Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 3 (2016), pp. 2219–2241

http://www.ripublication.com/gjpam.htm

# The inner limit process of slightly compressible multiphase equations[1]

**Hyeonseong Jin**[2]
*Department of Mathematics,*
*Jeju National University,*
*Jeju, 63243, South Korea.*

**Abstract**

We discuss the motion of the slightly compressible multiphase flow near the initial time. The formally uniformly valid inner limit expansions are derived for the solutions of compressible equations. Under the singular limit process the constitutive asymptotic expansions and conditions are examined as requirements for the derivation of asymptotic expansions for compressible solutions.

**AMS subject classification:** 76Txx, 76F25, 76N99, 76Axx.
**Keywords:** Multiphase flow models, Closure, Constitutive laws, Averaged equations.

## 1. Introduction

In this paper we consider a hyperbolic system of the compressible isentropic two-pressure two-phase flow equations [2, 6, 7, 12, 13] in an appropriate nondimensional form

$$\frac{\partial \beta_k^\lambda}{\partial t} + v^{*\lambda}\frac{\partial \beta_k^\lambda}{\partial z} = 0, \tag{1}$$

$$\beta_k^\lambda\left(\frac{\partial \rho_k^\lambda}{\partial t} + v_k^\lambda\frac{\partial \rho_k^\lambda}{\partial z}\right) + \beta_k^\lambda\rho_k^\lambda\frac{\partial v_k^\lambda}{\partial z} + \rho_k^\lambda(v_k^\lambda - v^{*\lambda})\frac{\partial \beta_k^\lambda}{\partial z} = 0, \tag{2}$$

$$\beta_k^\lambda\rho_k^\lambda\left(\frac{\partial v_k^\lambda}{\partial t} + v_k^\lambda\frac{\partial v_k^\lambda}{\partial z}\right) + \lambda^2\beta_k^\lambda\frac{\partial p_k^\lambda}{\partial z} + \lambda^2(p_k^\lambda - p^{*\lambda})\frac{\partial \beta_k^\lambda}{\partial z} = \beta_k^\lambda\rho_k^\lambda g(t), \tag{3}$$

[1]This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (No. NRF-2009-0069783).

[2]Corresponding author.

for the volume fraction $\beta_k^\lambda$, velocity $v_k^\lambda$, density $\rho_k^\lambda$, and pressure $p_k^\lambda$ of fluid $k$ depending on a large dimensionless parameter $\lambda$. Here $p_k^\lambda = p_k(\rho_k^\lambda)$ and an equation of state $p_k(\rho_k) = A_k \rho_k^{\gamma_k}$, $\gamma_k > 1$ is given with $\partial p_k / \partial \rho_k(\rho_k) > 0$ for $\rho_k > 0$ and the entropy $A_k$ assumed to be constant within each fluid but $A_1 \neq A_2$. The fluids are distinguished by a subscript $k$, where $k = 1$ and $k = 2$ denote the light and heavy fluids, respectively. Eqs. (1)–(3) have been nondimensionalized so that a typical mean fluid velocity, $|v_m|$, has been chosen as the ratio of time units to space units. Accordingly, the parameter $\lambda$ is essentially the reciprocal of the Mach number, $M = |v_m|(\gamma p(\rho_m)/\rho)^{-1/2}$, the ratio of fluid speed to sound speed, where $\rho_m$ is the mean density. In fact, $\lambda = M^{-1}(\gamma A)^{-1/2}$. The interface quantities $v^*$ and $p^*$ have been proposed [7,12,13] by closure relations

$$q^* = \mu_1^q q_2 + \mu_2^q q_1, \quad q = v, p, \tag{4}$$

where $\mu_1^q + \mu_2^q = 1$, $\mu_k^q \geq 0$ and that $\mu_k^q/\beta_k$is continuous on $0 \leq \beta_k \leq 1$ and for all $t$. Here the primed index $k'$ denotes the fluid complementary to fluid $k$, i.e., $k' = 3-k$. The $\mu_k^q$ thus depends on a single parameter $d_k^q$. The closure for the constitutive law $d_k^q(t)$ has been proposed [12, 13] and compared in a validation study based on simulation data [1, 17]. For all the data sets of the 3D Rayleigh-Taylor and circular 2D Richtmyer-Meshkov simulations, we have seen excellent validation agreement for the closures proposed. We denote $Z_k = Z_k(t)$ as the position of the mixing zone edge $k$, defined as the location of vanishing $\beta_k$ and $V_k = dZ_k/dt$ as the velocity of the edge $k$. At edge $k$, the following boundary data holds $v_k = V_k(t)$ at $z = Z_k(t)$.

On the other hand, the incompressible flow equations are a distinctly different system of the equations

$$\frac{\partial \beta_k^\infty}{\partial t} + v^{*\infty} \frac{\partial \beta_k^\infty}{\partial z} = 0, \tag{5}$$

$$\beta_k^\infty \frac{\partial v_k^\infty}{\partial z} + (v_k^\infty - v^{*\infty}) \frac{\partial \beta_k^\infty}{\partial z} = 0, \tag{6}$$

$$\beta_k^\infty \rho_k^\infty \left( \frac{\partial v_k^\infty}{\partial t} + v_k^\infty \frac{\partial v_k^\infty}{\partial z} \right) + \beta_k^\infty \frac{\partial p_k^\infty}{\partial z} + (p_k^\infty - p^{*\infty}) \frac{\partial \beta_k^\infty}{\partial z} = \beta_k^\infty \rho_k^\infty g(t) \tag{7}$$

for the volume fraction $\beta_k^\infty$, velocity $v_k^\infty$, and scalar pressure $p_k^\infty$, where $\rho_k^\infty$ is the constant density of phase $k$. We assume that all state variables are piecewise $C^1$ functions with discontinuous derivatives at the mixing zone edges $z = Z_k^\infty(t)$ of incompressible flow. Analytic solutions of the incompressible problem (5)–(7) have been obtained in closed form [5, 6, 7].

One expects, under appropriate conditions, that the compressible multiphase flow solutions $\beta_k^\lambda$, $v_k^\lambda$, $p_k^\lambda$ converge to the incompressible solutions $\beta_k^\infty$, $v_k^\infty$, $p_k^\infty$ as $\lambda \to \infty$. The zero Mach limit of the compressible multiphase flow equations is a time-singular and layer-type problem which requires advanced techniques in asymptotics. Jin et al. [4, 11] discussed the outer limit process of the weakly compressible multiphase flow equations describing the fluid motions away from the initial time. The slow variables in the outer limit asymptotic expansions have a slow scale of motion and they have been determined through second order in closed form. Information supplied from the weakly compressible theory resolves underdetermination of incompressible pressures. From a number of points of view [3, 8, 15, 16], the zero Mach limit of the single phase compressible Euler equations has been studied in higher space dimensions. In contrast, two-phase flow presents additional difficulties, so that even one dimensional two-phase flow is nontrivial.

In this paper we discuss the limiting behavior of the solutions $U_k^\lambda \equiv (\beta_k^\lambda, v_k^\lambda, \rho_k^\lambda, p_k^\lambda)^{tr}$ of the compressible equations (1)–(3) near the initial time as $\lambda \to \infty$. For simplicity, we suppress superscript $\lambda$'s of compressible variables from now on. We are concerned with the derivation of inner limit asymptotic expansions for the solutions of the compressible equations. In Sec. 4 the inner terms, uniformly valid in space, are evaluated in closed form through first order in the expansion of volume fraction and pressure, and zero-th order in the expansion of velocity. They are determined uniformly by matching the inner limit in the exterior and the incompressible mixing zone and the fast transition-layer expansions. The fast variables are defined as the inner expansion terms minus the common terms resulting from the outer expansion evaluated in the inner limit to each order. These variables oscillate on the fast time scale and are described by linear wave equations. They are also defined uniformly in space by matching. Specifically, in Sec. 4, the inner limit analysis is performed to show that there are no fast scale acoustical oscillations in the asymptotic expansions of $\beta_k$, $\rho_k$, $p_k$ through first order and in the zero-th order of the expansion of $v_k$. In Sec. 2 the constitutive asymptotic expansions and conditions are examined as requirements of the derivation of formal asymptotic expansions for compressible solutions.

The fluids are distinguished by a subscript $k$, where $k = 1$ and $k = 2$ denote the light and heavy fluids, respectively. We assume existence of rigid wall at the top of a finite but large domain $\mathcal{D}$ for compressible and incompressible two-phase flow. Then velocity is zero and the pressure is unknown there. At the bottom of this domain, we have conceptually an open container. This fixes the pressure at some ambient value, but not the velocity at the bottom of D. This leads to the boundary conditions

$$v_1(z^{+\infty}) = 0\,, \quad p_2(z^{-\infty}) = const, \tag{8}$$

where $z = z^{+\infty}(z = z^{-\infty})$ denotes the position of the upper (lower) wall of the domain $\mathcal{D}$.

## 2. Constitutive Asymptotic Assumptions

We consider the limiting behavior of the solutions $U_k \equiv (\beta_k, v_k, \rho_k, p_k)^{tr}$ of the compressible equations (1)–(3) as $\lambda \to \infty$. The asymptotic expansions of compressible solutions are introduced in the form

$$
\begin{aligned}
\beta_k &= \beta_k^{(0,s)} + \lambda^{-1}\beta_k^{(1,s)} + \lambda^{-2}\left(\beta_k^{(2,s)} + \beta_k^{(2,f)}\right) + O(\lambda^{-3}),\\
v_k &= v_k^{(0,s)} + \lambda^{-1}\left(v_k^{(1,s)} + v_k^{(1,f)}\right) + O(\lambda^{-2}),\\
\rho_k &= \rho_k^{(0,s)} + \lambda^{-1}\rho_k^{(1,s)} + \lambda^{-2}\left(\rho_k^{(2,s)} + \rho_k^{(2,f)}\right) + O(\lambda^{-3})\\
p_k &= p_k^{(0,s)} + \lambda^{-1}p_k^{(1,s)} + \lambda^{-2}\left(p_k^{(2,s)} + p_k^{(2,f)}\right) + O(\lambda^{-3}).
\end{aligned} \tag{9}
$$

The equation of state gives the relations

$$
\begin{aligned}
&p_k^{(0,s)} = p_k(\rho_k^{(0,s)}), \quad p_k^{(1,s)} = c_k^2(\rho_k^{(0,s)})\rho_k^{(1,s)}\\
&p_k^{(2,s)} = \frac{1}{2}\frac{\partial^2 p_k}{\partial \rho_k^2}(\rho_k^{(0,s)})\rho_k^{(1,s)2} + c_k^2(\rho_k^{(0,s)})\rho_k^{(2,s)}, \quad p_k^{(2,f)} = c_k^2(\rho_k^{(0,s)})\rho_k^{(2,f)},
\end{aligned} \tag{10}
$$

where $c_k^2(\rho) = \dfrac{\partial p_k}{\partial \rho_k}(\rho)$. We assume the initial conditions

$$
\begin{aligned}
\beta_k(z,0) &= \beta_k^{\infty}(z,0) + \lambda^{-1}\beta_k^{(1,s)}(z,0) + \lambda^{-2}\beta_k^{(2,s)}(z,0)\ ,\\
v_k(z,0) &= v_k^{\infty}(z,0) + \lambda^{-1}\left[v_k^{(1,s)}(z,0) + v_k^{(1)}(z)\right],\\
\rho_k(z,0) &= \rho_k^{\infty}(z,0) + \lambda^{-2}\left[c_k^{-2}(\rho_k^{\infty})\rho_k^{\infty}(z,0) + \rho_k^{(2)}(z)\right],\\
p_k(z,0) &= p_k(\rho_k^{\infty}) + \lambda^{-2}\left[p_k^{\infty}(z,0) + p_k^{(2)}(z)\right]
\end{aligned} \tag{11}
$$

for the compressible solutions, where $v_k^{(1)}(z)$, $\rho_k^{(2)}(z)$, and $p_k^{(2)}(z)$ belong to $C^1$ on $(-1)^k z \le (-1)^k Z_k(0)$ and $|||v_k^{(1)}(z)||| = O(1)$, $|||\rho_k^{(2)}(z)||| = O(1)$ and $|||p_k^{(2)}(z)||| = O(1)$.

The variables $U_k^{(m,s)} \equiv \left(\beta_k^{(m,s)}, v_k^{(m,s)}, \rho_k^{(m,s)}, p_k^{(m,s)}\right)^{tr}$, $m = 0, 1, 2$, have a slow scale of motion and solve linearized incompressible problems. They have been determined in closed form through second order [11]. The variables $v_k^{(1,f)}$, $\beta_k^{(2,f)}$, $\rho_k^{(2,f)}$, $p_k^{(2,f)}$ contain fast scale acoustical oscillations on the fast time scale $\tau \equiv \lambda t$ and they are solved in closed form in Sec. 4. We show by the inner limit analysis that there are no fast scale acoustical oscillations in the asymptotic expansions of $\beta_k$, $\rho_k$, $p_k$ through first order and in the zero-th order of the expansion of $v_k$.

The two phase flow model depends on the motions $Z_k$ of the mixing zone edges and closure for the interfacial averages with the constitutive law $d_k^q$, $q = v, p$. Since the $Z_k$ are not well characterized for compressible flows, the velocities or trajectories of the edges of the mixing zone must be provided as data. The compressible constitutive factor $d_k^q$ is asymptotically assumed with a specific limit term.

We assume a uniformly valid asymptotic expansion for the compressible mixing zone edge,

$$Z_k(t) = Z_k^{(0,s)}(t) + \lambda^{-1} Z_k^{(1,s)}(t) + \lambda^{-2}\left(Z_k^{(2,s)}(t) + Z_k^{(2,f)}(t,\lambda)\right) + O(\lambda^{-3}). \quad (12)$$

Here $\sum_{j=0}^{m} \lambda^{-j} Z_k^{(j,s)}$, $m = 0, 1, 2$, denotes the location of vanishing $\beta_k^{(m,s)}$. Thus $Z_k$ and each of the expansion coefficients $Z_k^{(m,s)}$ and $Z_k^{(2,f)}$ are input to the model equations. We assume that the compressible edge moves faster than the incompressible edge with no initial perturbation. A similar assumption is applied to any finite number of terms in the expansion (11). We assume that the zero-th order term in the expansion (11) equals to the incompressible edge trajectory $Z_k^{\infty}(t)$. Thus we require

$$\begin{gathered} Z_k^{(0,s)}(t) = Z_k^{\infty}(t), \ Z_k(0) = Z_k^{\infty}(0), \\ (-1)^k Z_k^{(m,t)}(t) \geq 0, \ m = 1, 2, t = s, f. \end{gathered} \quad (13)$$

The variables $Z_k^{(m,s)}$, $m = 0, 1, 2$, are the slow variables with a slow scale of motion while $Z_k^{(2,f)}$ is oscillatory on the fast time scale $\tau \equiv \lambda t$. We assume that the fast variable $Z_k^{(2,f)}$ decays exponentially in $\tau$ away from the initial curve $t = 0$. We observe [9] that the fast variables can appear in the second or higher order of the asymptotic expansion (12). The formally uniformly valid expansion (12) leads to the outer and inner expansion under the corresponding limit process. The reduced expansion is assumed in the derivation of inner and outer expansions for compressible solutions in Sec. 4 and [11, 9], respectively. Under the outer limit process, $\lambda \to \infty$ with $t$ fixed $\neq 0$, (12) leads to the outer expansion

$$Z_k(t) = Z_k^{(0,s)}(t) + \lambda^{-1} Z_k^{(1,s)}(t) + \lambda^{-2} Z_k^{(2,s)}(t) + O(\lambda^{-3}) \quad (14)$$

valid away from the initial curve $t = 0$. Specifically, following the inner limit process $\lambda \to \infty$ with $\tau$ fixed $\neq \infty$, (12) leads to the inner limit expansion, valid near $t = 0$:

$$\begin{aligned} Z_k(\tau) &= \hat{Z}_k^{(0)}(\tau) + \lambda^{-1}\hat{Z}_k^{(1)}(\tau) + \lambda^{-2}\hat{Z}_k^{(2)}(\tau) + O(\lambda^{-3}) \\ &= Z_k^{(0,s)}(0) + \lambda^{-1}\tau \dot{Z}_k^{(0,s)}(0) \\ &\quad + \lambda^{-2}\left(\frac{\tau^2}{2}\ddot{Z}_k^{(0,s)}(0) + \tau \dot{Z}_k^{(1,s)}(0) + Z_k^{(2,s)}(0) + Z_k^{(2,f)}(\tau)\right) + O(\lambda^{-3}). \end{aligned} \quad (15)$$

Refer to [9]. The initial conditions associated with (13) are

$$\hat{Z}_k^{(0)}(0) = Z_k^{\infty}(0), \ \hat{Z}_k^{(m)}(0) = 0, \ m \geq 1. \quad (16)$$

Notice that the fast variable $Z_k^{(2,f)}$ consist of the inner term $\hat{Z}_k^{(2)}$ minus common terms to order $\lambda^{-2}$. The assumptions (13) imply that $(-1)^k \hat{Z}_k^{(1)}$ and $(-1)^k \hat{Z}_k^{(2)}$ are nonnegative.

The edge velocity of the compressible flow satisfies $V_k = \dot{Z}_k = v_k(Z_k, t)$ and therefore, it must have an asymptotic expansion associated with the expansion (11) in the form

$$\begin{aligned} V_k(t) =& V_k^{(0,s)}(t) + \lambda^{-1}\left(V_k^{(1,s)}(t) + V_k^{(1,f)}(t,\lambda)\right) \\ &+ \lambda^{-2}\left(V_k^{(2,s)}(t) + V_k^{(2,f)}(t,\lambda)\right) + O(\lambda^{-3}). \end{aligned} \tag{17}$$

From the expansion of $v_k$ in (17), we see that the leading order term of the asymptotic expansion of $V_k$ must be $v_k^{(0,s)}(Z_k^{(0,s)}, t)$, where $Z_k^{(0,s)} = Z_k^{(0,s)}(t)$ denotes the location of vanishing $\beta_k^{(0,s)}(z,t)$. Thus $V_k^{(0,s)}(t) = v_k^{(0,s)}(Z_k^{(0,s)}, t)$. We obtain that the formal asymptotic expansion (17) leads to the outer limit expansion

$$V_k(t) = V_k^{(0,s)}(t) + \lambda^{-1}V_k^{(1,s)}(t) + \lambda^{-2}V_k^{(2,s)}(t) + O(\lambda^{-3}) \tag{18}$$

which is valid away from the initial curve $t = 0$ under the limit process $\lambda \to \infty$ with $t$ fixed $\neq 0$, and to the inner limit expansion

$$\begin{aligned} V_k(\tau) &= \hat{V}_k^{(0)}(\tau) + \lambda^{-1}\hat{V}_k^{(1)}(\tau) + O(\lambda^{-2}) \\ &= V_k^{(0,s)}(0) + \lambda^{-1}\left(\tau\dot{V}_k^{(0,s)}(0) + V_k^{(1,s)}(0) + V_k^{(1,f)}(\tau)\right) + O(\lambda^{-2}). \end{aligned} \tag{19}$$

which is valid near the initial curve $t = 0$ under the limit process $\lambda \to \infty$ with $\tau$ fixed $\neq \infty$.

Comparing (14), (15) with (18) and (19), we note that for $m \geq 0$,

$$\begin{aligned} &\frac{dZ_k^{(m,s)}(t)}{dt} = V_k^{(m,s)}(t), \quad \frac{d\hat{Z}_k^{(0)}(\tau))}{d\tau} = 0, \quad \frac{d\hat{Z}_k^{(m+1)}(\tau)}{d\tau} = \hat{V}_k^{(m)}(\tau), \\ &\frac{dZ_k^{(m+2,f)}(\tau)}{d\tau} = V_k^{(m+1,f)}(\tau), \quad \frac{dZ_k^{(m+2,f)}(t,\lambda)}{d\tau} = V_k^{(m+1,f)}(t,\lambda). \end{aligned} \tag{20}$$

From (20) it is reasonable to assume that the fast variables can appear in the second or higher order of the asymptotic expansion (12). If the expansion (12) has a fast variable $Z_k^{(1,f)}$ in the first order, it implies that a fast variable $V_k^{(0,f)}$ exists in the zero-th order of the asymptotic expansion (17). This contradicts the fact that the zero-th order fast variable $v_k^{(0,f)}$ is suppressed by the initialization (11), as discussed in Sec. 4.

We assume that the constitutive laws $d_k^v(t)$ and $d_k^p(t)$ have a formally uniformly valid asymptotic expansion as follows

$$d_k^q(t,\lambda) = d_k^{q(0,s)}(t) + \lambda^{-1}\left(d_k^{q(1,s)}(t) + d_k^{q(1,f)}(t,\lambda)\right) + O(\lambda^{-2}), q = v, p, \tag{21}$$

where $d_k^{q(m,s)}(t), d_k^{q(m,f)}(\tau) \in ([0,\infty))$, and we assume $d_k^{q(0,s)}(t) = d_k^{q\infty}(t)$. Similarly, we obtain [9] that the expansion (21) leads to the outer limit expansion

$$d_k^q(t) = d_k^{q(0,s)}(t) + \lambda^{-1}d_k^{q(1,s)}(t) + \lambda^{-2}d_k^{q(2,s)}(t) + O(\lambda^{-3}), q = v, p, \tag{22}$$

and to the inner limit asymptotic expansion

$$\begin{aligned} d_k^q(\tau) &= \hat{d}_k^{q(0)}(\tau) + \lambda^{-1}\hat{d}_k^{q(1)}(\tau) + (\lambda^{-2}) \\ &= d_k^{q(0,s)}(0) + \lambda^{-1}\left(\tau\frac{dd_k^q(0,s)}{dt}(0) + d_k^{q(1,s)}(0) + d_k^{q(1,f)}(\tau)\right) + O(\lambda^{-2}). \end{aligned} \tag{23}$$

## 3. Inner Expansions and Transitional Layers

To understand the motion of the fast variables, we make the change of variables to the fast time scale $\tau \equiv \lambda t$. The The compressible equations (1)–(3) become

$$\lambda\frac{\partial\beta_k}{\partial\tau} \quad + \quad v^*\frac{\partial\beta_k}{\partial z} = 0, \tag{24}$$

$$\beta_k\left(\lambda\frac{\partial\rho_k}{\partial\tau} + v_k\frac{\partial\rho_k}{\partial z}\right) \quad + \quad \beta_k\rho_k\frac{\partial v_k}{\partial z} + \rho_k(v_k - v^*)\frac{\partial\beta_k}{\partial z} = 0, \tag{25}$$

$$\beta_k\rho_k\left(\lambda\frac{\partial v_k}{\partial\tau} + v_k\frac{\partial v_k}{\partial z}\right) \quad + \quad \lambda^2\beta_k\frac{\partial p_k}{\partial z} + \lambda^2(p_k - p^*)\frac{\partial\beta_k}{\partial z} = \beta_k\rho_k g(t), \tag{26}$$

for $U_k(z,\tau)$. We assume the initial data (11) for the compressible solutions and the inner limit asymptotic expansions (15), (19) and (23) for $Z_k$, $V_k$ and $d_k^q$, $q = v, p$ to derive the inner limit asymptotic expansions, valid near $t = 0$, of the solutions of the compressible equations (24)-(26), uniformly valid in z. We introduce inner limit asymptotic expansions associated with the inner limit $\lambda \to \infty$ with $\tau$ fixed $\neq \infty$:

$$U_k(z,\tau) = \hat{U}_k^{(0)}(z,\tau) + \lambda^{-1}\hat{U}_k^{(1)}(z,\tau) + \lambda^{-2}\hat{U}_k^{(2)}(z,\tau) + \cdots, \tag{27}$$

where $\hat{U}_k^{(m)}(z,\tau) \equiv \left(\hat{\beta}_k^{(m)}, \hat{v}_k^{(m)}, \hat{\rho}_k^{(m)}, \hat{p}_k^{(m)}\right)^{tr}$, $m = 0, 1, 2, \ldots$. The equation of state has the expansion relations

$$\hat{p}_k^{(0)} = p_k(\hat{\rho}_k^{(0)}), \quad \hat{p}_k^{(1)} = c_k^2(\hat{\rho}_k^{(0)})\hat{\rho}_k^{(1)}, \quad \hat{p}_k^{(2)} = \frac{1}{2}\frac{\partial^2 p_k}{\partial\rho_k^2}(\hat{\rho}_k^{(0)})\hat{\rho}_k^{(1)2} + c_k^2(\hat{\rho}_k^{(0)})\hat{\rho}_k^{(2)}, \tag{28}$$

between the terms in the expansions of $\rho_k$, $p_k$. From (27) and (28), the inner limit expansion for the mixing coefficient $\mu_k^q(\beta_k, d_k^q)$, $q = v, p$,

$$
\begin{aligned}
\mu_k^q(\beta_k, d_k^q) =& \hat{\mu}_k^{q(0,s)} + \lambda^{-1}\hat{\mu}_k^{q(1,s)} + \lambda^{-2}\hat{\mu}_k^{q(2,s)} + O(\lambda^{-3}) \\
=& \mu_k^q(\hat{\beta}_k^{(0,s)}, \hat{d}_k^{q(0,s)}) \\
&+ \lambda^{-1}\left(\frac{\partial \mu_k^q}{\partial \beta_k}(\hat{\beta}_k^{(0,s)}, \hat{d}_k^{q(0,s)})\hat{\beta}_k^{(1,s)} + \frac{\partial \mu_k^q}{\partial d_k^q}(\hat{\beta}_k^{(0,s)}, \hat{d}_k^{q(0,s)})\hat{d}_k^{q(1,s)}\right) \\
&+ \lambda^{-2}\left(\frac{\partial \mu_k^q}{\partial \beta_k}(\hat{\beta}_k^{(0,s)}, \hat{d}_k^{q(0,s)})\hat{\beta}_k^{(2,s)} + \frac{1}{2}\frac{\partial^2 \mu_k^q}{\partial \beta_k^2}(\hat{\beta}_k^{(0,s)}, \hat{d}_k^{q(0,s)})\hat{\beta}_k^{(1,s)\,2}\right. \\
&+ \frac{\partial^2 \mu_k^q}{\partial \beta_k \partial d_k^q}(\hat{\beta}_k^{(0,s)}, \hat{d}_k^{q(0,s)})\hat{\beta}_k^{(1,s)}\hat{d}_k^{q(1,s)} + \frac{1}{2}\frac{\partial^2 \mu_k^q}{\partial d_k^{q\,2}}(\hat{\beta}_k^{(0,s)}, \hat{d}_k^{q(0,s)})\hat{d}_k^{q(1,s)\,2} \\
&\left. + \frac{\partial \mu_k^q}{\partial d_k^q}(\hat{\beta}_k^{(0,s)}, \hat{d}_k^{q(0,s)})\hat{d}_k^{q(2,s)}\right) + O(\lambda^{-3}).
\end{aligned}
\tag{29}
$$

The equations for the inner limit terms $\hat{U}_k^{(m)}(z,\tau) \equiv \left(\hat{\beta}_k^{(m)}, \hat{v}_k^{(m)}, \hat{\rho}_k^{(m)}, \hat{p}_k^{(m)}\right)^{tr}$, $m = 0, 1$, are derived by repeated application of the inner limit expansions (27) to the compressible equations (24)–(26), and by equating terms of the same order of $\lambda$. Within a single power of $\lambda$, the inner terms are defined as a solution of simple differential equations. We first solve the inner term $\hat{\beta}_k^{(0)}$, $\hat{\rho}_k^{(0)}$, $\hat{p}_k^{(0)}$ equations in Sec. 4.1. The inner terms $\hat{v}_k^{(0)}$, $\hat{\beta}_k^{(1)}$, $\hat{\rho}_k^{(1)}$, $\hat{p}_k^{(1)}$, uniformly valid in $z$ are found by solving a subsystem of hyperbolic differential equations and by matching at the outer and inner edges of regions in Sec. 4.2. In [10], we will completely determine the higher order terms $\hat{v}_k^{(1)}$, $\hat{\beta}_k^{(2)}$, $\hat{\rho}_k^{(2)}$, $\hat{p}_k^{(2)}$, uniformly valid in $z$.

In higher order in $\lambda^{-1}$, there exist fast transitional layers in the inner expansion similarily to the transition regions in the outer asymptotic expansion. The first order inner expansion is defined by five regions,

$$\hat{\mathcal{E}}_k^{(1)} \cup \hat{\mathcal{T}}_k^{(1)} \cup \hat{\mathcal{M}} \cup \hat{\mathcal{T}}_{k'}^{(1)} \cup \hat{\mathcal{E}}_{k'}^{(1)},$$

including two transition-layers through $z = \overline{\hat{Z}_i^{(1,s)}}$, $i = k, k'$. The regions are defined by

$$\hat{\mathcal{E}}_{k'}^{(1)} = \left\{(z,t) : (-1)^k \overline{\hat{Z}_k^{(1)}} \le (-1)^k z\right\}, \tag{30}$$

$$\hat{\mathcal{T}}_{k'}^{(n)} = \left\{(z,t) : (-1)^k \hat{Z}_k^{(0)} \le (-1)^k z < (-1)^k \overline{\hat{Z}_k^{(1)}}\right\}, \tag{31}$$

$$\hat{\mathcal{M}} = \left\{(z,t) : \hat{Z}_1^{(0)} < z < \hat{Z}_2^{(0)}\right\}, \tag{32}$$

where

$$\overline{\hat{Z}_k^{(1)}}(t) \equiv \sum_{j=0}^{1} \lambda^{-j} \hat{Z}_k^{(j,s)}. \tag{33}$$

denotes the position of boundaries of the fast transition-layers. With the new inner spatial variables

$$\hat{\zeta}_i^{(1)} = \lambda^n \left( z - \overline{\hat{Z}_i^{(1)}} \right), \quad i = k, k', \tag{34}$$

we make the change of variables from $(z, \tau)$ to $(\hat{\zeta}_i^{(1)}, \tau)$, reducing (24)-(26) to the equations

$$\lambda \frac{\partial \beta_k}{\partial \tau} - \lambda \left( \overline{\hat{V}_k^{(1)}} - v^* \right) \frac{\partial \beta_k}{\partial \hat{\zeta}_i^{(1)}} = 0, \tag{35}$$

$$\beta_k \left( \lambda \frac{\partial \rho_k}{\partial \tau} - \lambda \left( \overline{\hat{V}_k^{(1)}} - v_k \right) \frac{\partial \rho_k}{\partial \hat{\zeta}_i^{(1)}} \right) + \lambda \beta_k \rho_k \frac{\partial v_k}{\partial \hat{\zeta}_i^{(1)}} + \lambda \rho_k (v_k - v^*) \frac{\partial \beta_k}{\partial \hat{\zeta}_i^{(1)}} = 0, \tag{36}$$

$$\beta_k \rho_k \left( \lambda \frac{\partial v_k}{\partial \tau} - \lambda \left( \overline{\hat{V}_k^{(1)}} - v_k \right) \frac{\partial v_k}{\partial \hat{\zeta}_i^{(1)}} \right) + \lambda^3 \beta_k \frac{\partial p_k}{\partial \hat{\zeta}_i^{(1)}} + \lambda^3 (p_k - p^*) \frac{\partial \beta_k}{\partial \hat{\zeta}_i^{(1)}} = \beta_k \rho_k g. \tag{37}$$

Here the edge velocity of the transition-layers is defined by

$$\overline{\hat{V}_k^{(1)}}(t) \equiv \frac{d \overline{\hat{Z}}_k^{(1)}}{dt} = \sum_{j=0}^{1} \lambda^{-j} \hat{V}_k^{(j)}. \tag{38}$$

We assume fast transitional layer expansions of the form

$$U_k(\hat{\zeta}_i^{(1)}, \tau) = \hat{U}_k^{(0,ft)}(\hat{\zeta}_i^{(1)}, \tau) + \lambda^{-1} \hat{U}_k^{(1,ft)}(\hat{\zeta}_i^{(1)}, \tau) + \lambda^{-2} \hat{U}_k^{(2,ft)}(\hat{\zeta}_i^{(1)}, \tau) + O(\lambda^{-3}), \tag{39}$$

in the region $(-1)^{i+1} \hat{Z}_i^{(1)} \leq (-1)^i \hat{\zeta}_i^{(1)} \leq 0$. The fast transitional variables of each order of $\lambda^{-1}$ in the expansions (39) satisfy simple differential equations and they are solved in closed form in Sec. 4.2.2. The fast variables are matched continuously order by order at the boundaries of these layers in Sec. 4.2.4. This matching process of inner limit and fast transitional expansions determines uniformly valid outer expansions in $z$.

### 3.1. Transitional Effective Variable

In this section we solve a simple system of ODEs to be used in the analysis of the fast transitional terms $v_k^{(0,ft)}$, $p_k^{(0,ft)}$ in Sec. 4.2.2. Let the effective transitional variable $q_k^{(t)}$ satisfy the system

$$\beta_k^{(t)} \frac{\partial q_k^{(t)}}{\partial \zeta_i} + \mu_k^{q(t)} \left( q_k^{(t)} - q_{k'}^{(t)} \right) \frac{\partial \beta_k^{(t)}}{\partial \zeta_i} = 0 \tag{40}$$

n the region $(-1)^{i+1}Z_i^{(1)} \leq (-1)^i \zeta_i \leq 0$. Here $\beta_k^{(t)}(\zeta_i, t)$ is a given $C^1$ function in the region and

$$\mu_k^{q(t)}(\beta_k^{(t)}, d_k^{q(0,s)}) = \frac{\beta_k^{(t)}}{\beta_k^{(t)} + d_k^{q(0,s)} \beta_{k'}^{(t)}} \tag{41}$$

satisfying $\mu_k^{q(t)} + \mu_{k'}^{q(t)} = 1$. This system can be decoupled and solved in closed form [11] by the introduction of two linear combinations of the effective transitional variables as follows

**Proposition 3.1.** Let $q_k^{(t)}(\zeta_i, t)$ satisfy the system (40) in $(-1)^{i+1}Z_i^{(1)} \leq (-1)^i \zeta_i \leq 0$. Then the solution is

$$\begin{aligned} q_k^{(t)}(\zeta_i, t) =& \frac{\mu_k^{q(t)}}{\beta_k^{(t)}} \left\{ \left[ \beta_{k'}^{(t)}(-Z_i^{(1)}, t) - \beta_{k'}^{(t)} \right] q_{k'}^{(t)}(-Z_i^{(1)}, t) \right. \\ & \left. + \left[ \beta_k^{(t)}(-Z_i^{(1)}, t) + d_k^{q(t)} \beta_{k'}^{(t)} \right] q_k^{(t)}(-Z_i^{(1)}, t) \right\}. \end{aligned} \tag{42}$$

## 4. Expansion Procedure of the Inner Limit

We derive the inner terms in the inner limit asymptotic expansions (27) for the solutions of the compressible equations. The inner terms, uniformly valid in space, are determined by matching the inner limit in the exterior and the incompressible mixing zone and the fast transition-layer expansions. The fast variables are defined as the inner expansion terms minus the common terms resulting from the outer expansion evaluated in the inner limit to each order [14]. These variables oscillate on the fast time scale $\tau = \lambda t$ and are described by linear wave equations. They are also defined uniformly in space by matching. We substitute the inner limit asymptotic expansions (3.4) into the compressible equations (24)–(26), and equate powers of $\lambda$. Since $\lambda$ is arbitrary, the coefficient of $\lambda^m$ for each order $m$ must vanish, defining equations for the inner terms to each order. The zero-th order terms are determined by initial data of the incompressible flow. The inner terms of higher order in $\lambda^{-1}$ satisfy simple differential equations. Specifically the fast variables satisfy linearized compressible equations. The variables $v_k^{(0,f)}$ and $v_k^{(1,f)}$ solve a hyperbolic IBVP. They are solved by the method of characteristics and a Picard iteration. The remaining equations can thus be viewed as equations for $\beta_k^{(1,f)}$ alone. The solutions are expressed in terms of the initial and the boundary data. The boundary data is provided by matching at the outer and inner edges of regions which exist in order $\lambda^{-m}$ expansions. The first order inner terms in the expansion (3.4) of $\beta_k$ and $p_k$ and the zero-th order inner term of $v_k$ have the fast transitional layers in the region $\hat{\mathfrak{I}}_i^{(1)}$, $i = k, k'$, introduced in Sec. 3. The fast transitional variables for each order in $\lambda^{-1}$ satisfy simple differential equations. The inner terms are matched continuously order by order at the boundaries of these layers. This matching process for the inner limit and the fast transitional expansions determines uniformly valid inner expansions in $z$. The main result is summarized in the following theorem.

**Theorem 4.1.** Assume (8), (15), (19), (23), and the initial data (11). Then the compressible solutions have the uniformly valid inner expansions in $z$ to $O(\lambda^{-1})$ for $\beta_k$, $\rho_k$, $p_k$ and to $O(1)$ for $v_k$ of the form

$$\begin{aligned}
\beta_k &= \beta_k^\infty(z,0) + \lambda^{-1}\hat{\beta}_k^{(1)} + O(\lambda^{-2}),\\
v_k &= v_k^\infty(z,0) + O(\lambda^{-1}),\\
\rho_k &= \rho_k^\infty + O(\lambda^{-2}),\\
p_k &= p^{(0)} + O(\lambda^{-2}),
\end{aligned} \tag{43}$$

where $p^{(0)} = p_k(\rho_k^\infty)$. The inner term $\hat{\beta}_k^{(1)}(z,\tau)$ satisfies

$$\hat{\beta}_k^{(1)}(z,\tau) = \begin{cases}
0 & \text{in } \hat{\mathcal{E}}_1^{(1)},\ \hat{\mathcal{E}}_2^{(1)}\\
\lambda\left(z - Z_{k'}^\infty(0)\right)\dfrac{\partial\beta_k^\infty}{\partial z}(Z_{k'}^\infty(0) + (-1)^k 0, 0) & \\
+\tau\dfrac{\partial\beta_k^\infty}{\partial t}(Z_{k'}^\infty(0) + (-1)^k 0, 0) & \text{in } \hat{\mathcal{T}}_k^{(1)}\\
\beta_k^{(1,s)}(z,0) + \tau\dfrac{\partial\beta_k^\infty}{\partial t}(z,0) & \text{in } \hat{\mathcal{M}}\\
-\beta_{k'}^{(1,s)}(z,t) & \text{in } \hat{\mathcal{T}}_{k'}^{(1)}
\end{cases}, \tag{44}$$

### 4.1. The Zero-th Order Inner Terms

We show that the inner limit terms $\hat{\beta}_k^{(0)}$, $\hat{\rho}_k^{(0)}$, $\hat{p}_k^{(0)}$ in the expansions (27) describing the incompressible limit process near the initial time are determined by the incompressible background solution. To this order, there is no transition zone, and no contribution from fast transitional terms $U_k^{(m,ft)}$. Substituting the inner limit asymptotic expansions (3.4) into the compressible equations (24)–(26), we equate powers of $\lambda$. The coefficients of $\lambda$ in the interface and mass equations and the order $\lambda^2$ terms in the momentum equation define the equations

$$O(1) \quad : \quad \frac{\partial\hat{\beta}_k^{(0)}}{\partial\tau} = 0, \tag{45}$$

$$O(1) \quad : \quad \hat{\beta}_k^{(0)}\frac{\partial\hat{\rho}_k^{(0)}}{\partial\tau} = 0, \tag{46}$$

$$O(\lambda^2) \quad : \quad \hat{\beta}_k^{(0)}\frac{\partial\hat{p}_k^{(0)}}{\partial z} + \hat{\mu}_k^{p(0)}\left(\hat{p}_k^{(0)} - \hat{p}_{k'}^{(0)}\right)\frac{\partial\hat{\beta}_k^{(0)}}{\partial z} = 0 \tag{47}$$

for $\hat{\beta}_k^{(0)}$, $\hat{\rho}_k^{(0)}$, $\hat{p}_k^{(0)}$. We note that $\hat{v}_k^{(0)}$ does not contribute here and it is found in Sec. 4.2. The initial conditions associated with (11) are given as

$$\hat{\beta}_k^{(0)}(z,0) = \beta_k^\infty(z,0), \quad \hat{\rho}_k^{(0)}(z,0) = \rho_k^\infty, \quad \hat{p}_k^{(0)}(z,0) = p^{(0)}. \tag{48}$$

Using the initial data (48), we solve Eqs. (45)–(47).

**Proposition 4.2.** Assume (8), (15), (19), (23), and the initial data (11). Then the zero-th order terms in the expansions (27) satisfy

$$\hat{\beta}_k^{(0)}(z,0) = \beta_k^{\infty}(z,0), \quad \hat{\rho}_k^{(0)}(z,0) = \rho_k^{\infty}, \quad \hat{p}_k^{(0)}(z,0) = p^{(0)}, \tag{49}$$

where the universal constant $p^{(0)} = p_k(\rho_k^{\infty})$.

## 4.2. The First Order Inner Terms

Using the zero-th order inner terms (49), we find the inner terms $\hat{v}_k^{(0)}$, $\hat{\beta}_k^{(1)}$, $\hat{\rho}_k^{(1)}$, $\hat{p}_k^{(1)}$ in the expansions (27), uniformly valid in $z$. The initial conditions associated with (11) satisfy

$$\hat{v}_k^{(0)}(z,0) = v_k^{\infty}(z,0), \quad \hat{\beta}_k^{(1)}(z,0) = \beta_k^{(1,s)}(z,0), \quad \hat{\rho}_k^{(1)}(z,0) = 0 = \hat{p}_k^{(1)}(z,0). \tag{50}$$

Let us define the variables $v_k^{(0,f)}$, $\beta_k^{(1,f)}$, $p_k^{(1,f)}$ which are oscillatory part of $\hat{v}_k^{(0)}$, $\hat{\beta}_k^{(1)}$, and $\hat{p}_k^{(1)}$ as the following

$$\begin{aligned} v_k^{(0,f)}(z,\tau) &\equiv \hat{v}_k^{(0)} - v_k^{\infty}(z,0), \\ \beta_k^{(1,f)}(z,\tau) &\equiv \hat{\beta}_k^{(1)} - \left(\beta_k^{(1,s)}(z,0) + \tau\frac{\partial\beta_k^{\infty}}{\partial t}(z,0)\right), \\ p_k^{(1,f)}(z,\tau) &\equiv \hat{p}_k^{(1)}. \end{aligned} \tag{51}$$

The fast variables $v_k^{(0,f)}$ and $p_k^{(1,f)}$ satisfy a subsystem of hyperbolic PDFs with no source terms. The remaining equations are solved for $\beta_k^{(1,f)}$. Using the method of characteristics and the process of Picard iteration, we show that the variables $v_k^{(0,f)}$ and $p_k^{(1,f)}$ are suppressed under the initial condition (50).

We consider the transitional regions $\hat{\mathfrak{T}}_i^{(1)}$, $i = k, k'$, by discussion of the fast transition-layers through $z = \overline{\hat{Z}_i^{(1,s)}}$ in Sec. 4.2.2. The matching the inner edge of the inner limit expansions in the exterior with the outer edge of the transitional expansions and the inner edge of the transitional expansions with the edge of the incompressible mixing zone determines the first order inner terms uniformly in space in Sec. 4.2.4.

**Proposition 4.3.** Assume (8), (15), (19), (23), and the initial data (11). Then $\hat{\beta}_k^{(1)}(z,\tau)$ satisfies (44) and the inner terms satisfy

$$\hat{v}_k^{(0)}(z,\tau) = \begin{cases} v_k^{\infty}(z,0) & \text{in } \hat{\mathfrak{E}}_k^{(1)},\ \hat{\mathfrak{T}}_k^{(1)},\ \hat{\mathfrak{M}} \\ V_k^{\infty}(0) & \text{in } \hat{\mathfrak{T}}_{k'}^{(1)} \end{cases}, \tag{52}$$

$$\hat{p}_k^{(0)}(z,\tau) = \hat{\rho}_k^{(0)}(z,\tau) = 0 \quad in\ \hat{\mathfrak{E}}_k^{(1)},\ \hat{\mathfrak{T}}_k^{(1)},\ \hat{\mathfrak{M}},\ \hat{\mathfrak{T}}_{k'}^{(1)}. \tag{53}$$

### 4.2.1 The Inner Terms in the Exterior Domain

In this section, we find the inner terms $\hat{v}_k^{(0)}$ and $\hat{p}_k^{(1)}$ in the single phase region $\hat{\mathcal{E}}_k^{(1)}$. We substitute the asymptotic expansions (27) into (24)–(26) and equate powers of $\lambda$. Using (49), we isolate the order zero terms in the mass equations and the order $\lambda$ terms in the momentum equation. Since $\lambda$ is arbitrary, the coefficient of $\lambda$ must vanish, leading to the equations

$$O(1) \quad : \quad \frac{\partial \hat{\rho}_k^{(1)}}{\partial \tau} + \rho_k^\infty \frac{\partial \hat{v}_k^{(0)}}{\partial z} = 0, \tag{54}$$

$$O(\lambda) \quad : \quad \rho_k^\infty \hat{v}_k^{(0)} \frac{\partial \hat{\rho}_k^{(0)}}{\partial \tau} + \frac{\partial \hat{p}_k^{(1)}}{\partial z} = 0, \tag{55}$$

for $\hat{v}_k^{(0)}$, $\hat{\rho}_k^{(1)}$, $\hat{p}_k^{(1)}$ in $\hat{\mathcal{E}}_k^{(1)}$. We note that

$$\hat{\beta}_k^{(1)} = 0 \quad \text{in } \hat{\mathcal{E}}_k^{(1)} \; i = 1, 2. \tag{56}$$

Substituting (51) into (54), (55), and multiplying (54) by $a_k^2 \equiv dp_k/d\rho_k(\rho_k^\infty)$ and (55) by $1/\rho_k^\infty$, the equations reduce to the system of the wave equations,

$$\frac{\partial u_k^{(1,f)}}{\partial \tau} + A_k^0 \frac{\partial u_k^{(1,f)}}{\partial z} = 0 \tag{57}$$

for $u_k^{(1,f)} = \left(p_k^{(1,f)}, v_k^{(0,f)}\right)^{tr}$, where $A_k^0$ is the constant coefficient matrix

$$A_k^0 = \begin{pmatrix} 0 & \rho_k^\infty a_k^2 \\ 1/\rho_k^\infty & 0 \end{pmatrix} \tag{58}$$

From (50) the initial conditions follow

$$u_k^{(1,f)}(z,0) = \left(p_k^{(1,f)}(z,0), v_k^{(0,f)}(z,0)\right)^{tr} = 0. \tag{59}$$

We discuss the IBVP (57)-(59) in $\hat{\mathcal{E}}_k^{(1)}$ using the method of characteristics. The matrix ¥Á$_k^0$ has two distinct eigenvalues $\Lambda_{k,i}^0 \equiv (-1)^i a_k$, $i = 1, 2$. We define $\Gamma_k^0$ to be the nonsingular matrix whose column vectors consist of linearly independent eigenvectors of $A_k^0$,

$$\Gamma_k^0 = \begin{pmatrix} 1/2 & 1/2 \\ -1/(2\rho_k^\infty a_k) & 1/(2\rho_k^\infty a_k). \end{pmatrix} \tag{60}$$

The characteristic curves $C_{k,i}$, $i = 1, 2$, satisfy

$$C_{k,i} : \frac{dX}{ds} = \Lambda_{k,i}^0. \tag{61}$$

Therefore the backward characteristics $C_{k,i}$, $i = 1, 2$, through a point $(z, \tau)$ have equations for $0 \le s \le \tau$,

$$C_{k,i} : X(s) = \alpha_{k,i}(s, z, \tau) = (-1)^i a_k(s - \tau) + z. \tag{62}$$

Introducing a new unknown column vector $U_k^{(1,f)} = \left(U_{k,1}^{(1,f)}, U_{k,2}^{(0,f)}\right)^t \equiv (\Gamma_k^0)^{-1} u_k^{(1,f)}$ and multiplying (57) by $(\Gamma_k^0)^{-1}$, we obtain

$$\frac{\partial U_k^{(1,f)}}{\partial \tau} + \Lambda_k^0 \frac{\partial U_k^{(1,f)}}{\partial z} = 0 \tag{63}$$

$$U_k^{(1,f)}(z, 0) = (\Gamma_k^0)^{-1} u_k^{(1,f)}(z, 0) = 0. \tag{64}$$

By the method of characteristics we find the solution

$$\begin{aligned} U_{k,k}^{(1,f)}(z, \tau) &= U_{k,k}^{(1,f)}(\alpha_{k,k}(0, z, \tau), 0) \\ &= \left(p_k^{(1,f)} + (-1)^k \rho_k^\infty a_k v_k^{(0,f)}\right)(z + (-1)^{k'} a_k \tau, 0) = 0, \end{aligned} \tag{65}$$

$$U_{k,k'}^{(1,f)}(z, \tau) = \begin{cases} U_{k,k'}^{(1,f)}(\alpha_{k,k'}(0, z, \tau), 0) = 0 & \text{if } (-1)^{k'} \alpha_{k,k'}(0, z, \tau) \ge \hat{Z}_{k'}^{(0)} \\ U_{k,k'}^{(1,f)}\left(\overline{\hat{Z}_{k'}^{(1)}}(\tau_k^{(1)}), \tau_k^{(1)}\right) & \text{if } (-1)^{k'} \alpha_{k,k'}(0, z, \tau) \le \hat{Z}_{k'}^{(0)} \end{cases}, \tag{66}$$

in $\hat{\mathcal{E}}_k^{(1)}$, where $\tau_k^{(1)} = \tau_k^{(1)}(z, \tau)$ satisfies $\alpha_{k,k'}(\tau_k^{(1)}, z, \tau) = \overline{\hat{Z}_{k'}^{(1)}}(\tau_k^{(1)})$. From (65), we see that the solution $U_{k,k}^{(1,f)}$ is completely determined by initial data along the incoming characteristic $\alpha_{k,k}$. Furthermore, this solution provides the boundary data $U_{k,k}^{(1,f)}\left(\overline{\hat{Z}_{k'}^{(1)}}, \tau\right)$ which will be used to solve the related inner term system in $\hat{\mathcal{M}}$ in Sec. 4.2.3 once the variables $u_k^{(1,f)}$ are determined in the fast transitional regions $\hat{\mathcal{T}}_i^{(1)}$, $i = k, k'$. The solution $U_{k,k'}^{(1,f)}$ depends upon boundary data at the edge $z = \overline{\hat{Z}_{k'}^{(1)}}$ to be completely determined in the single phase region $\hat{\mathcal{E}}_k^{(1)}$ along the outgoing characteristic $\alpha_{k,k'}$. The boundary data

$$U_{k,k'}^{(1,f)}\left(\overline{\hat{Z}_{k'}^{(1)}}, \tau\right) = \left(p_k^{(1,f)} + (-1)^k \rho_k^\infty a_k v_k^{(0,f)}\right)\left(\overline{\hat{Z}_{k'}^{(1)}}, \tau\right) = 0 \tag{67}$$

will be shown later by matching in Sec. 4.2.4.

**Lemma 4.4.** Assume (8), (15), (19), (23), and the initial data (11). Assume the boundary data (67). Then the fast variables satisfy

$$v_k^{(0,f)} = p_k^{(1,f)} = 0 \tag{68}$$

in $\hat{\mathcal{E}}_k^{(1)}$. Thus, we obtain the inner terms

$$\hat{v}_k^{(0)} = v_k^\infty(z, 0), \quad \hat{\rho}_k^{(1)} = 0 = \hat{p}_k^{(1)}. \tag{69}$$

### 4.2.2 The Fast Transition-Layer Expansion

We discuss the fast transition-layer extending through $z = \overline{\hat{Z}_i^{(1)}}$, $i = k, k'$, which defines $\hat{v}_k^{(0)}$, $\hat{\beta}_k^{(1)}$, $\hat{p}_k^{(1)}$, continuously in $\hat{\mathfrak{I}}_i^{(1)}$. We substitute the fast transition-layer expansions (39) into the compressible equations (eq:comp-1-tran)–(eq:comp-3-tran) and equate powers of $\lambda$. Since $\lambda$ is arbitrary, the coefficient of $\lambda^n$ for each order n must vanish, defining differential equations for the fast transitional terms. We first determine the fast transitional limit solutions $U_k^{(0,ft)}$ in the expansions (3.14). We isolate the order $\lambda$ terms in the interface and mass equation and the order $\lambda^3$ terms in the momentum equation. Since the coefficients of $\lambda$ and $\lambda^3$ must vanish, the leading order terms satisfy the coupled ODEs

$$O(\lambda) \;:\; \frac{\partial \beta_k^{(0,ft)}}{\partial \tau} + \left(-\hat{V}_i^{(0)} + v^{*(0,ft)}\right) \frac{\partial \beta_k^{(0,ft)}}{\partial \hat{\zeta}_i^{(1)}} = 0, \tag{70}$$

$$\begin{aligned} O(\lambda) \;:\; & \beta_k^{(0,ft)} \left( \frac{\partial \rho_k^{(0,ft)}}{\partial \tau} + \left(-\hat{V}_i^{(0)} + v_k^{(0,ft)}\right) \frac{\partial \rho_k^{(0,ft)}}{\partial \hat{\zeta}_i^{(1)}} \right) + \beta_k^{(0,ft)} \rho_k^{(0,ft)} \frac{\partial v_k^{(0,ft)}}{\partial \hat{\zeta}_i^{(1)}} \\ & + \rho_k^{(0,ft)} \mu_k^{v(0,ft)} \left( v_k^{(0,ft)} - v_{k'}^{(0,ft)} \right) \frac{\partial \beta_k^{(0,ft)}}{\partial \hat{\zeta}_i^{(1)}} = 0, \end{aligned} \tag{71}$$

$$O(\lambda^3) \;:\; \beta_k^{(0,ft)} \frac{\partial p_k^{(0,ft)}}{\partial \hat{\zeta}_i^{(1)}} + \mu_k^{p(0,ft)} \left( p_k^{(0,ft)} - p_{k'}^{(0,ft)} \right) \frac{\partial \beta_k^{(0,ft)}}{\partial \hat{\zeta}_i^{(1)}} = 0 \tag{72}$$

for $\mathcal{U}_k^{(0,ft)}(\hat{\zeta}_i^{(1)}, t)$ in the region $(-1)^{i+1} \hat{Z}_i^{(1)} \leq (-1)^i \hat{\zeta}_i^{(1)} \leq 0$. We note that for $q = v, p$,

$$\mu_k^{q(0,ft)} = \frac{\beta_k^{(0,ft)}}{\beta_k^{(0,ft)} + \hat{d}_k^{q\infty} \beta_{k'}^{(0,ft)}}. \tag{73}$$

The matching conditions to $O(1)$ are

$$\begin{aligned} \beta_k^{(0,ft)} \left( \hat{\zeta}_i^{(1)} = 0, \tau \right) &= \hat{\beta}_k^{(0)} \left( \overline{\hat{Z}_i^{(1)}}, \tau \right) = \delta_{ik'}, \\ \beta_k^{(0,ft)} \left( \hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)}, \tau \right) &= \hat{\beta}_k^{(0)} \left( \hat{Z}_i^{(0)}, \tau \right) = \delta_{ik'}, \\ \rho_k^{(0,ft)} \left( \hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)}, \tau \right) &= \hat{\rho}_k^{(0)} \left( \hat{Z}_i^{(0)}, \tau \right) = \rho_k^{\infty} \\ p_k^{(0,ft)} \left( \hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)}, \tau \right) &= \hat{p}_k^{(0)} \left( \hat{Z}_i^{(0)}, \tau \right) = p^{(0)} \end{aligned} \tag{74}$$

by use of (49). We note that the zero-th order terms $\hat{v}_k^{(0)}$ and $v_k^{(0,ft)}$ are not discussed here. We will see later that $\hat{v}_k^{(0)}$ is coupled with the higher order term $\hat{p}_k^{(1)}$ and the boundary data of $\hat{v}_k^{(0)}$ and $v_k^{(0,ft)}$ is found by matching in Sec. 4.2.4.

Using (74), Eqs. (70)–(72) are solved as the following:

**Lemma 4.5.** Assume (8), (15), (19), (23), and the initial data (11). Assume the boundary data (74). Then the fast transitional limit in the expansions (39) satisfies

$$\beta_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)},\tau\right) = \delta_{ik'}, \tag{75}$$

$$\rho_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)},\tau\right) = \rho_k^{\infty}, \tag{76}$$

$$p_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)},\tau\right) = p^{(0)}, \tag{77}$$

$$v_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)},\tau\right) = v_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)},\tau\right) = v_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)} = 0,\tau\right), \tag{78}$$

where the universal constant $p^{(0)} = p_k(\rho_k^{\infty})$ and $\delta_{ik'}$ is the Kronecker symbol.

*Proof.* We first solve Eq. (72) for $p_k^{(0,ft)}$ by using Proposition 3.1 with $q_k^{(t)} = p_k^{(0,ft)}$ and the boundary data (74). Then we obtain the solution (77). The equation of state implies (76). Substituting (76) into (71) and multiplying the equation by $1/\rho_k^{(0,ft)}$, it reduces to

$$\beta_k^{(0,ft)}\frac{\partial v_k^{(0,ft)}}{\partial\hat{\zeta}_i^{(1)}} + \mu_k^{v(0,ft)}\left(v_k^{(0,ft)} - v_{k'}^{(0,ft)}\right)\frac{\partial\beta_k^{(0,ft)}}{\partial\hat{\zeta}_i^{(1)}} = 0. \tag{79}$$

To decouple $v_k^{(0,ft)}$ and $\beta_k^{(0,ft)}$, we use Proposition 3.1 with $q_k^{(t)} = v_k^{(0,ft)}$. Thus,

$$\begin{aligned} v_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)},\tau\right) =& \frac{\mu_k^{v(0,ft)}}{\beta_k^{(0,ft)}}\Big[\left(\beta_{k'}^{(0,ft)}(-\hat{Z}_i^{(1)},\tau) - \beta_{k'}^{(0,ft)}\right)v_{k'}^{(0,ft)}(-\hat{Z}_i^{(1)},\tau) \\ &+\left(\beta_k^{(0,ft)}(-\hat{Z}_i^{(1)},\tau) + \hat{d}_k^{v\infty}\beta_{k'}^{(0,ft)}\right)v_k^{(0,ft)}(-\hat{Z}_i^{(1)},\tau)\Big] \\ =& \begin{cases} \dfrac{\mu_k^{v(0,ft)}}{\beta_k^{(0,ft)}}\Big[-\beta_{k'}^{(0,ft)}v_{k'}^{(0,ft)}(-\hat{Z}_i^{(1)},\tau) + \left(1+\hat{d}_k^{v\infty}\beta_{k'}^{(0,ft)}\right)v_k^{(0,ft)}(-\hat{Z}_i^{(1)},\tau)\Big] & \text{if } i = k' \\ \mu_k^{v(0,ft)}v_{k'}^{(0,ft)}(-\hat{Z}_i^{(1)},\tau) + \mu_{k'}^{v(0,ft)}v_k^{(0,ft)}(-\hat{Z}_i^{(1)},\tau) & \text{if } i = k \end{cases} \end{aligned} \tag{80}$$

by (74). Substituting (refeq:v(0,ft)-cal) into (70), we solve the quasilinear PDE for $\beta_k^{(0,ft)}$. The characteristic speed satisfies $-\hat{V}_i^{(0)} + v^{*(0,ft)}$. By the boundary condition (74), the solution $\beta_k^{(0,ft)}$ is constant in space and time. Substitution of the $\beta_k^{(0,ft)}$ into (refeq:v(0,ft)-cal) implies the velocity $v_k^{(0,ft)}$ in (78 which is independent of space. ■

Using (75)–(78), we now define the fast transitional variables $U_k^{(1,ft)}$ in the expansions (39). We isolate the order zero terms in the interface equation, the order zero terms and the order $\lambda^{-1}$ terms in the continuity equation for $i = k'$ and $i = k$, and the order $\lambda^2$ terms and the order $\lambda$ terms in the momentum equation for $i = k'$ and $i = k$, respectively. Since $\lambda$ is arbitrary, the coefficients of $\lambda^2$, $\lambda$ and $\lambda^{-1}$ must vanish, leading to the equations

$$O(1) \;:\; \frac{\partial \beta_k^{(1,ft)}}{\partial \tau} + \left(-\hat{V}_i^{(1)} + v^{*(1,ft)}\right) \frac{\partial \beta_k^{(1,ft)}}{\partial \hat{\zeta}_i^{(1)}} = 0, \tag{81}$$

$$O(1) \;:\; \frac{\partial \rho_k^{(1,ft)}}{\partial \tau} + \left(-\hat{V}_i^{(1)} + v^{*(1,ft)}\right) \frac{\partial \rho_k^{(1,ft)}}{\partial \hat{\zeta}_i^{(1)}} + \rho_k^{\infty} \frac{\partial v_k^{(1,ft)}}{\partial \hat{\zeta}_i^{(1)}} + \rho_k^{\infty} \left(v_k^{(0,ft)} - v_{k'}^{(0,ft)}\right) \frac{\partial \beta_k^{(1,ft)}}{\partial \hat{\zeta}_i^{(1)}} = 0, \quad i = k', \tag{82}$$

$$O(\lambda^{-1}) \;:\; \beta_k^{(1,ft)} \left( \frac{\partial \rho_k^{(1,ft)}}{\partial \tau} + \left(-\hat{V}_i^{(1)} + v^{*(1,ft)}\right) \frac{\partial \rho_k^{(1,ft)}}{\partial \hat{\zeta}_i^{(1)}} \right) + \beta_k^{(1,ft)} \rho_k^{\infty} \frac{\partial v_k^{(1,ft)}}{\partial \hat{\zeta}_i^{(1)}} + \rho_k^{\infty} \mu_k^{v(1,ft)} \left(v_k^{(0,ft)} - v_{k'}^{(0,ft)}\right) \frac{\partial \beta_k^{(1,ft)}}{\partial \hat{\zeta}_i^{(1)}} = 0, \quad i = k, \tag{83}$$

$$O(\lambda^2) \;:\; \frac{\partial p_k^{(1,ft)}}{\partial \hat{\zeta}_i^{(1)}} = 0, \quad i = k', \tag{84}$$

$$O(\lambda) \;:\; \beta_k^{(1,ft)} \frac{\partial p_k^{(1,ft)}}{\partial \hat{\zeta}_i^{(1)}} = 0, \quad i = k \tag{85}$$

for $U_k^{(1,ft)}(\hat{\zeta}_i^{(1)}, t)$ in $(-1)^{i+1}\hat{Z}_i^{(1)} \le (-1)^i \hat{\zeta}_i^{(1)} \le 0$. Here we may assume that $\beta_k^{(1,ft)}$ is nontrivial.

Solving (84) and (85), we obtain

**Lemma 4.6.** Assume (8), (15), (19), (23), and the initial data (11). Then

$$p_k^{(1,ft)}(\hat{\zeta}_i^{(1)}, \tau) = p_k^{(1,ft)}(\hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1,s)}, \tau) = p_k^{(1,ft)}(\hat{\zeta}_i^{(1)} = 0, \tau). \tag{86}$$

From (78) and (86), the fast transitional terms $v_k^{(0,ft)}$ and $p_k^{(1,ft)}$ are constant in space and depend on boundary data at the boundaries of the regions $\hat{\mathfrak{T}}_i^{(1)}$. By the matching process in Sec. 4.2.4, the boundary data will be provided and therefore, $v_k^{(0,ft)}$ and $p_k^{(1,ft)}$ will be determined completely. Using the $v_k^{(0,ft)}$ and $p_k^{(1,ft)}$, the solution $\beta_k^{(1,ft)}$ of (81)–(83) is found in Sec. 4.2.4.

### 4.2.3 The Inner Terms within the Mixing Zone

We find the inner terms $\hat{v}_k^{(0)}$, $\hat{\beta}_k^{(1)}$, $\hat{\rho}_k^{(1)}$, $\hat{p}_k^{(1)}$ in the mixing zone $\hat{\mathfrak{M}}$. Using (49), we isolate the order zero terms in the interface and continuity equation and the order $\lambda$ terms in the momentum equation, defining the equations

$$O(1) \quad : \quad \frac{\partial \hat{\beta}_k^{(1)}}{\partial \tau} + \hat{v}^{*(0)} \frac{\partial \hat{\beta}_k^{(0)}}{\partial z} = 0, \tag{87}$$

$$O(1) \quad : \quad \hat{\beta}_k^{(0)} \frac{\partial \hat{\rho}_k^{(1)}}{\partial \tau} + \hat{\beta}_k^{(0)} \hat{\rho}_k^{(0)} \frac{\partial \hat{v}_k^{(0)}}{\partial z} + \hat{\rho}_k^{(0)} \hat{\mu}_k^{v(0)} \left( \hat{v}_k^{(0)} - \hat{v}_{k'}^{(0)} \right) \frac{\partial \hat{\beta}_k^{(0)}}{\partial z} = 0, \tag{88}$$

$$O(\lambda) \quad : \quad \hat{\beta}_k^{(0)} \hat{\rho}_k^{(0)} \frac{\partial \hat{v}_k^{(0)}}{\partial \tau} + \hat{\beta}_k^{(0)} \frac{\partial \hat{p}_k^{(1)}}{\partial z} + \hat{\mu}_k^{p(0)} \left( \hat{p}_k^{(1)} - \hat{p}_{k'}^{(1)} \right) \frac{\partial \hat{\beta}_k^{(0)}}{\partial z} = 0 \tag{89}$$

for $\hat{v}_k^{(0)}$, $\hat{\beta}_k^{(1)}$, $\hat{\rho}_k^{(1)}$, $\hat{p}_k^{(1)}$ in $\hat{\mathcal{M}}$.

Substitution of (51) into (87)-(89) reduces to the system

$$\frac{\partial \beta_k^{(1,f)}}{\partial \tau} + v^{*(0,f)} \frac{\partial \hat{\beta}_k^{(0)}}{\partial z} = 0, \tag{90}$$

$$\frac{\partial p_k^{(1,f)}}{\partial \tau} + \rho_k^\infty a_k^2 \frac{\partial v_k^{(0,f)}}{\partial z} + \rho_k^\infty a_k^2 f_k^v(z) \left( v_k^{(0,f)} - v_{k'}^{(0,f)} \right) \frac{\partial \hat{\beta}_k^{(0)}}{\partial z} = 0, \tag{91}$$

$$\frac{\partial v_k^{(0,f)}}{\partial \tau} + \frac{1}{\rho_k^\infty} \frac{\partial p_k^{(0,f)}}{\partial z} + \frac{f_k^p(z)}{\rho_k^\infty} \left( p_k^{(1,f)} - p_{k'}^{(1,f)} \right) = 0 \tag{92}$$

for $v_k^{(0,f)}$, $\beta_k^{(1,f)}$ and $p_k^{(1,f)}$. Here $a_k^2 = c^2(\rho_k^\infty)$ and for $q = \text{í}, p$,

$$f_k^q(z) \equiv \frac{\hat{\mu}_k^{q(0)}}{\hat{\beta}_k^{(0)}} \frac{\partial \hat{\beta}_k^{(0)}}{\partial z} = \left( \frac{\mu_k^{q\infty}}{\beta_k^\infty} \frac{\partial \beta_k^\infty}{\partial z} \right)(z, 0) = -d_{k'}^{q\infty}(0) f_{k'}^q(z). \tag{93}$$

Observe that $v_k^{(0,f)}$ and $p_k^{(1,f)}$ satisfy a subsystem of hyperbolic PDEs which are linearized compressible equations while $\beta_k^{(1,f)}$ depends on the solution $v_k^{(0,f)}$. The initial condition (11) implies the trivial initial data

$$\beta_k^{(1,f)}(z, 0) = v_k^{(0,f)}(z, 0) = p_k^{(1,f)}(z, 0) = 0 \tag{94}$$

for the fast variables. We prove that this initialization suppresses the fast variables $\beta_k^{(1,f)}$, $v_k^{(0,f)}$ and $p_k^{(1,f)}$. In this section, we assume the boundary data

$$\left( \hat{p}_k^{(1)} + (-1)^k \rho_k^\infty a_k \hat{v}_k^{(0)} \right) \left( \hat{Z}_{k'}^{(0)}, \tau \right) = \left( \hat{p}_k^{(1)} + (-1)^k \rho_k^\infty a_k \hat{v}_k^{(0)} \right) \left( \overline{\hat{Z}_{k'}^{(1)}}, \tau \right), \tag{95}$$

$$\hat{v}_k^{(0)} \left( \hat{Z}_k^{(0)}, \tau \right) = \hat{v}_k^{(0)} \left( \overline{\hat{Z}_{k'}^{(1)}}, \tau \right) = \hat{V}_k^{(0)}. \tag{96}$$

These data will be shown later by matching in Sec. 4.2.4. We see from (19) and (51) that the conditions (95), (96) are equivalent to the boundary data

$$\left( p_k^{(1,f)} + (-1)^k \rho_k^\infty a_k v_k^{(0,f)} \right) \left( \hat{Z}_{k'}^{(0)}, \tau \right) = \left( p_k^{(1,f)} + (-1)^k \rho_k^\infty a_k v_k^{(0,f)} \right) \left( \overline{\hat{Z}_{k'}^{(1)}}, \tau \right) = 0, \tag{97}$$

$$v_k^{(0,f)} \left( \hat{Z}_k^{(0)}, \tau \right) = v_k^{(0,f)} \left( \overline{\hat{Z}_k^{(1)}}, \tau \right) = 0. \tag{98}$$

Let us first solve the IBVP (91), (92), (94), (97), (98) for $p_k^{(1,f)}$ and $v_k^{(0,f)}$ in $\hat{\mathfrak{M}}$ by using the method of characteristics and by applying a Picard iteration. We can write (91), (92) for unknown column vector $u^{(1,f)} = (p_1^{(1,f)}, v_1^{(0,f)}, p_2^{(1,f)}, v_2^{(0,f)})^t$ in the form

$$\frac{\partial u^{(1,f)}}{\partial \tau} + A^0 \frac{\partial u^{(1,f)}}{\partial z} + B^0(z) u^{(1,f)} = 0 \tag{99}$$

where $A_k^0$ was defined in (58),

$$A^0 = \begin{pmatrix} A_1^0 & 0 \\ 0 & A_2^0 \end{pmatrix}, \; B^0(z) = \begin{pmatrix} B_1^0 & -B_1^0 \\ -B_2^0 & B_2^0 \end{pmatrix}, \; B_k^0(z) = \begin{pmatrix} 0 & \rho_k^\infty a_k^2 f_k^v(z) \\ -f_k^p(z)/\rho_k^\infty & 0 \end{pmatrix}. \tag{100}$$

The constant coefficient matrix $A^0$ has four distinct eigenvalues $\pm a_1, \pm a_2$. The nonsingular matrix $\Gamma^0$ whose column vectors consist of linearly independent eigenvector of $A^0$ has the form

$$\Gamma^0 = \begin{pmatrix} \Gamma_1^0 & 0 \\ 0 & \Gamma_2^0 \end{pmatrix}, \tag{101}$$

where $\Gamma_k^0$ was given in (60) and it satisfies

$$A^0 \Gamma^0 = \Gamma^0 \Lambda^0, \quad \Lambda^0 = \begin{pmatrix} \Lambda_1^0 & 0 \\ 0 & \Lambda_2^0 \end{pmatrix} \tag{102}$$

with the diagonal matrix $\Lambda_k^0$ defined in (63), whose diagonal entries are eigenvalues of $A^0$. The characteristics curves $C_{k,i}$, $i = 1, 2$, satisfy (61) and therefore, the backward characteristics $C_{k,i}$, $i = 1, 2$ through a point $(z, \tau)$ have the equations (62). Introducing a new unknown vector $U^{(1,f)} \equiv (\Gamma^0)^{-1} u^{(1,f)}$, we find that $U^{(1,f)}$ satisfies semilinear hyperbolic system

$$\frac{\partial U^{(1,f)}}{\partial \tau} + \Lambda^0 \frac{\partial U^{(1,f)}}{\partial z} + \overline{B}^{(0)}(z) U^{(1,f)} = 0 \tag{103}$$

with the initial condition

$$U^{(1,f)}(z, 0) = 0. \tag{104}$$

and the boundary data

$$U_1^{(1,f)}(\hat{Z}_2^{(0)}, \tau) = U_{1,1}^{(1,f)}(\overline{\hat{Z}_2^{(1)}}, \tau) = 0, \tag{105}$$

$$U_2^{(1,f)}(\hat{Z}_1^{(0)}, \tau) = \left( U_1^{(1,f)} + 2\rho_1^\infty a_1 v_1^{(0,f)} \right)(\hat{Z}_1^{(0)}, \tau) = U_1^{(1,f)}(\hat{Z}_1^{(0)}, \tau), \tag{106}$$

$$U_3^{(1,f)}(\hat{Z}_2^{(0)}, \tau) = \left( U_3^{(1,f)} - 2\rho_2^\infty a_2 v_2^{(0,f)} \right)(\hat{Z}_2^{(0)}, \tau) = U_4^{(1,f)}(\hat{Z}_2^{(0)}, \tau), \tag{107}$$

$$U_4^{(1,f)}(\hat{Z}_1^{(0)}, \tau) = U_{2,2}^{(1,f)}(\overline{\hat{Z}_1^{(1)}}, \tau) = 0 \tag{108}$$

in $\hat{\mathcal{M}}$. Here $\overline{B}^{(0)} = (\Gamma^0)^{-1}B^0\Gamma^0$, $U_j^{(1,f)}$ denotes the $j-$th component of the vector $U^{(1,f)}$ and $U_{k,k}^{(1,f)}$ was given in (65)

By the method of characteristics, the IBVP for $U^{(1,f)}$ is solved implicitly. In this case a Picard iteration converges to a solution in closed form.

**Proposition 4.7.** Assume (8), (15), (19), (23), and the initial data (11). Assume the boundary data (95) and (96). Then the fast variables are

$$v_k^{(0,f)} = p_k^{(1,f)} = 0, \tag{109}$$

in $\hat{\mathcal{M}}$. Thus the inner terms satisfy

$$\hat{v}_k^{(0)} = v_k^{\infty}(z,0), \quad \hat{p}_k^{(1)} = 0. \tag{110}$$

*Proof.* We solve the IBVP (103)-(108) for $U^{(1,f)}$ in $\hat{\mathcal{M}}$. By the method of characteristics, the solution is determined implicitly,

$$U_j^{(1,f)} = -\int_0^{\tau}\sum_{k=1}^{4}\overline{B}^0_{j,k}\left(\alpha_{i,l}(s,z,\tau)\right)U_k^{(1,f)}\left(\alpha_{i,l}(s,z,\tau),s\right)ds,\ j=1,2,3,4. \tag{111}$$

Here $\overline{B}^0_{j,k}$ denotes the $(j,k)$ component of the matrix $B^0$, $\alpha_{i,l}(s,z,\tau)$ was defined in (62) and $(i,l) = (1,1), (1,2), (2,1), (2,2)$ corresponding to $j = 1,2,3,4$. Notice that there is no contribution of the initial data (eq:sys-initial). The solution (111) can be written symbolically as

$$U^{(1,f)} = SU^{(1,f)} \tag{112}$$

Here $S$ is the integral operator taking a vector $U^{(1,f)}$ into a vector $SU^{(1,f)}$ and the $j-$th component of $SU^{(1,f)}$. Using the process of a Picard iteration

$$\begin{aligned} U^{(1,f)[m+1]} &= SU^{(1,f)[m]}, m \geq 0, && (113)\\ U^{(1,f)[0]} &= U^{(1,f)}(z,0) = 0, && (114) \end{aligned}$$

we obtain the explicit solution

$$U^{(1,f)} = \sum_{m=0}^{\infty} U^{(1,f)[m]} = 0. \tag{115}$$

Therefore the fast variables satisfy $u^{(1,f)} = \Gamma^0 U^{(1,f)} = 0$. ■

Using (109), Eq. (90) is solved for $\beta_k^{(1,f)}$ under the initial data (94). Thus the inner term $\hat{\beta}_k^{(1)}$ is determined in closed form. We note that $\hat{v}^{*(0)} = v^{*\infty}(z,0)$ from (49), (110).

**Proposition 4.8.** Assume (8), (15), (19), (23), and the initial data (11). Assume the boundary data (95) and (96). We obtain the fast variable

$$\beta_k^{(1,f)} = 0, \tag{116}$$

and therefore, the inner term satisfies

$$\hat{\beta}_k^{(1)} = \beta_k^{\infty}(z,0) + \tau \frac{\partial \beta_k^{\infty}}{\partial t}(z,0) \quad \text{in } \hat{\mathcal{M}}. \tag{117}$$

#### 4.2.4 Matching Process

We match the inner asymptotic in the exterior domain with the outer edge of the fast transition layer and we match the inner edge of the fast transition layer with the edge of the incompressible mixing zone $\hat{\mathcal{M}}$. The matching conditions determine the unknown boundary data of the inner terms in the exterior and the incompressible mixing zone and the fast transition-layer variables.

The matching conditions to $O(\lambda^{-1})$ for $\beta_k$ and $p_k$ and to $O(1)$ for $v_k$ are

$$\beta_k^{(1,ft)}\left(\hat{\zeta}_i^{(1)} = 0, \tau\right) = \hat{\beta}_k^{(1)}\left(\overline{\hat{Z}_i^{(1)}}, \tau\right) = 0, \tag{118}$$

$$v_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)} = 0, \tau\right) = \hat{v}_k^{(0)}\left(\overline{\hat{Z}_i^{(1)}}, \tau\right) = \delta_{ik}\hat{V}_k^{(0)}, \tag{119}$$

$$p_k^{(1,ft)}\left(\hat{\zeta}_i^{(1)} = 0, \tau\right) = \hat{p}_k^{(1)}\left(\overline{\hat{Z}_i^{(1)}}, \tau\right) = 0, \tag{120}$$

$$\beta_k^{(1,ft)}\left(\hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)}, \tau\right) = \hat{\beta}_k^{(1)}\left(\hat{Z}_i^{(0)}, \tau\right), \tag{121}$$

$$v_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)}, \tau\right) = \hat{v}_k^{(0)}\left(\hat{Z}_i^{(0)}, \tau\right), \tag{122}$$

$$p_k^{(1,ft)}\left(\hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)}, \tau\right) = \hat{p}_k^{(1)}\left(\hat{Z}_i^{(0)}, \tau\right). \tag{123}$$

The conditions (118)–(120) match the inner limit expansions in the exterior domain and fast transition-layer expansions while (121)–(123) match the inner limit expansions in the mixing zone $\hat{\mathcal{M}}$ and the fast transition-layer expansions. These matching conditions give boundary data for the inner terms $\hat{\beta}_k^{(1)}$, $\hat{v}_k^{(0)}$, $\hat{p}_k^{(1)}$ and fast transition-layer variables $\beta_k^{(1,ft)}$, $v_k^{(0,ft)}$, $p_k^{(1,ft)}$. Using (78), (119) and (122), we obtain the boundary condition

$$\begin{aligned}\hat{v}_k^{(0)}\left(\hat{Z}_i^{(0)}, \tau\right) &= v_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)}, \tau\right)\\ &= v_k^{(0,ft)}\left(\hat{\zeta}_i^{(1)} = 0, \tau\right) = \hat{v}_k^{(0)}\left(\overline{\hat{Z}_i^{(1)}}, \tau\right) = \delta_{ik}\hat{V}_k^{(0)}.\end{aligned} \tag{124}$$

Also,

$$\hat{p}_k^{(1)}\left(\hat{Z}_i^{(0)}, \tau\right) = p_k^{(1,ft)}\left(\hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)}, \tau\right) = p_k^{(1,ft)}\left(\hat{\zeta}_i^{(1)} = 0, \tau\right) = \hat{p}_k^{(1)}\left(\overline{\hat{Z}_i^{(1)}}, \tau\right) \tag{125}$$

by use of (86), (120) and (123). Therefore the boundary data (95) and (96) are proved. So we can use Proposition 4.7 to evaluate $\hat{v}_k^{(0)}$, $\hat{p}_k^{(1)}$ in $\hat{\mathfrak{M}}$. Then the boundary data satisfy

$$\hat{v}_k^{(0)}\left(\overline{\hat{Z}_{k'}^{(1)}}, \tau\right) = \hat{v}_k^{(0)}\left(\hat{Z}_{k'}^{(0)}, \tau\right) = v_k^{\infty}\left(Z_{k'}^{\infty}(0), 0\right) = 0, \tag{126}$$

$$\hat{p}_k^{(1)}\left(\overline{\hat{Z}_i^{(1)}}, \tau\right) = \hat{p}_k^{(1)}\left(\hat{Z}_i^{(0)}, \tau\right) = 0. \tag{127}$$

These boundary data determine the fast transitional variables $v_k^{(0,ft)}$ and $p_k^{(1,ft)}$ completely in $\hat{\mathfrak{T}}_i^{(1)}$. Using (121) and (116), we solve Eq. (83) for $\beta_k^{(1,ft)}(\hat{\zeta}_i^{(1)}, \tau)$.

**Proposition 4.9.** Assume (8), (15), (19), (23), and the initial data (11). Assume (118)–(123). Then the fast transitional variables are

$$v_k^{(0,ft)}(\hat{\zeta}_i^{(1)}, \tau) = \delta_{ik}\hat{V}_k^{(0)}, \tag{128}$$

$$p_k^{(1,ft)}(\hat{\zeta}_i^{(1)}, \tau) = 0, \tag{129}$$

$$\beta_k^{(1,ft)}(\hat{\zeta}_i^{(1)}, \tau) = \hat{\zeta}_i^{(1)}\frac{\partial \beta_k^{\infty}}{\partial z}(Z_i^{\infty}(0) + (-1)^{i+1}0, 0). \tag{130}$$

*Proof.* Since $v^{*(0,ft)} = \hat{V}_i^{(0)}$, Eq. (83) reduces to

$$\frac{\partial \beta_k^{(1,ft)}}{\partial \tau} = 0 \tag{131}$$

in $(-1)^{i+1}\hat{Z}_i^{(1)} \leq (-1)^i\hat{\zeta}_i^{(1)} \leq 0$. Using (121) and (116), evaluation of the boundary data yields

$$\begin{aligned}\beta_k^{(1,ft)}\left(\hat{\zeta}_i^{(1)} = -\hat{Z}_i^{(1)}, \tau\right) = \hat{\beta}_k^{(1)}\left(\hat{Z}_i^{(0)}, \tau\right) &= -\tau\hat{V}_i^{(0)}\frac{\partial \beta_k^{\infty}}{\partial z}(Z_i^{\infty}(0) + (-1)^{i+1}0, 0)\\ &= -\hat{Z}_i^{(1)}\frac{\partial \beta_k^{\infty}}{\partial z}(Z_i^{\infty}(0) + (-1)^{i+1}0, 0).\end{aligned} \tag{132}$$

Therefore the solution $\beta_k^{(1,ft)}$ satisfies (130). ∎

Proposition 4.3 is proved by summarizing Propositions 4.7-4.9 that determine the inner terms $\hat{\beta}_k^{(1)}$, $\hat{v}_k^{(0)}$, $\hat{p}_k^{(1)}$ uniformly in space. The proof of Theorem 4.1 is complete from Propositions 4.2 and 4.3. Thus the inner limit expansions of $\beta_k$ and $p_k$ up to $O(\lambda^{-1})$ and of $v_k$ up to $O(1)$ are nonoscillatory and depend on the initial data only.

## References

[1] W. Bo, H. Jin, D. Kim, X. Liu, H. Lee, N. Pestieau, Y. Yan, J. Glimm, and J. Grove. Comparison and validation of multi phase closure models *Computers & Mathematics with Applications*, 56:1291–1302, 2008.

[2] Y. Chen, J. Glimm, D. H. Sharp, and Q. Zhang. A two-phase flow model of the Rayleigh-Taylor mixing zone. *Phys. Fluids*, 8(3):816–825, 1996.

[3] D. Ebin. *Motion of slightly compressible fluids in a bounded domain I. Comm. on Pure Appl. Math.*, 35:451–485, 1982.

[4] J. Glimm and H. Jin. An asymptotic analysis of two-phase fluid mixing. *Bol. Soc. Bras. Mat.*, 32:213–236, 2001.

[5] J. Glimm, H. Jin, M. Laforest, F. Tangerman, and Y. Zhang. A two pressure numerical model of two fluid mixing. *Multiscale Model. Simul.*, 1:458–484, 2003.

[6] J. Glimm, D. Saltz, and D. H. Sharp. Two-pressure two-phase flow. In G.-Q. Chen, Y. Li, and X. Zhu, editors, *Nonlinear Partial Differential Equations*, pages 124–148. World Scientific, Singapore, 1998.

[7] J. Glimm, D. Saltz, and D. H. Sharp. Two-phase modeling of a fluid mixing layer. *J. Fluid Mech.*, 378:119–143, 1999.

[8] D. Hoff. *The zero-mach limit of compressible flows*. *Commu. Math. Phys.*, 192:543–554, 1998.

[9] H. Jin. *The singular limit of weakly compressible multiphase flow*. *Global J. Pure Appl. Math.*, In Press, 2016.

[10] H. Jin. *The singular perturbations of compressible multiphase flow equations*. In preparation, 2016.

[11] H. Jin, and J. Glimm *Weakly compressible two-pressure two-phase flow*. *Acta. Math. Sci*, 29B(6):1497–1540, 2009.

[12] H. Jin, J. Glimm, and D. H. Sharp. Compressible two-pressure two-phase flow models. *Phys. Lett. A*, 353:469–474, 2006.

[13] H. Jin, J. Glimm, and D. H. Sharp. Entropy of averaging for compressible two-pressure two-phase flow models. *Phys. Lett. A*, 360:114–121, 2006.

[14] J. Kevorkian and J.D. Cole. Perturbation methods in applied mathematics. Springer-Verlag, New York, 1985.

[15] S. Klainerman and A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl.*, 34:481–524, 1981.

[16] S. Klainerman and A. Majda. Compresible and incompressible fluids. *Comm. Pure Appl.*, 35:629–651, 1982.

[17] H. Lee, H. Jin, Y. Yu and J. Glimm. On the validation of turbulent mixing simulations of Rayleigh-Taylor mixing. *Phys. Fluids*, 20:012102, 2008.