb{N}$. Therefore, we obtain that

$$g(\lambda x) \leq \max\left(|\lambda|, |\lambda|^{\alpha/M}, |\lambda|^{\beta/M}\right) g(x)$$

using this, one can prove that $\lambda x \to \theta$, whenever x is fixed and $\lambda \to 0$, (or) $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$. This completes the proof. ■

3.4. Theorem

Let $0 < p_{mnk} \leq q_{mnk} \leq 1$ for all $m, n, k \in \mathbb{N}$, then

(i) $\Lambda^3 \left(\Delta_v^m, s, p\right) \subseteq \Lambda^3 \left(\Delta_v^m, s, q\right)$

(ii) $\Gamma^3 \left(\Delta_v^m, s, p\right) \subseteq \Gamma^3 \left(\Delta_v^m, s, q\right)$.

Proof. (i) Let $x \in \Lambda^3 \left(\Delta_v^m, s, p\right)$. Then there exists a constant $M > 1$ such that

$$(mnk)^{-s} \left|\Delta_v^m x_{mn}\right|^{q_{mnk}/m+n+k} \leq M \text{ for all } m, n, k$$

and so

$$(mnk)^{-s} \left|\Delta_v^m x_{mnk}\right|^{q_{mnk}/m+n+k} \leq M \text{ for all } m, n, k$$

suppose that $x^i \in \Lambda^3 \left(\Delta_v^m, s, q\right)$ and $x^i \to x \in \Lambda^3 \left(\Delta_v^m, s, p\right)$. Then for every $o < \epsilon < 1$, there exist N such that for all m, n, k

$$(mnk)^{-s} \left|\Delta_v^m \left(x_{mnk}^{(i)} - x_{mnk}\right)\right|^{p_{mnk}/m+n+k} < \epsilon \text{ for all } i > N$$

Now,

$$\begin{aligned} &(mnk)^{-s} \left|\Delta_v^m \left(x_{mnk}^{(i)} - x_{mnk}\right)\right|^{q_{mnk}/m+n+k} \\ &< (mnk)^{-s} \left|\Delta_v^m \left(x_{mnk}^{(i)} - x_{mnk}\right)\right|^{p_{mnk}/m+n+k} \\ &< \epsilon \text{ (for all } i > N) \end{aligned}$$

Therefore $x \in \Lambda^3 \left(\Delta_v^m, s, q\right)$. This completes the proof. (ii) It is easy. Therefore omit the proof. ■

3.5. Proposition

For $X = \Gamma^3$ and Λ^3, then we obtain (i) $X \left(\Delta_v^m, s, p\right)$ is not sequence algebra, in general (ii) $X \left(\Delta_v^m, s, p\right)$ is not solid, in general.

Proof. (i) This result is clear from the following example:

Example (1) Let

$$p_{mnk} = 1, v_{mnk} = \frac{1}{(mnk)^{2(m+n+k)}}, x_{mnk} = (mnk)^{2(m+n+k)}$$

and

$$y_{mnk} = (mnk)^{2(m+n+k)}$$

for all m, n, k. Then we have $x, y \in \Gamma^3 (\Delta, 0, p)$ but $x, y \notin \Gamma^3 (\Delta, 0, p)$ with $m = 1$ and $s = 0$.

Proof. (ii) This result is clear from the following example

Example (2) Consider

$$x_{mnk} = \begin{bmatrix} 1 & 1 & ...1 & 1 & 0... \\ 1 & 1 & ...1 & 1 & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1 & 1 & ...1 & 1 & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix} \in \Gamma^3 \left(\Delta_v^m, s, p\right)$$

Let $p_{mnk} = 1, \alpha_{mnk} = (-1)^{m+n+k}$, then $\alpha_{mnk} x_{mnk} \notin \Gamma^3 \left(\Delta_v^m, s, p\right)$ with $m = 1$ and $s = 0$. The following proposition's proof is a routine verification. ■

3.6. Proposition

For $X = \Gamma^3$ and Λ^3, then we obtain

(i) $s_1 < s_2$ implies $X \left(\Delta_v^m, s_1, p\right) \subset X \left(\Delta_v^m, s_2, p\right)$,

(ii) Let $0 < inf p_{mnk} < p_{mnk} \leq 1$ then $X \left(\Delta_v^m, s, p\right) \subset X \left(\Delta_v^m, s\right)$,

(iii) Let $1 \leq p_{mnk} \leq sup_{mnk} p_{mnk} < \infty$, then $X \left(\Delta_v^m, s\right) \subset X \left(\Delta_v^m, s, p\right)$,

(iv) Let $0 < p_{mnk} \leq q_{mnk}$ and $\left(\frac{q_{mnk}}{p_{mnk}}\right)$ be bounded, then $X \left(\Delta_v^m, s, q\right) \subset X \left(\Delta_v^m, s, p\right)$.

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