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Some results on near-rings with soft properties

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Abstract

The notions of DFS-sets and DFS-near-rings is introduced, characterizations of DFS-near-rings are provided and establish a new DFS-near-ring from old one. Finally, some conditions for a DFS-set to be a DFS-near-ring are considered.

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1. Introduction

In [4], Molodtsov introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly.

In this paper, using the notion of DFS-sets which is introduced in [2], we introduce the notions of double-framed soft near-rings, and investigate their properties. We provide characterizations of DFS-near-rings and establish a new DFS-near-ring from old one. We consider conditions for a DFS-set to be a DFS-near-ring.

2. Results on double-framed soft near-rings

In 1999, Molodtsov [4] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe.* Let $\mathscr{P}(U)$ denotes the power set of U and $A \subseteq E$.

Definition 2.1. [1] A pair $(\tilde{\alpha}, A)$ is called a *soft set* over U, where $\tilde{\alpha}$ is a mapping given by

$$\tilde{\alpha} : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\tilde{\alpha}(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set $(\tilde{\alpha}, A)$. For illustration, Molodtsov considered several examples in [4].

From now on, we will take N, which is a near-ring (see [5]) unless otherwise specified. In the following, consider that $E = N$, as a set of parameters.

Definition 2.2. [3] A double-framed pair $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is called a *double-framed soft set* (briefly, *DFS-set*) over U where $\tilde{\alpha}$ and $\tilde{f}$ are mappings from A to $\mathscr{P}(U)$ and $A \subseteq N$.

For a DFS-set $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ over U and two subsets γ and δ of U, the γ-*inclusive set* and the δ-*exclusive set* of $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$, denoted by $i_A(\tilde{\alpha}; \gamma)$ and $e_A(\tilde{f}; \delta)$, respectively, are defined as follows:

$$i_A(\tilde{\alpha}; \gamma) := \{x \in A \mid \gamma \subseteq \tilde{\alpha}(x)\}$$

and

$$e_A(\tilde{f}; \delta) := \left\{x \in A \mid \delta \supseteq \tilde{f}(x)\right\},$$

respectively. The set

$$DF_A\left(\tilde{\alpha}, \tilde{f}\right)_{(\gamma,\delta)} := \left\{x \in A \mid \gamma \subseteq \tilde{\alpha}(x),\ \delta \supseteq \tilde{f}(x)\right\}$$

is called a *double-framed including set* of $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$. It is clear that

$$DF_A\left(\tilde{\alpha}, \tilde{f}\right)_{(\gamma,\delta)} = i_A(\tilde{\alpha}; \gamma) \cap e_A(\tilde{f}; \delta).$$

Definition 2.3. A DFS-set $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ over U is called a *double-framed soft near-ring* (briefly, *DFS-near-ring*) over U if it satisfies:

(1) $\tilde{\alpha}(x + y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(x + y) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$,

(2) $\tilde{\alpha}(-x) = \tilde{\alpha}(x)$ and $\tilde{f}(-x) = \tilde{f}(x)$,

(3) $\tilde{\alpha}(xy) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(xy) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$,

for all $x, y \in N$.

Example 2.4. Let $N = \{0, 1, 2, 3\}$ be a set with Cayley tables which are given in Tables 1 and 2.
Then $(N, +, \cdot)$ is a near-ring. Let U be a set of 2×2 matrices of the form $\begin{bmatrix} x & x \\ y & y \end{bmatrix}$ with entries in N, that is,

$$U := \left\{\begin{bmatrix} x & x \\ y & y \end{bmatrix} \mid x, y \in N\right\}.$$

Table 1: Cayley table for the "+"-operation

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Table 2: Cayley table for the "$\cdot$"-operation

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	1
2	0	2	0	2
3	0	3	2	3

Let $\left\langle (\tilde{\alpha}, \tilde{f}); N \right\rangle$ be a double-framed soft set over U given by

$$\tilde{\alpha} : N \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{a, b, c, d\} & \text{if } x = 0, \\ \{a, c\} & \text{if } x \in \{1, 3\}, \\ \{a, b, c\} & \text{if } x = 2, \end{cases}$$

and

$$\tilde{f} : N \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{a, c\} & \text{if } x \in \{0, 2\}, \\ \{a, c, d\} & \text{otherwise}, \end{cases}$$

where $a = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $c = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$, and $d = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix}$. Then it is easy to verify that $\left\langle (\tilde{\alpha}, \tilde{f}); N \right\rangle$ is a DFS-near-ring over U.

Proposition 2.5. [3] Let $\left\langle (\tilde{\alpha}, \tilde{f}); N \right\rangle$ be a DFS-near-ring over U. Then

(1) $\tilde{\alpha}(0) \supseteq \tilde{\alpha}(x)$ and $\tilde{f}(0) \subseteq \tilde{f}(x)$ for all $x \in N$.

(2) If $x, y \in N$ satisfies $\tilde{\alpha}(x - y) = \tilde{\alpha}(0)$ and $\tilde{f}(x - y) = \tilde{f}(0)$, then $\tilde{\alpha}(x) = \tilde{\alpha}(y)$ and $\tilde{f}(x) = \tilde{f}(y)$, respectively.

(3) For a fixed element $x \in N$, we have

$$\tilde{\alpha}(x) = \tilde{\alpha}(0) \;\Leftrightarrow\; \tilde{\alpha}(x + y) = \tilde{\alpha}(y + x) = \tilde{\alpha}(y),$$

and

$$\tilde{f}(x) = \tilde{f}(0) \Leftrightarrow \tilde{f}(x+y) = \tilde{f}(y+x) = \tilde{f}(y).$$

for all $y \in N$.

We consider a characterization of a DFS-near-ring.

Theorem 2.6. [3] A double-framed soft set $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ over U is a DFS-near-ring over U if and only if the following assertions are valid.

(1) $\tilde{\alpha}(x-y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(x-y) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$,

(2) $\tilde{\alpha}(xy) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(xy) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$,

for all $x, y \in N$.

We consider conditions for a double-framed soft set to be a DFS-near-ring.

Theorem 2.7. Let $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ be a double-framed soft set over U. If N is a near-field and the following assertion is valid

$$(\forall x \in N)\left(x \neq 0 \Rightarrow \tilde{\alpha}(0) \supseteq \tilde{\alpha}(1) = \tilde{\alpha}(x),\ \tilde{f}(0) \subseteq \tilde{f}(1) = \tilde{f}(x)\right), \tag{1}$$

then $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a DFS-near-ring over U.

Proof. Assume that N is a near-field and the condition (1) is valid. Then

$$(\forall x \in N)\left(\tilde{\alpha}(0) \supseteq \tilde{\alpha}(x),\ \tilde{f}(0) \subseteq \tilde{f}(x)\right). \tag{2}$$

Let $x, y \in N$. If $x - y = 0$, then $\tilde{\alpha}(x-y) = \tilde{\alpha}(0) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(x-y) = \tilde{f}(0) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$. If $x - y \neq 0$, then we have three cases:

Case 1. $x \neq 0$ and $y = 0$, Case 2. $x = 0$ and $y \neq 0$, Case 3. $x \neq 0 \neq y$ and $x \neq y$. Case 1 implies that $\tilde{\alpha}(0) \supseteq \tilde{\alpha}(1) = \tilde{\alpha}(x)$ and $\tilde{f}(0) \subseteq \tilde{f}(1) = \tilde{f}(x)$ by (1). Hence $\tilde{\alpha}(x-y) = \tilde{\alpha}(1) = \tilde{\alpha}(x) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(x-y) = \tilde{f}(1) = \tilde{f}(x) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$. For the Case 2, the proof is similar to the Case 1. Case 3 implies that $\tilde{\alpha}(x-y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(x-y) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$.

If $x \neq 0 \neq y$, then $xy \neq 0$, and so $\tilde{\alpha}(xy) = \tilde{\alpha}(1) = \tilde{\alpha}(x)$, $\tilde{f}(xy) = \tilde{f}(1) = \tilde{f}(x)$, $\tilde{\alpha}(xy) = \tilde{\alpha}(1) = \tilde{\alpha}(y)$, and $\tilde{f}(xy) = \tilde{f}(1) = \tilde{f}(y)$. Hence $\tilde{\alpha}(xy) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(xy) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$. If any one of x and y is 0, say $x = 0$ and $y \neq 0$, then $xy = 0$ since N is a near-field. It follows from the condition (1) that $\tilde{\alpha}(xy) = \tilde{\alpha}(0) \supseteq \tilde{\alpha}(1) = \tilde{\alpha}(x)$, $\tilde{f}(xy) = \tilde{f}(0) \subseteq \tilde{f}(1) = \tilde{f}(x)$. Since $\tilde{\alpha}(xy) = \tilde{\alpha}(0) \supseteq \tilde{\alpha}(y)$ and $\tilde{f}(xy) = \tilde{f}(0) \subseteq \tilde{f}(y)$, we have $\tilde{\alpha}(xy) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(xy) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$. If $x = 0 = y$, then clearly $\tilde{\alpha}(xy) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y)$ and $\tilde{f}(xy) \subseteq \tilde{f}(x) \cup \tilde{f}(y)$. Therefore $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a DFS-near-ring over U by Theorem 2.6. ■

Theorem 2.8. For a double-framed soft set $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ over U, the following are equivalent:

(1) $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a DFS-near-ring over U.

(2) The nonempty γ-inclusive set and δ-exclusive set of $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ are subnear-rings of N for any subsets γ and δ of U.

Proof. Suppose that $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a DFS-near-ring over U. Let γ and δ be subsets of U such that $i_N(\tilde{\alpha}; \gamma) \neq \emptyset \neq e_N(\tilde{f}; \delta)$. Let $x, y \in i_N(\tilde{\alpha}; \gamma)$ and $x, y \in e_N(\tilde{f}; \delta)$. Then $\tilde{\alpha}(x) \supseteq \gamma$, $\tilde{\alpha}(y) \supseteq \gamma$, $\tilde{f}(x) \subseteq \delta$ and $\tilde{f}(y) \subseteq \delta$. It follows from Theorem 2.6 that $\tilde{\alpha}(x-y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y) \supseteq \gamma$, $\tilde{f}(x-y) \subseteq \tilde{f}(x) \cup \tilde{f}(y) \subseteq \delta$, $\tilde{\alpha}(xy) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y) \supseteq \gamma$, $\tilde{f}(xy) \subseteq \tilde{f}(x) \cup \tilde{f}(y) \subseteq \delta$. Hence $x - y \in i_N(\tilde{\alpha}; \gamma)$, $xy \in i_N(\tilde{\alpha}; \gamma)$, $x - y \in e_N(\tilde{f}; \delta)$ and $xy \in e_N(\tilde{f}; \delta)$. Therefore $i_N(\tilde{\alpha}; \gamma)$ and $e_N(\tilde{f}; \delta)$ are subnear-rings of N.

Conversely, suppose that the nonempty γ-inclusive set and δ-exclusive set of $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ are subnear-rings of N for any subsets γ and δ of U. Let $x, y \in N$ be such that $\tilde{\alpha}(x) = \gamma_x$, $\tilde{\alpha}(y) = \gamma_y$, $\tilde{f}(x) = \delta_x$ and $\tilde{f}(y) = \delta_y$. Taking $\gamma = \gamma_x \cap \gamma_y$ and $\delta = \delta_x \cup \delta_y$ imply that $x, y \in i_N(\tilde{\alpha}; \gamma)$ and $x, y \in e_N(\tilde{f}; \delta)$. Hence $x - y \in i_N(\tilde{\alpha}; \gamma)$, $xy \in i_N(\tilde{\alpha}; \gamma)$, $x - y \in e_N(\tilde{f}; \delta)$, and $xy \in e_N(\tilde{f}; \delta)$, which imply that

$$\begin{aligned}
&\tilde{\alpha}(x-y) \supseteq \gamma = \gamma_x \cap \gamma_y = \tilde{\alpha}(x) \cap \tilde{\alpha}(y), \\
&\tilde{f}(x-y) \subseteq \delta = \delta_x \cup \delta_y = \tilde{f}(x) \cup \tilde{f}(y), \\
&\tilde{\alpha}(xy) \supseteq \gamma = \gamma_x \cap \gamma_y = \tilde{\alpha}(x) \cap \tilde{\alpha}(y), \\
&\tilde{f}(xy) \subseteq \delta = \delta_x \cup \delta_y = \tilde{f}(x) \cup \tilde{f}(y).
\end{aligned}$$

It follows from Theorem 2.6 that $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a DFS-near-ring of N. ■

Corollary 2.9. If $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a DFS-near-ring over U, then the double-framed including set of $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a subnear-ring N.

Theorem 2.10. If $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a DFS-near-ring over U, then the set

$$N_0 := \{x \in N \mid \tilde{\alpha}(x) = \tilde{\alpha}(0),\ \tilde{f}(x) = \tilde{f}(0)\}$$

is a subnear-ring of N.

Proof. Obviously, $0 \in N_0$. Let $x, y \in N_0$. Then $\tilde{\alpha}(x) = \tilde{\alpha}(0) = \tilde{\alpha}(y)$ and $\tilde{f}(x) = \tilde{f}(0) = \tilde{f}(y)$. It follows from Theorem 2.6 that

$$\tilde{\alpha}(x-y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y) = \tilde{\alpha}(0), \tilde{\alpha}(xy) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y) = \tilde{\alpha}(0),$$

$$\tilde{f}(x-y) \subseteq \tilde{f}(x) \cup \tilde{f}(y) = \tilde{f}(0), \tilde{f}(xy) \subseteq \tilde{f}(x) \cup \tilde{f}(y) = \tilde{f}(0).$$

On the other hand, Proposition 2.5 (1) implies that

$$\tilde{\alpha}(x-y) \subseteq \tilde{\alpha}(0), \tilde{\alpha}(xy) \subseteq \tilde{\alpha}(0),$$

$$\tilde{f}(x-y) \supseteq \tilde{f}(0), \tilde{f}(xy) \supseteq \tilde{f}(0).$$

Therefore $\tilde{\alpha}(x-y) = \tilde{\alpha}(0) = \tilde{\alpha}(xy)$ and $\tilde{f}(x-y) = \tilde{f}(0) = \tilde{f}(xy)$, that is, $x-y \in N_0$ and $xy \in N_0$. Hence N_0 is a subnear-ring of N. ■

Now, we construct a new DFS-near-ring from old one.

Theorem 2.11. For any double-framed soft set $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ over U, let $\left\langle(\tilde{\alpha}^*, \tilde{f}^*); N\right\rangle$ be a double-framed soft set over U defined by

$$\tilde{\alpha}^* : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{\alpha}(x) & \text{if } x \in i_N(\tilde{\alpha}; \gamma), \\ \eta & \text{otherwise,} \end{cases}$$

$$\tilde{f}^* : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{f}(x) & \text{if } x \in e_N(\tilde{f}; \delta), \\ \rho & \text{otherwise,} \end{cases}$$

where γ, δ, η and ρ are subsets of U with

$$\eta \subsetneq \bigcap_{x \notin i_N(\tilde{\alpha}; \gamma)} \tilde{\alpha}(x)$$

and

$$\rho \supsetneq \bigcup_{x \notin e_N(\tilde{f}; \delta)} \tilde{f}(x).$$

If $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a DFS-near-ring over U, then so is $\left\langle(\tilde{\alpha}^*, \tilde{f}^*); N\right\rangle$.

Proof. Assume that $\left\langle(\tilde{\alpha}, \tilde{f}); N\right\rangle$ is a DFS-near-ring over U. Then $i_N(\tilde{\alpha}; \gamma)$ and $e_N(\tilde{f}; \delta)$ are subnear-rings of N for every subsets γ and δ of U with $i_N(\tilde{\alpha}; \gamma) \neq \emptyset$ and $e_N(\tilde{f}; \delta) \neq \emptyset$. Let $x, y \in N$. If $x, y \in i_N(\tilde{\alpha}; \gamma)$, then $x-y \in i_N(\tilde{\alpha}; \gamma)$ and $xy \in i_N(\tilde{\alpha}; \gamma)$. Thus

$$\tilde{\alpha}^*(x-y) = \tilde{\alpha}(x-y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y) = \tilde{\alpha}^*(x) \cap \tilde{\alpha}^*(y),$$

$$\tilde{\alpha}^*(xy) = \tilde{\alpha}(xy) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y) = \tilde{\alpha}^*(x) \cap \tilde{\alpha}^*(y).$$

If $x \notin i_N(\tilde{\alpha}; \gamma)$ or $y \notin i_N(\tilde{\alpha}; \gamma)$, then $\tilde{\alpha}^*(x) = \eta$ or $\tilde{\alpha}^*(y) = \eta$. Hence

$$\tilde{\alpha}^*(x-y) \supseteq \eta = \tilde{\alpha}^*(x) \cap \tilde{\alpha}^*(y), \tilde{\alpha}^*(xy) \supseteq \eta = \tilde{\alpha}^*(x) \cap \tilde{\alpha}^*(y).$$

Now, if $x, y \in e_N(\tilde{f}; \delta)$, then $x-y \in e_N(\tilde{f}; \delta)$. Thus

$$\tilde{f}^*(x-y) = \tilde{f}(x-y) \subseteq \tilde{f}(x) \cup \tilde{f}(y) = \tilde{f}^*(x) \cup \tilde{f}^*(y),$$

$$\tilde{f}^*(xy) = \tilde{f}(xy) \subseteq \tilde{f}(x) \cup \tilde{f}(y) = \tilde{f}^*(x) \cup \tilde{f}^*(y).$$

If $x \notin e_N(\tilde{f}; \delta)$ or $y \notin e_N(\tilde{f}; \delta)$, then $\tilde{f}^*(x) = \rho$ or $\tilde{f}^*(y) = \rho$. Hence

$$\tilde{f}^*(x-y) \subseteq \rho = \tilde{f}^*(x) \cup \tilde{f}^*(y),\ \tilde{f}^*(xy) \subseteq \rho = \tilde{f}^*(x) \cup \tilde{f}^*(y).$$

Therefore $\left\langle (\tilde{\alpha}^*, \tilde{f}^*); N \right\rangle$ is a DFS-near-ring over U from Theorem 2.6. ■

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