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Exact Solutions and Lattice Boltzmann Method Modelling for Shallow Water Equations

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Abstract

In this study, lattice Boltzmann Method (LBM) is applied for shallow water flows problem in order to handle two dimensional flows. The LBM and its implementation in solving Shallow Water Equation (SWE)for physically significant nonlinear partial differential equations were also analysed.An analytical solution of steady and two dimensional SWE was derived from a cubic algebraic form of SWE, and obtained the analytical solution of unsteady and two-dimensional SWE from sub inertial and super inertial frequency quadratic algebraic form of SWE. The numerical solution of the two dimensional SWE was generated by applying the LBM. The LBM numerical result of SWE was then compared graphically with the SWE result obtained via the finite difference method (FDM). Our findings revealed that the simulation via LBM is more stable than FDM in the microscopic sense.

AMS subject classification:
Keywords: Shallow water equations, Lattice Boltzmann method, Two dimensional flows.

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1. Introduction

Partial differential equations have been utilized as practical tools to framework many environmental problems from real life and also they have been used to foretell and approximate the dynamics of related problems. The shallow water equations are fundamentally used to model physical phenomenal of water flows like tidal flows in an estuary and coastal water regions, dam breaks, flood waves, and bore wave propagation in rivers, among others [1, 2]. The mathematical theoretical accounts can be very useful to understand the dynamics of water flows. Water quality modelling has obtained significance as a research area because it also involves some problems that are related to public safety. On the other hand, numerical simulation of shallow water flows on a rotating sphere has an important role on atmospheric sciences [2, 3].

The shallow water equation has general characteristics which show that the vertical dimension is much smaller than the typical horizontal scale. As a result, the depth can be average to get rid of the vertical dimension. The SWE is an important equation which can be used in predicting tides, storm surge levels and coastline changes from hurricanes, ocean currents, and studying dredging feasibility [1, 3, 4]. The general analytical solution of one-dimensional shallow water equation is discussed but the analytical investigation of two-dimensional shallow water equations is much more complex [5, 6]. Up till date, there are very few exact solutions to the SWE and most of them only describe simple flows in idealized environments but in this study, shape of the eddy (shallow water) would be determined. This study investigates the application of the LBM formulation for the shallow water flow in order to handle two dimensional flows, and to study the stability and accuracy of the method.Furthermore, a simple and practical numerical model that can help to resolve problems and issues that are related to shallow water flows shall be presented in this study. This involves looking into how the Lattice Boltzmann Method (LBM traditional method) will perform under the same condition. This research focuses on shallow water equation with the aim of applying the proposed method for numerical simulation of shallow water equations.

2. Analytical Solutions to the Shallow Water Equations

In the rotating fluid, the set of shallow water equations has been applied which is nonetheless severely limited and it regularly excludes the baroclinic-instability mechanism. In a closed form, for exact analytical solutions, the power of the present contribution is the quest and is often used as stepping stones toward further analytical investigations. For testing the accuracy of numerical models also it can be used. Wherever the coriolis parameter is a constant, the choice is made to limit the search within the context of the shallow water equations because such solutions are rare and quite difficult to obtain. Considering Goldsbrough's [4] inclusion of rotation, there has been a rediscovery and the shallow water equations has been forgotten. This admits a special class of solutions in the case that the height field is a quadratic expression and the velocity components are linear functions in the horizontal coordinate variables. Perhaps it was further discovered by Kirchhoff [5, 7] for the 2-dimensional Euler equations. The pressure field is a quadratic expression and the velocity components are linear functions of the coordinate variables, therefore, this solution is known as the Kirchhoff vortex. After Goldsbrough, in order to study oscillatory modes in shallow, Ball [8, 9] took advantage of this solution for the spatial structure; he rotated based on reference to tides. Ball's work became forgotten and obsolete, as the interest in tides vanished. Cushman-Roisin [10] found an exact analytical solution for an elliptical vortex with oceanic warm-core rings in mind which steadily rotates without deformation. This solution is referred to as the Rodon and Cushman-Roisin et al. [11], which was further analyzed and placed in a context.

3. Rodon Solutions of the Shallow Water Equations

Ball [9] found in an elliptic paraboloid the exact nonlinear analytical solutions of the SWE for steady flow [6]. Young, reported for unsteady elliptic vortices. Zero divergence and spatially constant vorticity characterize these flows. The analysis procedure for both proceeds is similar to each other. Rodon is the time-dependent solutions of Cushman-Roisin et al. [11] which correspond to steadily rotating elliptic eddies. In order to exist in a circular paraboloid and to reduce to the Rodon solution as the bottom topographic variations vanish, unsteady motion with dynamics similar to Rodon is therefore suggested Mathematical formulation of the problem. The model has been verified by solving both steady and unsteady flow problems.

3.1. Analytical Solutions of the Shallow Water Equations (1-D)

The classical system of one-dimensional shallow water equations has the form,

$$u_t + uu_x + gh_x = 0, \tag{1}$$

$$h_t + uh_x + hu_x = 0 \tag{2}$$

where $u(t, x)$ is the velocity and $h(t, x)$ is the depth of the horizontal boundary, and g is the acceleration due to gravity. The suffixes in equation (1) correspond to the differentiation over time t and position x. The more symmetric form of the system will be used

$$w = 2(gh)^{\frac{1}{2}} \tag{3}$$

$$w_x = gh_x(gh)^{-\frac{1}{2}} \implies h_x = w_x g^{-\frac{1}{2}} h^{-\frac{1}{2}} \tag{4}$$

$$\frac{1}{2} ww_x = gh_x \tag{5}$$

$$w_t = gh_t(gh)^{-\frac{1}{2}} \implies h_t = w_t g^{-\frac{1}{2}} h^{\frac{1}{2}}. \tag{6}$$

By using transformation equations (3), (4), (5) and (6), the equation (1) became as

$$u_t + uu_x + \frac{1}{2} ww_x = 0 \tag{7}$$

and the equation (2) became as

$$w_t + uw_x + \frac{1}{2}wu_x = 0. \tag{8}$$

By introducing "hodograph" transformation and considering (t, x) as functions of (u, w). The Jacobian should be non-zero. Using (Reimann-invariant)

$$u_t = Dx_w \tag{9}$$

$$w_t = -Dx_u \tag{10}$$

$$u_x = -Dt_w \tag{11}$$

$$w_x = Dt_u. \tag{12}$$

Equations (7) and (8) take the form of linear equation

$$Dx_w - uDt_w + \frac{1}{2}wDt_u = 0, \tag{13}$$

$$-Dx_u + uDt_u - \frac{1}{2}wDt_w = 0. \tag{14}$$

Linear equations

$$x_u - ut_u + wt_w = 0 \tag{15}$$

$$x_w - ut_w + \frac{1}{2}wt_u = 0. \tag{16}$$

By cross-differentiation x is eliminated from above and we get, equation (15) as follows

$$x_{uw} - ut_{uw} + \frac{1}{2}t_w + \frac{1}{2}wt_{ww} = 0 \tag{17}$$

and equation (16) as follows

$$-x_{uw} + t_w + ut_{wu} - \frac{1}{2}wt_{uu} = 0, \tag{18}$$

$$3w^{-1}t_w + t_{ww} - t_{uu} = 0. \tag{19}$$

Using the Fourier-transformation,

$$t(u, w) = \int_{-\infty}^{\infty} e^{iku} \cdot \tau(k, w)\, dk, \tag{20}$$

$$\delta\{T_{ww}\} + \delta\{3w^{-1}T_w\} - \delta\{T_{uu}\} = 0, \tag{21}$$

$$\tau_{ww} + 3w^{-1}\tau_w + k^2\tau = 0, \tag{22}$$

$$\varphi = w\tau, \quad \tau_w = \frac{\varphi_w \cdot w - \varphi}{w^2}, \quad \tau_{ww} = \frac{1}{w^3}(\varphi_{ww} \cdot w^2 - 2(\varphi_w \cdot w - \varphi)), \tag{23}$$

and we get,

$$w^2\varphi_{ww} + w\varphi_w + (k^2w^2 - 1)\varphi = 0. \tag{24}$$

The above equation is the Bessel equation, then the solution of equation (24) became as

$$\tau(u, w) = c_1 J_{\frac{1}{k}}(w) + c_2 J_{\frac{1}{k}}(w) \tag{25}$$

or

$$\tau(u, w) = c_1 J_1^{(1)}(kw) + c_2 J_1^{(2)}(kw) \tag{26}$$

where c_1, c_2 are the arbitrary functions and $J_1^{(1)}$, $J_1^{(2)}$ are two independent Bessel functions of the first order. Finally, we have

$$\tau(u, w) = w^{-1} \int_{-\infty}^{\infty} e^{iku} \left\{ c_1 J_1^{(1)}(kw) + c_2 J_1^{(2)}(kw) \right\}. \tag{27}$$

3.2. Analytical Solutions to the Shallow Water Equations (2-D)

The rotating fluid flows governed by the inviscid shallow water equation (SWE) on an f-plan are focused on. Dimensionless variables related to those in [12] are used so that in Cartesian coordinates (x, y) the continuity and momentum equations are;

$$\varepsilon F \eta_t + (hu)_x + (hu)_y = 0, \tag{28}$$

$$\varepsilon u_t + \varepsilon(uu_x + vu_y) - v = -\eta_x, \tag{29}$$

$$\varepsilon v_t + \varepsilon(uv_x + vv_y) - u = -\eta_y, \tag{30}$$

$$h = \varepsilon F \eta + 1 - h_B. \tag{31}$$

$\varepsilon F \eta$ is the height of the free surface relative to the undisturbed depth of the fluid $H = 1 - h_B$, where $h_B = (x, y)$ is the height of the bottom topography and (u, v) are velocity components in the (x, y) direction and t is the time.

Following Ball [9], the motion in an elliptic paraboloid with bottom topography is considered.

$$h_B = \frac{1}{2}\alpha x^2 + \frac{1}{2}\beta y^2, \quad \alpha, \ \beta \geq 0, \tag{32}$$

$$u = U_1 x + U_2 y, \tag{33}$$

$$v = V_1 x + V_2 y, \tag{34}$$

$$\eta = \eta_0 + \frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2, \tag{35}$$

where the coefficients U_1, U_2, V_1, V_2, η_0, A, B and C are functions of time. Substitution of equations (32)-(35) in equations (28)-(31), gives eight nonlinear ordinary differential equations for the eight coefficients.

$$\varepsilon F\left(\eta_0+\frac{1}{2}Ax^2+Bxy+\frac{1}{2}Cy^2\right)+\left(\varepsilon F\left(\eta_0+\frac{1}{2}Ax^2+Bxy+\frac{1}{2}Cy^2\right)+1\right.$$
$$\left.-\left(\frac{1}{2}\alpha x^2+\frac{1}{2}\beta y^2\right)(U_1x+U_2y)_x\right)+\left(\varepsilon F\left(\eta_0+\frac{1}{2}Ax^2+Bxy+\frac{1}{2}Cy^2\right)+1\right.$$
$$\left.-\left(\frac{1}{2}\alpha x^2+\frac{1}{2}\beta y^2\right)(U_1x+U_2y)_y\right)=0 \tag{36}$$

$$\varepsilon(U_1x+U_2y)_t+\varepsilon((U_1x+U_2y)(U_1x+U_2y)_x+(V_1x+V_2y)(U_1x+U_2y)_y)$$
$$-(V_1x+V_2y)=-\left(\eta_0+\frac{1}{2}Ax^2+Bxy+\frac{1}{2}Cy^2\right)_x \tag{37}$$

$$\varepsilon(V_1x+V_2y)_t+\varepsilon((U_1x+U_2y)(V_1x+V_2y)_x+(V_1x+V_2y)(V_1x+V_2y)_y)$$
$$-(U_1x+U_2y)=-\left(\eta_0+\frac{1}{2}Ax^2+Bxy+\frac{1}{2}Cy^2\right)_y \tag{38}$$

$$h=\varepsilon F\left(\eta_0+\frac{1}{2}Ax^2+Bxy+\frac{1}{2}Cy^2\right)+1-\left(\frac{1}{2}\alpha x^2+\frac{1}{2}\beta y^2\right). \tag{39}$$

The following variables are more convenient to work with;

$$\xi=v_x-u_y=V_1-U_2, \tag{40}$$

$$D=u_x+v_y=U_1+V_2, \tag{41}$$

$$M=v_x+u_y=V_1+U_2, \tag{42}$$

$$L=u_x-v_y=U_1-V_2, \tag{43}$$

$$R=\eta_{xx}+\eta_{yy}=A+C, \tag{44}$$

$$S=2\eta_{xy}=2B, \tag{45}$$

$$Q=\eta_{xx}-\eta_{yy}=A-C. \tag{46}$$

The solutions of the equations for the variables defined above are;

$$\varepsilon\xi_t+(\varepsilon\xi+1)D=0 \tag{47}$$

$$\varepsilon D_t+\frac{1}{2}\varepsilon(D^2+L^2+M^2-\xi^2)-\xi=-R \tag{48}$$

$$\varepsilon M_t+\varepsilon MD+L=-S \tag{49}$$

$$\varepsilon L_t+\varepsilon LD-M=-Q \tag{50}$$

$$R_t + SM + 2D(R - \tilde{\theta}) + L(Q - \tilde{\varphi}) = 0 \tag{51}$$

$$Q_t + S\xi + 2D(Q - \tilde{\varphi}) + L(R - \tilde{\theta}) = 0 \tag{52}$$

$$S_t + 2SD + M(R - \tilde{\theta}) - \xi(Q - \tilde{\varphi}) = 0 \tag{53}$$

$$\varepsilon F\eta_{0t} + (\varepsilon F\eta_0 + 1)D = 0 \tag{54}$$

$$\tilde{\theta} = \frac{\theta}{\varepsilon F} \implies \theta = \alpha + \beta \implies \tilde{\theta} = \frac{\theta}{F} \tag{55}$$

$$\tilde{\varphi} = \frac{\varphi}{\varepsilon F} \implies \varphi = \alpha - \beta \implies \tilde{\varphi} = \frac{\varphi}{F}. \tag{56}$$

(The stability of the solutions is not taken into consideration here).

3.3. Steady Flow

For steady solutions of equations (47)-(54) for $\tilde{\varphi} \neq 0$ are given by

$$D = 0, \quad L = -S = 0, \quad M = Q, \quad \tilde{\varphi} \neq 0 \implies \tilde{\varphi} = \frac{\varphi}{F} = \frac{\alpha - \beta}{F} \neq 0, \quad \varepsilon\xi_t = 0, \tag{57}$$

$$\frac{1}{2}\varepsilon(M^2 - \xi^2) - \xi = -R \implies R = \xi + \frac{1}{2}\varepsilon(\xi^2 - Q^2) \tag{58}$$

$$\varepsilon M_t = 0 \tag{59}$$

$$M = Q \tag{60}$$

$$R_t = 0 \tag{61}$$

$$Q_t = 0 \tag{62}$$

$$M(R - \tilde{\theta}) - \xi(Q - \tilde{\varphi}) = 0 \implies \xi Q - QR + Q\tilde{\theta} = \xi\tilde{\varphi} \implies Q = \frac{\xi\tilde{\varphi}}{[\tilde{\theta} - \varepsilon(R - \xi)]} \tag{63}$$

$$\varepsilon F\eta_{0t} = 0. \tag{64}$$

The solutions above are also valid in the limit $\tilde{\varphi} \to 0$. But for $\tilde{\varphi} = 0$ additional steady solutions with $L \neq 0$, $S \neq 0$ are possible. Substituting equation (63) in equation (58) gives the following cubic equation for, $T = \varepsilon(R - \xi)$

$$T^3 - \left(2\tilde{\theta} + \frac{1}{2}\varepsilon^2\xi^2\right)T^2 + (\tilde{\theta}^2 + \tilde{\theta}\varepsilon^2\xi^2)T - \frac{1}{2}\varepsilon^2\xi^2(\tilde{\theta}^2 - \tilde{\varphi}^2) = 0. \tag{65}$$

To insure physically realizable solutions, we require

$$\varepsilon FA < \alpha, \quad \varepsilon FC < \beta \quad \text{or} \quad \frac{1}{2}(R + Q) < \alpha, \quad \frac{1}{2}(R - Q) < \beta. \tag{66}$$

We consider the steady flow in an elliptic paraboloid for $0 \leq \varepsilon < 1$ with either positive or negative vorticity ($\xi \pm 1$) (and with variable values of the bottom topographic

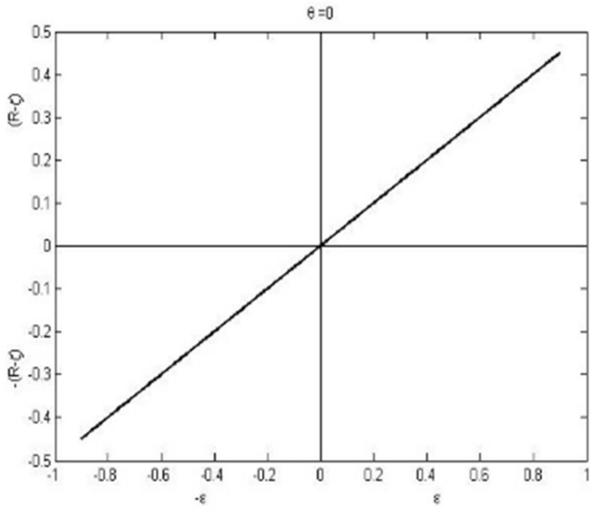

Figure 1: Exact solutions for steady in an elliptic paraboloid with $\theta = 0$.

parameters and F, such that $0 \le \tilde{\theta} < \infty$ and $0 \le \dfrac{\tilde{\varphi}}{\tilde{\theta}} \le 1$. For SWE, T is obtained from the numerical solution of the cubic equation. First we assume $\tilde{\theta} = 0$, $\tilde{\varphi} = 0$, and $\xi = 1$ then

$$T^3 - \left(\frac{1}{2}\varepsilon^2\xi^2\right)T^2 = 0 \implies T = 0,\ T = \frac{1}{2}\varepsilon^2, \quad R = 1 + \frac{1}{2}\varepsilon. \tag{67}$$

Figure 1 shows the graph T against ε. Second we assume $\tilde{\theta} = 1$, $\tilde{\varphi} = 0.5$, and $\xi = 1$ then

$$T^3 - \left(2 + \frac{1}{2}\varepsilon^2\right)T^2 + (1 + \varepsilon^2)T - \frac{3}{8}\varepsilon^2 = 0. \tag{68}$$

Figure 2 shows the graph T against ε.

Third we assume $\tilde{\theta} = 2000$, $\tilde{\varphi} = 1000$, and $\xi = \pm 1$ then the effect of an increased value of $\tilde{\theta}$ in the steady flow in an elliptic paraboloid is shown in below graph. Figure 3 shows the graph T against ε. Fourth, the effect of a decreased value of $\tilde{\theta}$ in the steady flow in an elliptic paraboloid is shown in below graph where the parameters are $\tilde{\theta} = 0.02$, $\tilde{\varphi} = 0.01$, and $\xi = -1$. Figure 4 shows the graph T against ε.

3.4. Unsteady Flow

For unsteady case, the class of motions, governed by equations (47)-(54) for which $D = 0$ is considered

$$\varepsilon\xi_t + (\varepsilon\xi + 1)D = 0 \implies \xi_t = 0 \tag{69}$$

$$\varepsilon D_t + \frac{1}{2}\varepsilon(D^2 + L^2 + M^2 - \xi^2) - \xi = -R \implies \frac{1}{2}\varepsilon(L^2 + M^2 - \xi^2) - \xi = -R \tag{70}$$

$$\varepsilon M_t + \varepsilon MD + L = -S \implies \varepsilon M_t + L = -S \tag{71}$$

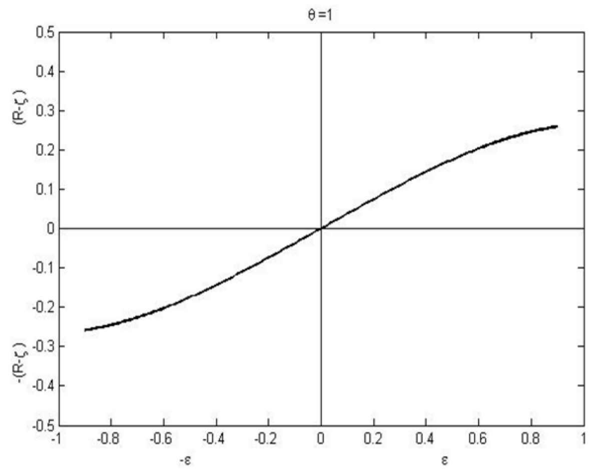

Figure 2: Exact solutions for steady in an elliptic paraboloid with $\theta = 1$ [9].

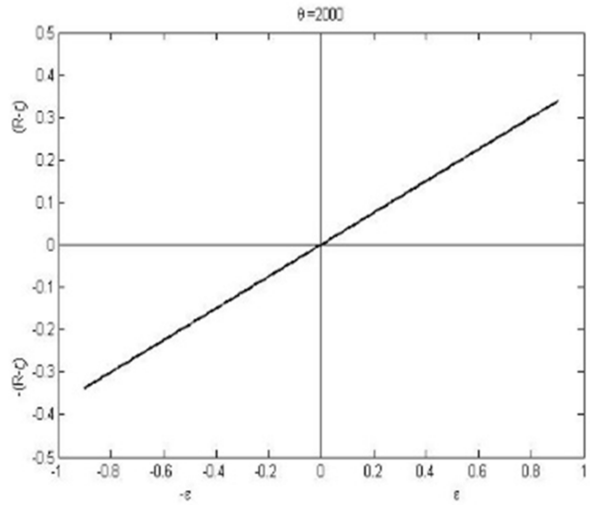

Figure 3: Exact solutions for steady in an elliptic paraboloid with $\theta = 2000$ [9].

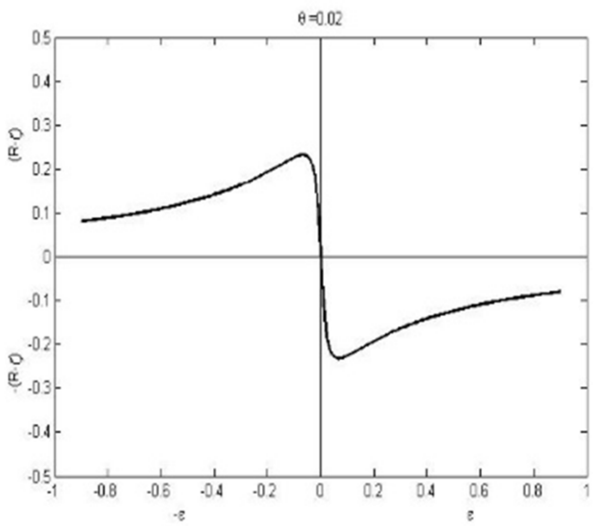

Figure 4: Exact solutions for steady in an elliptic paraboloid with $\theta = 0.02$ [9].

$$\varepsilon L_t + \varepsilon LD - M = -Q \implies \varepsilon L_t - M = -Q \tag{72}$$

$$R_t + SM + 2D(R - \tilde{\theta}) + L(Q - \tilde{\varphi}) = 0 \implies R_t + SM + L(Q - \tilde{\varphi}) = 0 \tag{73}$$

$$Q_t + S\xi + 2D(Q - \tilde{\varphi}) + L(R - \tilde{\theta}) = 0 \implies Q_t + S\xi + L(R - \tilde{\theta}) = 0 \tag{74}$$

$$S_t + 2SD + M(R - \tilde{\theta}) - \xi(Q - \tilde{\varphi}) = 0 \implies S_t + M(R - \tilde{\theta}) - \xi(Q - \tilde{\varphi}) = 0 \tag{75}$$

$$\varepsilon F\eta_{0t} + (\varepsilon F\eta_0 + 1)D = 0 \implies \varepsilon F\eta_{0t} = 0. \tag{76}$$

Substituting equations (69), (71)-(73), in the time derivative of equation (70), gives

$$\frac{1}{2}\varepsilon(L^2 + M^2 - \xi^2) - \xi = 0 \implies \frac{1}{2}\varepsilon(2L_tL + 2M_tM - 2\xi_t\xi) - \xi_t = -R_t$$

$$\varepsilon L_tL + \varepsilon M_tM - \varepsilon\xi_t\xi - \xi_t = -R_t \implies \xi_t = 0,$$

$$-LQ + LM - SM - ML = SM + LQ - L\tilde{\varphi} \implies SM + LQ - \frac{1}{2}L\tilde{\varphi} = 0 \tag{77}$$

The substitution of equation (77) in equation (73) gives

$$R_t = \frac{1}{2}L\tilde{\varphi}. \tag{78}$$

When the topography is a circular paraboloid, the analytical solutions in terms of simple functions maybe determined, $\tilde{\varphi} = \alpha - \beta = 0$, we reduce equation (75) to

$$S_t + M(R - \tilde{\theta}) - \xi(Q - \tilde{\varphi}) = 0 \implies S_t + M(R - \tilde{\theta}) - \xi Q = 0,$$

equation (77) to

$$SM + LQ - \frac{1}{2}L\tilde{\varphi} = 0 \implies LQ + MS = 0,$$

equation (78) to

$$R_t = \frac{1}{2}L\tilde{\varphi} \implies R_t = 0.$$

Solution to equation (69)

$$\xi_t = 0,$$

equation (70)

$$\frac{1}{2}\varepsilon(L^2 + M^2 - \xi^2) - \xi = -R,$$

equation (71)

$$\varepsilon M_t + L = -S,$$

equation (72)

$$\varepsilon L_t - M = -Q,$$

equation (75)

$$S_t + M(R - \tilde{\theta}) - \xi Q = 0,$$

equation (77)

$$LQ + MS = 0,$$

and equation (78)

$$R_t = 0,$$

may be found [11] in the form

$$\xi_0 = \xi,\ R_0 = R,\ (L, S) = (L_0, Q_0)\sin(wt+\theta_0),\ (M, Q) = (-L_0, Q_0)\cos(wt+\theta_0),$$

$$\begin{aligned} &\frac{1}{2}\varepsilon(L^2 + M^2 - \xi^2) - \xi = -R \\ \implies &\frac{1}{2}\varepsilon\left[L_0^2\sin^2(wt+\theta_0)\right] + \left[L_0^2\cos^2(wt+\theta_0)\right] - \xi_0^2 - \xi_0 = -R_0 \\ \implies &-R_0 = \frac{1}{2}\varepsilon L_0^2 - \xi_0^2 - \xi_0 \end{aligned} \tag{79}$$

$$\begin{aligned} \varepsilon M_t + L = -S &\implies \varepsilon L_0 w\sin(wt+\theta_0) + L_0\sin(wt+\theta_0) = -\xi_0 \\ &\implies \sin(wt+\theta_0)[\varepsilon L_0 w + L_0] = -\xi_0 \end{aligned} \tag{80}$$

$$\begin{aligned} &\varepsilon L_t - M = -Q \\ \implies &\varepsilon L_0 w\cos(wt+\theta_0) + L_0\cos(wt+\theta_0) = -Q_0\cos(wt+\theta_0) \\ \implies &\varepsilon L_0 w + L_0 = -Q_0 \end{aligned} \tag{81}$$

$$
\begin{aligned}
& LQ + MS = 0 \\
\Longrightarrow\ & L_0 \sin(wt + \theta_0) Q_0 \cos(wt + \theta_0) - L_0 \cos(wt + \theta_0) Q_0 \sin(wt + \theta_0) = 0 \\
\Longrightarrow\ & L_0 Q_0 = 0
\end{aligned} \tag{82}
$$

$$
\begin{aligned}
& S_t + M(R - \tilde{\theta}) - \xi Q = 0 \\
\Longrightarrow\ & Q_0 w \cos(wt + \theta_0) + R_0[-L_0 \cos(wt + \theta_0)] \\
& + L_0 \tilde{\theta} \cos(wt + \theta_0) = \xi_0 Q_0 \cos(wt + \theta_0) \\
\Longrightarrow\ & Q_0 w - R_0 L_0 + L_0 \tilde{\theta} = \xi_0 Q_0 \\
\Longrightarrow\ & Q_0 w + L_0 \left(\frac{1}{2} \varepsilon L_0^2 - \xi_0^2 - \xi_0 \right) + L_0 \tilde{\theta} = \xi_0 (-\varepsilon L_0 w - L_0).
\end{aligned} \tag{83}
$$

With reference to equation (78)

$$
(-\varepsilon L_0 w - L_0) w + \frac{1}{2} \varepsilon L_0^3 - L_0 \xi_0^2 + L_0 \tilde{\theta} = -\varepsilon L_0 w \xi_0.
$$

Recall that

$$
-R_0 = \frac{1}{2} \varepsilon L_0^2 - \xi_0^2 - \xi_0,
$$

then equation (79) becomes

$$
-\varepsilon w^2 - w + \frac{1}{2} \varepsilon L_0^2 - \xi_0^2 + \tilde{\theta} = -\varepsilon w \xi_0 \implies -\varepsilon w^2 + (\varepsilon \xi_0 - 1) w - (R - \xi_0 - \tilde{\theta}) = 0 \tag{84}
$$

and we have,

$$
R = R_0 = \xi + \frac{1}{2} \varepsilon (\xi^2 - L_0^2) \tag{85}
$$

$$
Q_0 = -L_0 (1 + \varepsilon w). \tag{86}
$$

For $\tilde{\theta} = 0$, $-\varepsilon w^2 + (\varepsilon \xi_0 - 1) w - (R - \xi_0) = 0$. The two roots to give are sub inertial and super inertial frequency. The sub inertial frequency of interest here is given by

$$
w_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad w = \frac{1}{2\varepsilon} \left[-(1 - \varepsilon w) \pm \sqrt{1 - \xi^2 \varepsilon^2 - 2\varepsilon \xi - 4\varepsilon R + 4\varepsilon \xi} \right],
$$

$$
\varepsilon w = -\frac{1}{2} (1 - \varepsilon \xi) + \frac{1}{2} \left[(1 + \varepsilon \xi)^2 - 4\varepsilon R + 4\tilde{\theta} \right]^{\frac{1}{2}}. \tag{87}
$$

For $1 - \tilde{\theta} = 0$, this solution reduces to the steady rotating elliptical eddy Rodon of Cushman-Roisin. For $2 - \tilde{\theta} = 0$, the dynamics is similar, but the frequency is altered by the presence of the topography.

Substituting (85) into (87) gives

$$
\varepsilon w = -\frac{1}{2} (1 - \varepsilon \xi) + \frac{1}{2} \left[2(1 + \varepsilon^2 L_0^2 + 2\tilde{\theta}) - (1 + \varepsilon \xi)^2 \right]^{\frac{1}{2}}. \tag{88}
$$

It is known according equation (79) $R = R_0$ so $R = \xi + \frac{1}{2}\varepsilon(\xi^2 - L_0^2)$ by multiplying $\times 2\varepsilon$ then we obtained

$$2\varepsilon R = (1 + \varepsilon\xi)^2 - (1 + \varepsilon^2 L_0^2) \tag{89}$$

and $Q_0 = -L_0(1 + \varepsilon w)$ and we obtained

$$4\varepsilon^2[(R-\tilde{\theta})^2 - Q_0^2] = (1+2\tilde{\theta})^2 - \left\{(1 + \varepsilon\xi)\left[2(1 + \varepsilon^2 L_0^2 + 2\tilde{\theta}) - (1 + \varepsilon\xi)^2\right]^{\frac{1}{2}} + \varepsilon^2 L_0^2\right\}^2 \tag{90}$$

Real value of εw are found from (88) provided

$$2(1 + \varepsilon^2 L_0^2 + 2\tilde{\theta}) \geq (1 + \varepsilon\xi)^2 \tag{91}$$

for physical reliability. It is required that

$$Q_0^2 < (R - \tilde{\theta})^2, \quad R < \tilde{\theta}. \tag{92}$$

From (89) and (92) requires $2(1 + \varepsilon^2 L_0^2 + 2\tilde{\theta}) > (1 + \varepsilon\xi)^2$ which, if satisfied, implies that (91) maybe found in terms of ξ and L_0 from (90) and reduces to

$$1 + \varepsilon^2 L_0^2 + 2\tilde{\theta} - 2\varepsilon|L_0|(1 + 2\tilde{\theta})^{\frac{1}{2}} > (1 + \varepsilon\xi)^2$$

$$\left[(1 + 2\tilde{\theta})^{\frac{1}{2}} - \varepsilon|L_0|\right]^2 > (1 + \varepsilon\xi)^2.$$

From (90) that $4\varepsilon^2[(R - \tilde{\theta})^2 - Q_0^2] \leq (1 + 2\tilde{\theta})^2$ which gives a minimum value of the eddy mean radius. When $\tilde{\theta} = 0$, the equations reduce to the proper existence conditions for the Rodon (Cushman-Roisin).

First, we consider the unsteady flow problem flow problem for $0 \leq \varepsilon < 1$ with $\xi = -1$ and L_0^2 specified so that physical reliability are satisfied. Results for the rotating elliptical eddy solution with no topography $\tilde{\theta} = 0$ with $L_0^2 = 0.1$. The frequency εw is plotted in Figure 5 shows the graph εw against ε.

3.5. Lattice Boltzmann Method

The Austrian physicist, Ludwig Eduard Boltzmann (1844–1906) clarifies and calculates the way of determination of the atoms and molecules' properties (microscopic properties) in the phenomenological (macroscopic) properties of matter such as the viscosity, diffusion coefficient, and thermal conductivity [13, 14]. The function of distribution replaces tagging each particle, this process also happened in molecular dynamic simulations. Significantly, this method designs to saves the computer resources. One of distinct computational method is lattice Boltzmann method. The base on this method is upon the lattice gas automata which are simplified, fabricated molecular model. This model is consists of three basic tasks: lattice pattern, lattice Boltzmann equation, and local equilibrium distribution function. The lattice Boltzmann equation, lattice pattern are standard, these two are the same for fluid flows. The local equilibrium distribution determines which lattice Boltzmann model solves flow equations.

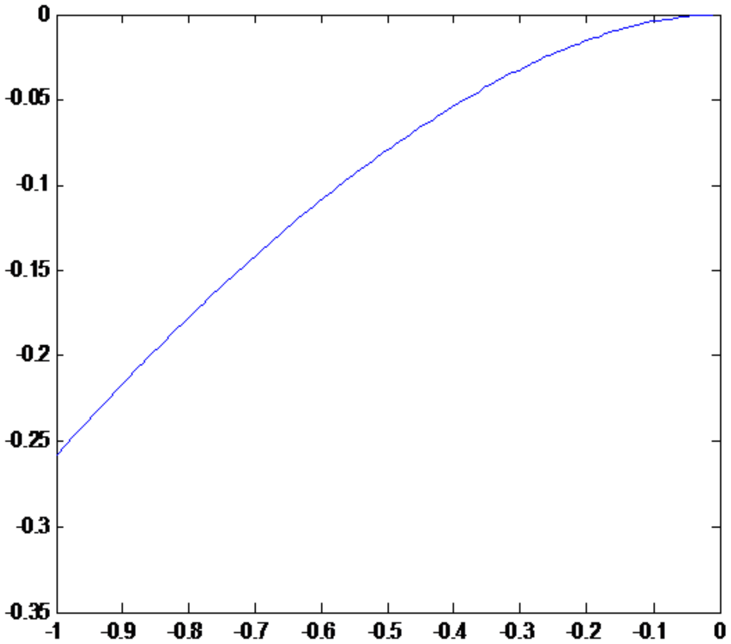

Figure 5: Exact solutions for unsteady rotating elliptic eddy [13] with $\tilde{\theta} = 0$.

3.5.1 Lattice Boltzmann Equation

A distribution function $f(r, c, t)$ can be used to explain the statistical description of the system. For a system without an external force, the Boltzmann equation can be written as,

$$\frac{\partial f}{\partial t} + e \cdot \nabla f = \Omega \tag{93}$$

where e and ∇f are vectors. The rate of change between final and initial condition of the distribution function is termed collision operator, Ω. The Ω is a function of f and should be determined in order to solve the Boltzmann equation. The complication of collision term makes the solution of Boltzmann equation difficult [9, 13, 15]. The outcome of collision in two bodies does not have a considerable influence on the value of many measured quantities. Therefore, the collision operator with simple operator is possible to approximate. This simple operator is without major error to the outcome of the solution. A simplified model for collision operator [14], were introduced. Simultaneously, at that time, Welander, 1954 in [2], introduced a similar operator. The collision operator is replaced as;

$$\Omega = \omega(f^{eq} - f) = \frac{1}{\tau}(f^{eq} - f) \tag{94}$$

where $\omega = \frac{1}{\tau}$. The coefficient ω is known as the collision frequency and τ is called the relaxation factor. The local equilibrium distribution function is represented by f^{eq}. After the introduction of BGKW approximation [14], the Boltzmann equation without external forces, can be approximated as;

$$\frac{\partial f_i}{\partial t} + e_i \cdot \nabla f_i = \frac{1}{\tau}(f_i^{eq} - f_i). \tag{95}$$

The left-hand side terms denote the streaming process, when the distribution function streams along the lattice link with velocity $e_i = \frac{\Delta x}{\Delta t}$.

$$f_i(r + e_i \Delta t, t + \Delta t) = f_i(r, t) + \frac{\Delta t}{\tau}\left[f_i^{eq}(r, t) - f_i(r, t)\right]. \tag{96}$$

3.5.2 Equilibrium Distribution Function

Finding out a suitable local equilibrium function plays a vital role in using the Lattice Boltzmann method. This function decides which flow equations are solved through the Lattice Boltzmann equation. To apply this equation, (96) for solution of the 2D shallow water equations

$$\frac{\partial h}{\partial t} + \frac{\partial (hu_i)}{\partial x_i} = 0, \tag{97}$$

$$\frac{\partial (hu_i)}{\partial t} + \frac{\partial (hu_i u_j)}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\frac{gh^2}{2}\right) = \gamma \left[\frac{\partial^2 (hu_i)}{\partial x_i x_j}\right] + F_i. \tag{98}$$

The derivation of local equation f_α^{eq} is done in this part. Considering the theory of the lattice gas automata, the equilibrium function is the Maxwell-Boltzmann equilibrium distribution function. This Maxwell-Boltzmann equilibrium distribution function is often expanded as a Taylor series in macroscopic velocity to its second order. On the other hand, the use of such equilibrium function in the lattice Boltzmann equation can recover the Navier-Stokes equation. This equation severely limits the capability of the method in order to solve flow equations. Therefore, a powerful and alternative way is to assume that an equilibrium function can be expressed as a power series in macroscopic velocity i.e.

$$f_i^{eq} = A_\alpha + B_\alpha e_{\alpha i} u_i + C_\alpha e_{\alpha i} e_{\alpha j} u_i u_j + D_\alpha u_i u_j. \tag{99}$$

This turns out to be a general approach, which is effectively used for solution of various flow problems and demonstrating its accuracy and suitability. For this reason, it is used. Since the equilibrium function has the same symmetry as the lattice, there must be

$$A_1 = A_3 = A_5 = A_7 = \bar{A}, \quad A_2 = A_4 = A_6 = A_8 = \tilde{A} \tag{100}$$

and similar expressions for B_α, C_α and D_α. Accordingly, it is convenient to write equation above in the following form,

$$f_\alpha^{eq} = \begin{cases} A_0 + D_0 u_i u_i, & \alpha = 0 \\ \bar{A} + \bar{B} e_{\alpha i} + \bar{C} e_{\alpha i} e_{\alpha j} u_i u_j + \bar{D} u_i u_i, & \alpha = 1, 3, 5, 7 \\ \tilde{A} + \tilde{B} e_{\alpha i} u_i + \tilde{C} e_{\alpha i} e_{\alpha j} u_i u_j + \tilde{D} u_i u_j, & \alpha = 2, 4, 6, 8. \end{cases} \tag{101}$$

The coefficients of A_0, $\bar{A}$ and $\tilde{A}$ can be determined based on the limitations of the equilibrium distribution function. The following three conditions must be satisfied by the local equilibrium distribution function in shallow water equation.

$$\sum_{\alpha} f_{\alpha}^{eq}(X,t) = h(X,t), \tag{102}$$

$$\sum_{\alpha} e_{\alpha i} f_{\alpha}^{eq}(X,t) = h(X,t)u_i(X,t), \tag{103}$$

$$\sum_{\alpha} e_{\alpha i} e_{\alpha j} f_{\alpha}^{eq}(X,t) = \frac{1}{2} g h^2(X,t)\delta_{ij} + h(X,t)u_i(X,t)u_j(X,t). \tag{104}$$

The calculation of the lattice Boltzmann equation leads to the solution of the 2D shallow water equations if the local equilibrium function could be determined under the above constraint. Substituting equation (101) into equation (102) yields

$$\begin{aligned} A_0 + D_0 u_i u_i + 4\bar{A} + \sum_{\alpha=1,3,5,7} \bar{B} e_{\alpha i} u_i + \sum_{\alpha=1,3,5,7} \bar{C} e_{\alpha i} e_{\alpha j} u_i u_j + 4\bar{D} u_i u_i + 4\tilde{A} \\ + \sum_{\alpha=2,4,6,8} \tilde{B} e_{\alpha i} u_i + \sum_{\alpha=2,4,6,8} \tilde{C} e_{\alpha i} e_{\alpha j} u_i u_j + 4\tilde{D} u_i u_i = h, \end{aligned} \tag{105}$$

after evaluating the terms in the above equation with equation

$$e_\alpha = \begin{cases} (0,0), & \alpha = 0 \\ e\left[\cos\frac{(\alpha-1)\pi}{4}, \sin\frac{(\alpha-1)\pi}{4}\right], & \alpha = 1,3,5,7 \\ \sqrt{2}e\left[\cos\frac{(\alpha-1)\pi}{4}, \sin\frac{(\alpha-1)\pi}{4}\right], & \alpha = 2,4,6,8. \end{cases} \tag{106}$$

and equating the coefficients of h and $u_i u_i$ respectively, we have

$$A_0 + 4\bar{A} + 4\tilde{A} = h \quad \text{and} \quad D_0 + 2e^2\bar{C} + 4e^2\tilde{C} + 4\bar{D} + 4\tilde{D} = 0. \tag{107}$$

Place in equations (101) to equation (103) leads to

$$\begin{aligned} A_0 e_{\alpha i} + D_0 e_{\alpha i} u_j u_j + \sum_{\alpha=1,3,5,7} \left(\bar{A} e_{\alpha i} + \bar{B} e_{\alpha i} e_{\alpha j} u_j + \bar{C} e_{\alpha i} e_{\alpha j} e_{\alpha k} u_j u_k + \bar{D} e_{\alpha i} u_j u_j\right) \\ + \sum_{\alpha=2,4,6,8} \left(\tilde{A} e_{\alpha i} + \tilde{B} e_{\alpha i} e_{\alpha j} u_j + \tilde{C} e_{\alpha i} e_{\alpha j} e_{\alpha k} u_j u_k + \tilde{D} e_{\alpha i} u_j u_j\right) = h u_i. \end{aligned} \tag{108}$$

In which we can obtain

$$2e^2\bar{B} + 4e^2\tilde{B} = h. \tag{109}$$

Substituting equation (101) into equation (104) results in

$$\sum_{\alpha=1,3,5,7} \left(\bar{A}e_{\alpha i}e_{\alpha j} + \bar{B}e_{\alpha i}e_{\alpha j}e_{\alpha k}u_k + \bar{C}e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha l}u_k u_l + \bar{D}e_{\alpha i}e_{\alpha j}u_k u_k\right) + \sum_{\alpha=2,4,6,8} \left(\tilde{A}e_{\alpha i}e_{\alpha j} + \tilde{B}e_{\alpha i}e_{\alpha j}e_{\alpha k}u_k + \tilde{C}e_{\alpha i}e_{\alpha j}e_{\alpha k}e_{\alpha l}u_k u_l + \tilde{D}e_{\alpha i}e_{\alpha j}u_k u_k\right) = \frac{1}{2}gh^2\delta_{ij} + hu_i u_j. \tag{110}$$

By using equation (106) to simplify the above equation, we have

$$2\bar{A}e^2\delta_{ij} + 2\bar{C}e^4u_iu_i + 2\bar{D}e^2u_iu_i + 4\tilde{A}e^2\delta_{ij} + 8\tilde{C}e^4u_iu_j + 4\tilde{C}e^4u_iu_i + 4\tilde{D}e^2u_iu_i = \frac{1}{2}gh^2\delta_{ij} + hu_iu_j. \tag{111}$$

This provides the following four relations,

$$2e^2\bar{A} + 4e^2\tilde{A} = \frac{1}{2}gh^2, \qquad \left.\begin{matrix} 8e^4\tilde{C} = h \\ 2e^4\bar{C} = h \end{matrix}\right\} \Longrightarrow \bar{C} = 4\tilde{C}, \quad 2e^2\bar{D} + 4e^2\tilde{D} = 4e^2\tilde{C} = 0. \tag{112}$$

From the symmetry of the lattice, based on $\bar{C} = 4\tilde{C}$, it is reasonable to assume three additional relations as follows,

$$A_0 = h - \frac{5gh^2}{6e^2}, \quad \bar{A} = \frac{gh^2}{6e^2}, \quad \tilde{A} = \frac{gh^2}{24e^2}, \quad \bar{B} = \frac{h}{3e^2}, \quad \tilde{B} = \frac{h}{12e^2},$$
$$\bar{C} = \frac{h}{2e^4}, \quad \tilde{C} = \frac{h}{8e^4}, \quad \bar{D} = -\frac{h}{6e^2}, \quad \tilde{D} = -\frac{h}{24e^2}, \quad D_0 = -\frac{2h}{3e^2}, \tag{113}$$

results in,

$$f_\alpha^{eq} = \begin{cases} h - \dfrac{5gh^2}{6e^2} - \dfrac{2hu_iu_i}{3e^2}, & \alpha = 0 \\ \dfrac{gh^2}{6e^2} + \dfrac{he_{\alpha i}u_i}{3e^2} + \dfrac{he_{\alpha i}e_{\alpha j}u_iu_j}{2e^4} - \dfrac{hu_iu_j}{6e^2}, & \alpha = 1, 3, 5, 7 \\ \dfrac{gh^2}{24e^2} + \dfrac{he_{\alpha i}u_i}{12e^2} + \dfrac{he_{\alpha i}e_{\alpha j}u_iu_j}{8e^4} - \dfrac{hu_iu_j}{24e^2}, & \alpha = 2, 4, 6, 8. \end{cases} \tag{114}$$

3.5.3 Recovery of the Shallow Water Equations

To prove that the calculated depth and velocities from equation (102) and (96) are the solution to the shallow water equations, we perform the Chapmann-Enskog expansion to the lattice Boltzmann equation that recovers the macroscopic equations. Supposing Δt is small and is equal to ε, $\Delta t = \varepsilon$, the equation (96) is expressed as

$$f_\alpha(X + e_\alpha\varepsilon, t + \varepsilon) - f_\alpha(X, t) = -\frac{1}{\tau}\left(f_\alpha - f_\alpha^{eq}\right) + \frac{\varepsilon}{6e^2}e_{\alpha j}F_j. \tag{115}$$

Putting a Taylor expansion to the first term on the left-hand side of the above equation in time and space around point (X, t) results in

$$\begin{aligned} f_\alpha(X + e_\alpha\varepsilon, t+\varepsilon) - f_\alpha(X,t) &= \varepsilon\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha \\ &\quad + \frac{1}{2}\varepsilon^2\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)^2 f_\alpha + O(\varepsilon^2) \\ &= -\frac{1}{\tau}(f_\alpha - f_\alpha^{eq}) + \frac{\varepsilon}{6e^2}e_{\alpha j}F_j. \end{aligned} \tag{116}$$

f_α around $f_\alpha^{(0)}$, can also be expanded to have

$$f_\alpha = f_\alpha^{(0)} + \varepsilon f_\alpha^{(1)} + \varepsilon^2 f_\alpha^{(2)} + O(\varepsilon^2) \tag{117}$$

where $f_\alpha^{(0)} = f_\alpha^{eq}$. The equation above to order ε is

$$\begin{aligned} \varepsilon\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha &= -\frac{1}{\tau}\varepsilon f_\alpha^{(1)} + \frac{\varepsilon}{6e^2}e_{\alpha j}F_j \\ \Longrightarrow \left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha &= -\frac{1}{\tau}f_\alpha^{(1)} + \frac{1}{6e^2}e_{\alpha j}F_j \end{aligned} \tag{118}$$

and to order ε^2 is

$$\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha^{(1)} + \frac{1}{2}\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)^2 f_\alpha^{(0)} = -\frac{1}{\tau}f_\alpha^{(2)}. \tag{119}$$

Substituting equation order ε in equation order ε^2 after rearrangement, leads to

$$\left(1 - \frac{1}{2\tau}\right)\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha^{(1)} = -\frac{1}{\tau}f_\alpha^{(2)} - \frac{1}{2}\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)\left(\frac{1}{6e^2}e_{\alpha k}F_k\right). \tag{120}$$

Taking

$$\begin{aligned} \sum_\alpha \Bigg[\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha &= -\frac{1}{\tau}f_\alpha^{(1)} + \frac{1}{6e^2}e_{\alpha j}F_j \\ + \varepsilon \times \Bigg(\left(1 - \frac{1}{2\tau}\right)&\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha^{(1)} \\ = -\frac{1}{\tau}f_\alpha^{(2)} - \frac{1}{2}&\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)\left(\frac{1}{6e^2}e_{\alpha k}F_k\right)\Bigg)\Bigg] \end{aligned}$$

about α gives

$$\frac{\partial}{\partial t}\left(\sum_\alpha f_\alpha^{(0)}\right) + \frac{\partial}{\partial x_j}\left(\sum_\alpha f_\alpha^{(1)}\right) = -\varepsilon\frac{1}{12e^2}\frac{\partial}{\partial x_j}\left(\sum_\alpha e_{\alpha j}e_{\alpha k}F_k\right). \tag{121}$$

By applying the first-order accuracy for the force term, the evaluation of the other terms in the above equation using (115) and (114) results in

$$\frac{\partial h}{\partial t} + \frac{\partial (hu_j)}{\partial x_j} = 0 \tag{122}$$

which is the continuity equation for shallow water flows. From

$$\begin{aligned}
&\sum_\alpha e_{\alpha i}\Bigg[\left(\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha = -\frac{1}{\tau} f_\alpha^{(1)} + \frac{1}{6e^2} e_{\alpha j} F_j\right) \\
&\quad + \varepsilon\left(\left(1 - \frac{1}{2\tau}\right)\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha^{(1)}\right. \\
&\quad \left. = -\frac{1}{\tau} f_\alpha^{(2)} - \frac{1}{2}\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)\left(\frac{1}{6e^2} e_{\alpha k} F_k\right)\right)\Bigg]
\end{aligned} \tag{123}$$

about α, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\sum_\alpha e_{\alpha i} f_\alpha^{(0)}\right) + \frac{\partial}{\partial x_j}\left(\sum_\alpha e_{\alpha i} e_{\alpha j} f_\alpha^{(0)}\right) + \varepsilon\left(1 - \frac{1}{2\tau}\right)\frac{\partial}{\partial x_j}\left(\sum_\alpha e_{\alpha i} e_{\alpha j} f_\alpha^{(1)}\right) \\
&= F_j \delta_{ij} - \varepsilon \frac{1}{2} \sum_\alpha e_{\alpha i}\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)\left(\frac{1}{6e^2} e_{\alpha j} F_j\right).
\end{aligned} \tag{124}$$

Again, by using the first-order accuracy for the force term, the other terms can be simplified with equations (106) and (114), and above equation becomes

$$\frac{\partial (hu_i)}{\partial t} + \frac{\partial (hu_i u_j)}{\partial x_j} = g\frac{\partial}{\partial x_i}\left(\frac{h^2}{2}\right) - \frac{\partial}{\partial x_j}\Lambda_{ij} + F_i, \tag{125}$$

$$\Lambda_{ij} = \frac{\varepsilon}{2\tau}(2\tau - 1)\sum_\alpha e_{\alpha i} e_{\alpha j} f_\alpha^{(1)}. \tag{126}$$

Considering equation (118) using equation (114) after some algebra, we obtain

$$\Lambda_{ij} \approx -\gamma\left[\frac{\partial (hu_i)}{\partial x_j} + \frac{\partial (hu_j)}{\partial x_i}\right]. \tag{127}$$

Substituting equation (127) into equation (125) gives

$$\frac{\partial (hu_i)}{\partial t} + \frac{\partial (hu_i u_j)}{\partial x_j} = -g\frac{\partial}{\partial x_i}\left(\frac{h^2}{2}\right) + \gamma\frac{\partial^2 (hu_i)}{\partial x_j \partial x_j} + F_i \tag{128}$$

with the kinematic viscosity γ defined as $\gamma = \frac{c^2 \Delta t}{6}(2\tau - 1)$ and the force F_i expressed as

$$F_i = -gh\frac{\partial z_b}{\partial x_i} + \frac{\tau\omega_i}{\rho} - \frac{\tau b_i}{\rho} + E_i. \tag{129}$$

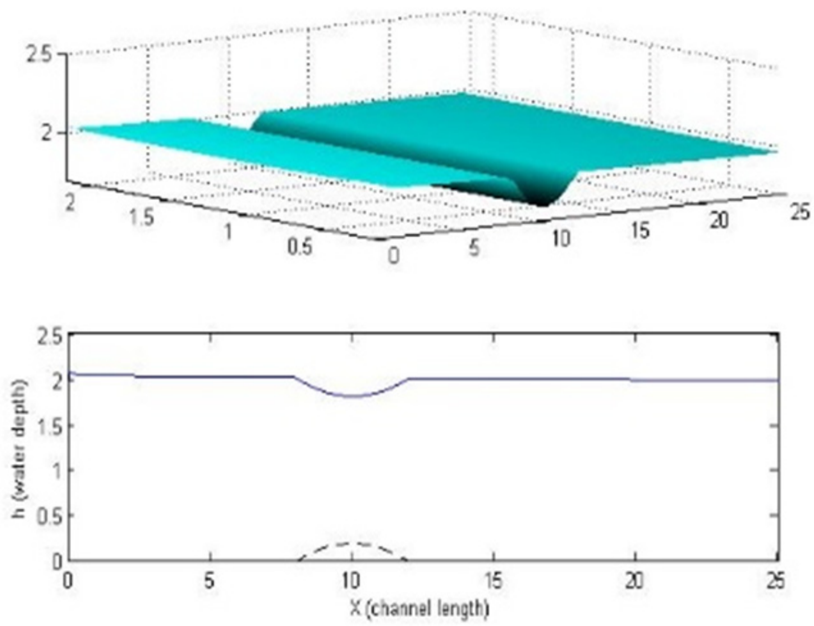

Figure 6: LBM solutions to the SWE.

3.5.4 Boundary and Initial Conditions

Suitable boundary and initial conditions must be provided to simulate shallow water flow problems. Initial and boundary conditions in the lattice Boltzmann formulation rely on connecting the macroscopic boundary conditions in the physical problem to macroscopic boundary conditions on the distribution functions f_α. Using the periodic boundary conditions in the lattice Boltzmann formulation is accomplished by setting the unknown distribution functions, f_1, f_5 and f_8. At the inflow boundary to the corresponding known distribution functions at the out flow boundary;

$$f_\alpha(i = 1, j, t) = f_\alpha(i = 1, j = 1, t), \quad \alpha = 1, 5, 8 \tag{130}$$

and the unknown distribution functions f_3, f_6 and f_7 at the inflow boundary,

$$f_\alpha(i = 1, j, t) = f_\alpha(i = 1, j = 1, t), \quad \alpha = 3, 6, 7. \tag{131}$$

Boundary conditions for solid boundaries as an example to structures in the flow region or impermeable boundaries are set down to apply no-slip or free-slip at these boundaries to set down zero velocity or zero normal velocity at the boundary, respectively. However, in this research the author made use of bounce back conditions, open boundary conditions and initial conditions. Furthermore, as opined by previous authors [15] for physical problems to be modelled, it is given in form of macroscopic variables which are normal practice in traditional numerical methods. Hence, the lattice Boltzmann formulation is

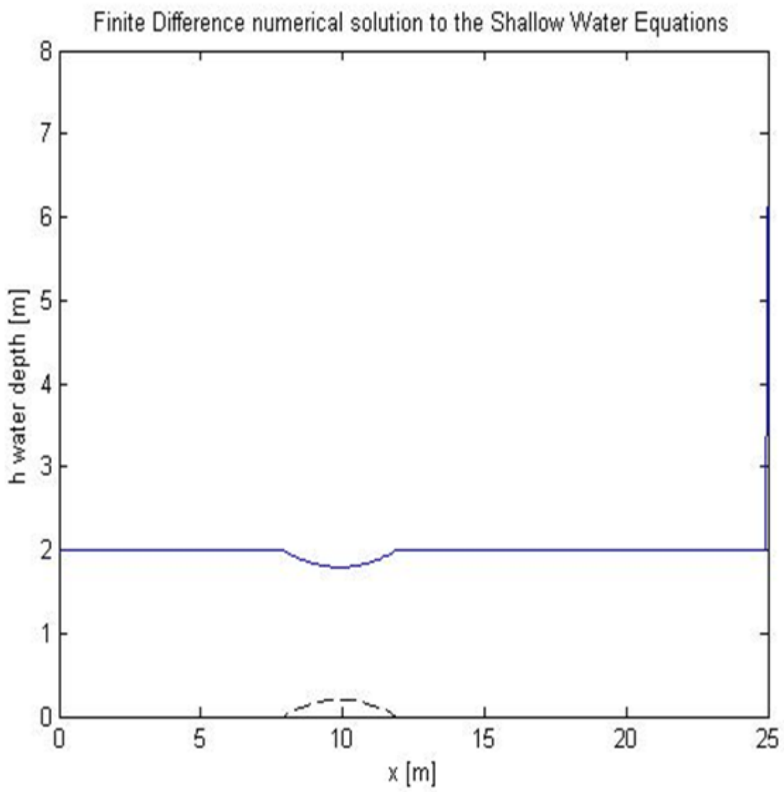

Figure 7: Finite Difference (FD) numerical solutions to the SWE.

based on solving equations, so that the initial conditions must be written in terms of the distribution function f_α. If the initial macroscopic boundary conditions h, u_x and u_y, are given, the EDF, f_α^{eq}, can be computed and used as initial conditions for f_α as follow

$$f_\alpha = f_\alpha^{eq}\left(h(x, t = 0),\ u_x(x, t = 0),\ u_y(x, t = 0)\right). \tag{132}$$

4. Numerical Result

In this section, the Lattice Boltzmann Method for Shallow Water Equation is used to solve flow problem. Simulation of steady flow over a bump is now presented. The LBM results arising are then compared with finite difference method (FDM) solutions. The channel is a rectangular reservoir of area 2.5×2.5 m^2 with a bump defined by

$$z_b(x) = \begin{cases} 0.2 - 0.05(x - 10)^2, & 8 < x < 12 \\ 0, & \text{otherwise.} \end{cases}$$

In the numerical computations, the water depth inside the channel is $h = 2m$, $e = 15\dfrac{m}{s}$, and $\tau = 1.5$. The domain is covered by 500×50 lattices or cells for numerical simulations

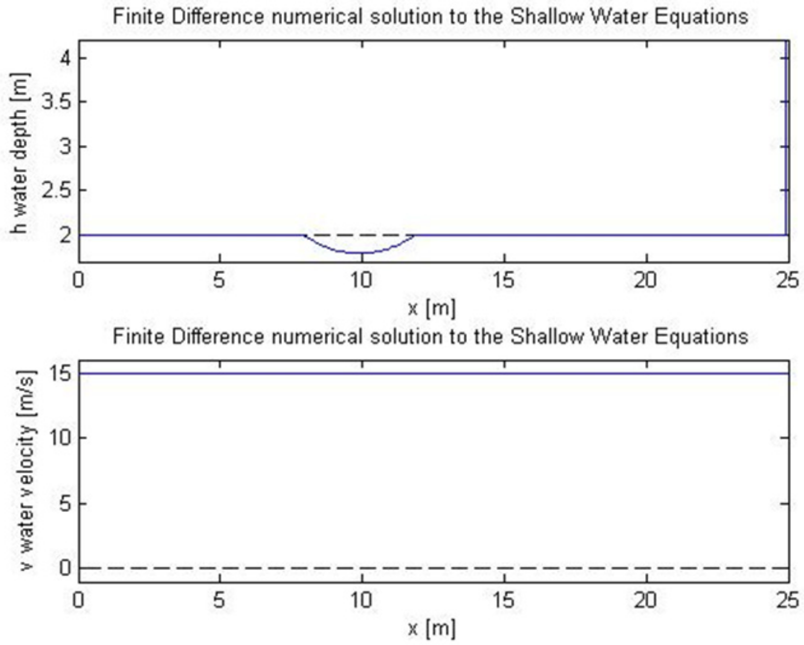

Figure 8: Finite Difference (FD) numerical solutions to the SWE.

and $\Delta x = \Delta y = 0.05$. Figure 6 shows the profile of the water surface along channel with Lattice Boltzmann solutions where else Figure 7 shows the Finite Difference numerical solution to the shallow water equations. The comparison of the water surface is shows as in Figure 8.

5. Conclusions and Recommendations

This work focuses systematically on the study, validation, and demonstration of the lattice Boltzmann method as an invaluable numerical modelling and simulation tool, for two dimensional flows in the shallow water regime in high performance computing environments. An accurate, simple and conservative LABSWE model is discussed in this research. The model has a wide range of applications, most especially in solving steady and unsteady flow problems. Additionally, it is a numerical technique used for an indirect solution of flow equations through a microscopic approach to macroscopic phenomena. From the result of the Numerical test, it is evident that this method can provide accurate solution, making it an ideal model for shallow water flow simulation. The new analytical method which has two-dimensional steady-state solutions to the rotating shallow water equations over variable topography.Exploiting a drastic simplification drive this analytical method that occurs for non-divergent flows. Set up and implementing Multi-layer shallow water equations with MRT collision operator, and also it is recommended that to perform a higher order recovery of the multi-layer shallow water equations. In order to perform a stability analysis, the MRT collision operator should be used to provide a systematic basis for choosing MRT parameters.

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