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The singular limit of weakly compressible multiphase flow[1]

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Abstract

We discuss the formally uniformly valid expansions describing the motion of the weakly compressible multi-phase flow, which lead to the outer and inner expansion under the corresponding limit process. We examine the reduced expansions and conditions required in the derivation of outer and inner expansions for compressible solutions.

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1. Introduction

We discuss the motion of the weakly compressible multi-phase flow model proposed in [2, 6, 7, 11, 12]. The limiting behavior of a compressible multi-phase flow model is analyzed as the Mach number goes to zero. The equations of compressible isentropic ideal fluid flow in appropriate nondimensional form are a nonlinear hyperbolic system depending on a large nondimensional parameter λ, the reciprocal of the Mach number. The zero Mach limit of the compressible multiphase flow equations is a time-singular and layer-type problem which requires advanced techniques in asymptotics. From a number of points of view [3, 8, 14, 13], the zero Mach limit of the single phase compressible Euler equations has been studied in higher space dimensions. In contrast, two-phase flow presents additional difficulties, so that even one dimensional two-phase flow is

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nontrivial. We are here concerned with the derivation of uniformly valid asymptotic expansions describing the singular limiting process which consists of the outer limit describing the fluid motions away from the initial time, and the inner limit describing the fluid near the initial time, respectively. The slow variables in the outer limit asymptotic expansions have a slow scale of motion and solve linearized incompressible problems. They have been determined in closed form [10] and are summarized in Sec. 4. The information supplied from the weakly compressible theory resolves underdetermination of incompressible pressures. The incompressible pressures are uniquely specified by certain details of the compressible fluids from which they are derived as a limit. This aspect of two phase flow in the incompressible limit appears to be unique, and results basically from closures which satisfy single phase boundary conditions at the edges of the mixing zone. The fast variables in the inner limit asymptotic expansions contain fast scale acoustical oscillations. In Sec. 3, we examine formal asymptotic expansions and conditions required to describe the singular limit process of compressible isentropic equations. The two phase flow model depends on the motions of the mixing zone edges and closure for the interfacial averages with the constitutive law. Since the mixing zone edges are not well characterized for compressible flows, the velocities or trajectories of the edges of the mixing zone must be provided as data. The compressible constitutive factor is asymptotically assumed with a specific limit term. Measurement of the trajectory of the mixing zone edge is provided as data with asymptotic conditions in Sec. 3.

2. Flow Equations in Dimensionless Form

The first step in understanding the singular limit of incompressible flow is through the nondimensionalization of the compressible fluid equations. The equations of compressible isentropic ideal fluid flow proposed in [2, 6, 7, 11, 12] are written in nondimensional form depending on a large nondimensional parameter λ. Since the equation of state disappears from the equations in this limit, we assume a simple equation of state for the compressible equations, that of a γ-law isentropic gas,

$$p(\rho, \lambda) \equiv \lambda^2 p(\rho) = \lambda^2 A\rho^\gamma = A_\lambda \rho^\gamma, \quad \gamma > 1, \tag{1}$$

where A is the entropy of the fluid. An isentropic gas does not have temperature, energy, or entropy as a dynamic variable, but has a (constant) entropy $A_\lambda = \lambda^2 A$ which becomes large as λ increases. The above formula defines $p(\rho)$ in terms of $p(\rho, \lambda)$: $p(\rho) = \lambda^{-2} p(\rho, \lambda)$. The pressure $p(\rho)$ is bounded as $\lambda \to \infty$. The sound speed calculated from (1) is

$$c \equiv \left(\frac{dp(\rho^\lambda, \lambda)}{d\rho^\lambda}\right)^{1/2} = \left(\gamma \frac{p(\rho^\lambda, \lambda)}{\rho^\lambda}\right)^{1/2} = \lambda \left(\frac{dp(\rho^\lambda)}{(d\rho^\lambda}\right)^{1/2}. \tag{2}$$

In particular, the Mach number is defined by the ratio of the typical fluid speed $|v_m|$ to the typical sound speed c_m,

$$M = \frac{|v_m|}{c_m} = \frac{|v_m|}{(\gamma p(\rho_m, \lambda)/\rho_m)^{1/2}}. \tag{3}$$

In this physical model, entropy and temperature increase while velocity and density are fixed. Therefore, $c \to \infty$ and $M \to 0$. On the other hands, a λ-dependent change of length scales would keep the global system entropy and the temperature fixed and send all v^{λ}'s to zero when a length scale tends to zero as $\lambda \to \infty$. Also pressure, which is force per unit area, would remain fixed while the extensive quantity, force, tends to zero due to the change in units for measurement of area. Then c is fixed and also $M \to 0$. This is the normal version of the incompressible limit.

The dimensionless compressible equations are a hyperbolic system in the volume fraction β_k^{λ}, velocity v_k^{λ}, density ρ_k^{λ}, and pressure p_k^{λ} of fluid k defined by the equations

$$\frac{\partial \beta_k^{\lambda}}{\partial t} + v^{*\lambda}\frac{\partial \beta_k^{\lambda}}{\partial z} = 0\,, \tag{4}$$

$$\beta_k^{\lambda}\left(\frac{\partial \rho_k^{\lambda}}{\partial t} + v_k^{\lambda}\frac{\partial \rho_k^{\lambda}}{\partial z}\right) + \beta_k^{\lambda}\rho_k^{\lambda}\frac{\partial v_k^{\lambda}}{\partial z} + \rho_k^{\lambda}(v_k^{\lambda} - v^{*\lambda})\frac{\partial \beta_k^{\lambda}}{\partial z} = 0, \tag{5}$$

$$\beta_k^{\lambda}\rho_k^{\lambda}\left(\frac{\partial v_k^{\lambda}}{\partial t} + v_k^{\lambda}\frac{\partial v_k^{\lambda}}{\partial z}\right) + \lambda^2\beta_k^{\lambda}\frac{\partial p_k^{\lambda}}{\partial z} + \lambda^2(p_k^{\lambda} - p^{*\lambda})\frac{\partial \beta_k^{\lambda}}{\partial z} = \beta_k^{\lambda}\rho_k^{\lambda} g(t), \tag{6}$$

where $p_k^{\lambda} = p_k(\rho_k^{\lambda})$ and an equation of state $p_k(\rho_k) = A_k\rho_k^{\gamma_k}$, $\gamma_k > 1$ is given with $\dfrac{\partial p_k}{\partial \rho_k}(\rho_k) > 0$ for $\rho_k > 0$ and the entropy A_k assumed to be constant within each fluid but $A_1 \neq A_2$. The fluids are distinguished by a subscript k, where $k = 1$ and $k = 2$ denote the light and heavy fluids, respectively. For simplicity, we suppress superscript λ's of compressible variables from now on. The interface quantities v^* and p^* have been proposed [4, 11] by closure relations

$$q^* = \mu_1^q q_2 + \mu_2^q q_1, \quad q = v, p, \tag{7}$$

where the mixing coefficients have the fractional linear form

$$\mu_k^q(\beta_k, d_k^q) = \frac{\beta_k}{\beta_k + d_k^q\beta_{k'}}, \tag{8}$$

where $\mu_1^q + \mu_2^q = 1$, $\mu_k^q \geq 0$ and that $\mu_k^q/\beta_k$is continuous on $0 \leq \beta_k \leq 1$ and for all t. Here the primed index k' denotes the fluid complementary to fluid k, i.e., $k' = 3-k$. The μ_k^q thus depends on a single parameter d_k^q. The closure for the constitutive law $d_k^q(t)$ has been proposed [11, 12] and compared in a validation study based on simulation data [1, 15]. For all the data sets of the 3D Rayleigh-Taylor and circular 2D Richtmyer-Meshkov simulations, we have seen excellent validation agreement for the closures proposed. We denote $Z_k = Z_k(t)$ as the position of the mixing zone edge k, defined as the location of vanishing β_k and $V_k = dZ_k/dt$ as the velocity of the edge k. At edge k, the following boundary data holds

$$v_k = V_k(t) \quad \text{at} \;\; z = Z_k(t). \tag{9}$$

Incompressible flow is discussed as a background flow of the incompressible limit, $\lambda \to \infty$, described by the equations

$$\frac{\partial \beta_k^\infty}{\partial t} + v^{*\infty} \frac{\partial \beta_k^\infty}{\partial z} = 0, \tag{10}$$

$$\beta_k^\infty \frac{\partial v_k^\infty}{\partial z} + (v_k^\infty - v^{*\infty}) \frac{\partial \beta_k^\infty}{\partial z} = 0, \tag{11}$$

$$\beta_k^\infty \rho_k^\infty \left(\frac{\partial v_k^\infty}{\partial t} + v_k^\infty \frac{\partial v_k^\infty}{\partial z} \right) + \beta_k^\infty \frac{\partial p_k^\infty}{\partial z} + (p_k^\infty - p^{*\infty}) \frac{\partial \beta_k^\infty}{\partial z} = \beta_k^\infty \rho_k^\infty g(t) \tag{12}$$

for the volume fraction β_k^∞, velocity v_k^∞, and scalar pressure p_k^∞, where ρ_k^∞ is the constant density of phase k. We assume that all state variables are piecewise C^1 functions with discontinuous derivatives at the mixing zone edges $z = Z_k^\infty(t)$ of incompressible flow. Analytic solutions of the incompressible problem (10)–(12) have been obtained in closed form [5, 6, 7].

3. Asymptotic Expansions and Conditions

We specify boundary conditions for compressible and incompressible flow. We assume a container with a slab of heavy fluid of density ρ_2 lying beneath a slab of light fluid of density ρ_1 and separated by an interface. This configuration is then accelerated downwards with an acceleration larger than the earth gravity, reversing the direction of gravity. We assume existence of rigid wall at the top of a finite but large domain $\mathcal{D}$. Then velocity is zero and the pressure is unknown there. At the bottom of this domain, we have conceptually an open container. This fixes the pressure at some ambient value, but not the velocity at the bottom of $\mathcal{D}$. This leads to the boundary conditions

$$v_1(z^{+\infty}) = 0, \quad p_2(z^{-\infty}) = const, \tag{13}$$

where $z = z^{+\infty} (z = z^{-\infty})$ denotes the position of the upper (lower) wall of the domain $\mathcal{D}$.

We discuss the limiting behavior of the solutions $U_k \equiv (\beta_k, v_k, \rho_k, p_k)^{tr}$ of the compressible equations (4)-(6) as $\lambda \to \infty$. We introduce asymptotic expansions

$$\begin{aligned}
\beta_k &= \beta_k^{(0,s)} + \lambda^{-1} \beta_k^{(1,s)} + \lambda^{-2} \left(\beta_k^{(2,s)} + \beta_k^{(2,f)} \right) + O(\lambda^{-3}), \\
v_k &= v_k^{(0,s)} + \lambda^{-1} \left(v_k^{(1,s)} + v_k^{(1,f)} \right) + O(\lambda^{-2}), \\
\rho_k &= \rho_k^{(0,s)} + \lambda^{-1} \rho_k^{(1,s)} + \lambda^{-2} \left(\rho_k^{(2,s)} + \rho_k^{(2,f)} \right) + O(\lambda^{-3}).
\end{aligned} \tag{14}$$

The equation of state gives the expansion

$$
\begin{aligned}
p_k &= p_k^{(0,s)} + \lambda^{-1} p_k^{(1,s)} + \lambda^{-2}\left(p_k^{(2,s)} + p_k^{(2,f)}\right) + O(\lambda^{-3}) \\
&= p_k(\rho_k^{(0,s)}) + \lambda^{-1} c_k^2(\rho_k^{(0,s)})\rho_k^{(1,s)} \\
&\quad + \lambda^{-2}\left(\frac{1}{2}\frac{\partial^2 p_k}{\partial \rho_k^2}(\rho_k^{(0,s)})\rho_k^{(1,s)2} + c_k^2(\rho_k^{(0,s)})\rho_k^{(2,s)} + c_k^2(\rho_k^{(0,s)})\rho_k^{(2,f)}\right) + O(\lambda^{-3}),
\end{aligned}
\tag{15}
$$

where $c_k^2(\rho) = \dfrac{\partial p_k}{\partial \rho_k}(\rho)$. We assume the initial conditions

$$
\begin{aligned}
\beta_k(z,0) &= \beta_k^\infty(z,0) + \lambda^{-1}\beta_k^{(1,s)}(z,0) + \lambda^{-2}\beta_k^{(2,s)}(z,0), \\
v_k(z,0) &= v_k^\infty(z,0) + \lambda^{-1}\left[v_k^{(1,s)}(z,0) + v_k^{(1)}(z)\right], \\
\rho_k(z,0) &= \rho_k^\infty(z,0) + \lambda^{-2}\left[c_k^{-2}(\rho_k^\infty)\rho_k^\infty(z,0) + \rho_k^{(2)}(z)\right], \\
p_k(z,0) &= p_k(\rho_k^\infty) + \lambda^{-2}\left[p_k^\infty(z,0) + p_k^{(2)}(z)\right]
\end{aligned}
\tag{16}
$$

for the compressible solutions, where $v_k^{(1)}(z)$, $\rho_k^{(2)}(z)$, and $p_k^{(2)}(z)$ belong to C^1 on $(-1)^k z \le (-1)^k Z_k(0)$ and $|||v_k^{(1)}(z)||| = O(1)$, $|||\rho_k^{(2)}(z)||| = O(1)$ and $|||p_k^{(2)}(z)||| = O(1)$. The variables $U_k^{(m,s)} \equiv \left(\beta_k^{(m,s)}, v_k^{(m,s)}, \rho_k^{(m,s)}, p_k^{(m,s)}\right)^{tr}$, $m = 0, 1, 2$, have a slow scale of motion and solve linearized incompressible problems. They have been determined in closed form through second order [10] and are summarized in Sec. 4. The variables $v_k^{(1,f)}$, $\beta_k^{(2,f)}$, $\rho_k^{(2,f)}$, $p_k^{(2,f)}$ contain fast scale acoustical oscillations on the fast time scale $\tau \equiv \lambda t$. As discussed in [9], the inner limit analysis shows that there are no fast scale acoustical oscillations up to first order in the asymptotic expansions of β_k, ρ_k, p_k and up to zero-th order in the expansion of v_k. The two phase flow model depends on the motions Z_k of the mixing zone edges and closure for the interfacial averages with the constitutive law d_k^q, $q = v, p$. Since the Z_k are not well characterized for compressible flows, the velocities or trajectories of the edges of the mixing zone must be provided as data. The compressible constitutive factor d_k^q is asymptotically assumed with a specific limit term. Measurement of the trajectory of the mixing zone edge must be provided as data.

We assume a uniformly valid asymptotic expansion for the compressible mixing zone edge,

$$
Z_k(t) = Z_k^{(0,s)}(t) + \lambda^{-1} Z_k^{(1,s)}(t) + \lambda^{-2}\left(Z_k^{(2,s)}(t) + Z_k^{(2,f)}(t,\lambda)\right) + O(\lambda^{-3}). \tag{17}
$$

Here $\sum_{j=0}^{m} \lambda^{-j} Z_k^{(j,s)}$, $m = 0, 1, 2$, denotes the location of vanishing $\beta_k^{(m,s)}$. Thus Z_k and each of the expansion coefficients $Z_k^{(m,s)}$ and $Z_k^{(2,f)}$ are input to the model equations.

We assume that the compressible edge moves faster than the incompressible edge with no initial perturbation. A similar assumption is applied to any finite number of terms in the expansion (49). We assume that the zero-th order term in the expansion (49) equals to the incompressible edge trajectory $Z_k^{\infty}(t)$. Thus we require

$$\begin{aligned} &Z_k^{(0,s)}(t) = Z_k^{\infty}(t),\, Z_k(0) = Z_k^{\infty}(0), \\ &(-1)^k Z_k^{(m,t)}(t) \geq 0,\ m = 1, 2,\ t = s, f. \end{aligned} \tag{18}$$

The variables $Z_k^{(m,s)}, m = 0, 1, 2$, are the slow variables with a slow scale of motion while $Z_k^{(2,f)}$ is oscillatory on the fast time scale $\tau \equiv \lambda t$. We assume that the fast variable $Z_k^{(2,f)}$ decays exponentially in τ away from the boundary $t = 0$. Here we examine that this fast variable can appear in the second or higher order of the asymptotic expansion (17). The formally uniformly valid expansion (17) leads to the outer and inner expansion under the corresponding limit process. The reduced expansion is assumed in the derivation of outer and inner expansions for compressible solutions in Sec. 4 and [9], respectively. We first show that (17) reduces to the outer expansion under the outer limit process. One calculation shows the outer limit process

$$\lim_{\lambda\to\infty,\ t \text{ fixed}} Z_k(t) = Z_k^{(0,s)}(t), \tag{19}$$

$$\lim_{\lambda\to\infty,\ t \text{ fixed}} \lambda^m \left[Z_k(t) - \sum_{j=0}^{m-1} \lambda^{-j} Z_k^{(j,s)}(t) \right] = Z_k^{(m,s)}(t), \quad m \geq 1. \tag{20}$$

Therefore (17) leads to the outer limit expansion

$$Z_k(t) = Z_k^{(0,s)}(t) + \lambda^{-1} Z_k^{(1,s)}(t) + \lambda^{-2} Z_k^{(2,s)}(t) + O(\lambda^{-3}) \tag{21}$$

valid away from the initial curve $t = 0$. Now we consider the inner limit process $\lambda \to \infty$ with τ fixed $\neq \infty$ to find the inner limit expansion, valid near $t = 0$:

$$Z_k(\tau) = \hat{Z}_k^{(0)}(\tau) + \lambda^{-1} \hat{Z}_k^{(1)}(\tau) + \lambda^{-2} \hat{Z}_k^{(2)}(\tau) + O(\lambda^{-3}). \tag{22}$$

The initial conditions associated with (18) are

$$\hat{Z}_k^{(0)}(0) = Z_k^{\infty}(0),\ \hat{Z}_k^{(m)}(0) = 0,\ m \geq 1. \tag{23}$$

We rewrite the formally uniformly valid expansion (17) in terms of the fast time scale $\tau = \lambda t$ as follows

$$\begin{aligned} Z_k(\tau) =& Z_k^{(0,s)}(\lambda^{-1}\tau) + \lambda^{-1} Z_k^{(1,s)}(\lambda^{-1}\tau) \\ &+ \lambda^{-2} \left(Z_k^{(2,s)}(\lambda^{-1}\tau) + Z_k^{(2,f)}(\tau) \right) + O(\lambda^{-3}). \end{aligned} \tag{24}$$

The inner limit

$$\hat{Z}_k^{(0)}(\tau) = Z_k^{(0,s)}(0) \tag{25}$$

follows from the limit process

$$\begin{aligned}\hat{Z}_k^{(0)}(\tau) &= \lim_{\lambda\to\infty,\ t\ \text{fixed}} Z_k(\tau)\\ &= \lim_{\lambda\to\infty,\ t\ \text{fixed}} \Big[Z_k^{(0,s)}(\lambda^{-1}\tau) + \lambda^{-1} Z_k^{(1,s)}(\lambda^{-1}\tau)\\ &+\lambda^{-2}(Z_k^{(2,s)}(\lambda^{-1}\tau) + Z_k^{(2,f)}(\tau)) + O(\lambda^{-3})\Big] = Z_k^{(0,s)}(0).\end{aligned} \tag{26}$$

The first order term yields

$$\hat{Z}_k^{(1)}(\tau) = \tau \dot{Z}_k^{(0,s)}(0) = \tau V_k^{(0,s)}(0), \tag{27}$$

from the calculation

$$\begin{aligned}\hat{Z}_k^{(1)}(\tau) &= \lim_{\lambda\to\infty,\ t\ \text{fixed}} \lambda[Z_k(\tau) - \hat{Z}_k^{(0)}(\tau)]\\ &= \lim_{\lambda\to\infty,\ t\ \text{fixed}} \left[\tau \frac{Z_k^{(0,s)}(\lambda^{-1}\tau) - Z_k^{(0,s)}(0)}{\lambda^{-1}\tau} + Z_k^{(1,s)}(\lambda^{-1}\tau) \right.\\ &\left. +\lambda^{-1}\left(Z_k^{(2,s)}(\lambda^{-1}\tau) + Z_k^{(2,f)}(\tau)\right) + O(\lambda^{-2})\right]\\ &= \tau \dot{Z}_k^{(0,s)}(0) + Z_k^{(1,s)}(0),\end{aligned} \tag{28}$$

where $\dot{Z} = dZ/dt$ and $Z_k^{(1,s)}(0) = 0$ from (18). Similarly, we obtain the second inner term

$$\begin{aligned}\hat{Z}_k^{(2)}(\tau) &= \lim_{\lambda\to\infty,\ t\ \text{fixed}} \lambda^2[Z_k(\tau) - \hat{Z}_k^{(0)}(\tau) - \lambda^{-1}\hat{Z}_k^{(1)}(\tau)]\\ &= \lim_{\lambda\to\infty,\ t\ \text{fixed}} \left[\frac{Z_k^{(0,s)}(\lambda^{-1}\tau) - Z_k^{(0,s)}(0) - \lambda^{-1}\tau\dot{Z}_k^{(0,s)}(0)}{\lambda^{-2}} \right.\\ &\left. + \frac{Z_k^{(1,s)}(\lambda^{-1}\tau) - Z_k^{(1,s)}(0)}{\lambda^{-1}} + Z_k^{(2,s)}(\lambda^{-1}\tau) + Z_k^{(2,f)}(\tau) + O(\lambda^{-1}\right]\\ &= \frac{\tau^2}{2}\ddot{Z}_k^{(0,s)}(0) + \tau\dot{Z}_k^{(1,s)}(0) + Z_k^{(2,s)}(0) + Z_k^{(2,f)}(\tau),\end{aligned} \tag{29}$$

where $\ddot{Z} = d^2Z/dt^2$. From the initial condition (18), we require

$$Z_k^{(2,s)}(0) = Z_k^{(2,f)}(0) = 0. \tag{30}$$

Therefore the formally uniformly valid expansion (17) implies the inner limit expansion of the compressible edge Z_k as the following

$$\begin{aligned}Z_k(\tau) &= \hat{Z}_k^{(0)}(\tau) + \lambda^{-1}\hat{Z}_k^{(1)}(\tau) + \lambda^{-2}\hat{Z}_k^{(2)}(\tau) + O(\lambda^{-3})\\ &= Z_k^{(0,s)}(0) + \lambda^{-1}\tau\dot{Z}_k^{(0,s)}(0)\\ &+ \lambda^{-2}\left(\frac{\tau^2}{2}\ddot{Z}_k^{(0,s)}(0) + \tau\dot{Z}_k^{(1,s)}(0) + Z_k^{(2,s)}(0) + Z_k^{(2,f)}(\tau)\right) + O(\lambda^{-3}).\end{aligned} \tag{31}$$

Notice that the fast variable $Z_k^{(2,f)}$ consist of the inner term $\hat{Z}_k^{(2)}$ minus common terms to order λ^{-2}. The assumptions (18) imply that $(-1)^k \hat{Z}_k^{(1)}$ and $(-1)^k \hat{Z}_k^{(2)}$ are nonnegative:

$$(-1)^k \dot{Z}_k^{(0,s)}(0) \geq 0, \tag{32}$$

$$(-1)^k \left(\frac{\tau^2}{2} \ddot{Z}_k^{(0,s)}(0) + \tau \dot{Z}_k^{(1,s)}(0) + Z_k^{(2,f)}(\tau) \right) \geq 0. \tag{33}$$

The edge velocity of the compressible flow satisfies $V_k = \dot{Z}_k = v_k(Z_k, t)$ and therefore, it must have an asymptotic expansion associated with the expansion (49) in the form

$$\begin{aligned} V_k(t) =& V_k^{(0,s)}(t) + \lambda^{-1} \left(V_k^{(1,s)}(t) + V_k^{(1,f)}(t,\lambda) \right) \\ &+ \lambda^{-2} \left(V_k^{(2,s)}(t) + V_k^{(2,f)}(t,\lambda) \right) + O(\lambda^{-3}). \end{aligned} \tag{34}$$

From the expansion of v_k in (14), we see that the leading order term of the asymptotic expansion of V_k must be $v_k^{(0,s)}(Z_k^{(0,s)}, t)$, where $Z_k^{(0,s)} = Z_k^{(0,s)}(t)$ denotes the location of vanishing $\beta_k^{(0,s)}(z,t)$. Thus $V_k^{(0,s)}(t) = v_k^{(0,s)}(Z_k^{(0,s)}, t)$. Following the similar limit calculations, we obtain that the formal asymptotic expansion (34) leads to the outer limit expansion

$$V_k(t) = V_k^{(0,s)}(t) + \lambda^{-1} V_k^{(1,s)}(t) + \lambda^{-2} V_k^{(2,s)}(t) + O(\lambda^{-3}) \tag{35}$$

which is valid away from the initial curve $t = 0$ under the limit process $\lambda \to \infty$ with t fixed $\neq 0$, and to the inner limit expansion

$$\begin{aligned} V_k(\tau) &= \hat{V}_k^{(0)}(\tau) + \lambda^{-1} \hat{V}_k^{(1)}(\tau) + O(\lambda^{-2}) \\ &= V_k^{(0,s)}(0) + \lambda^{-1} \left(\tau \dot{V}_k^{(0,s)}(0) + V_k^{(1,s)}(0) + V_k^{(1,f)}(\tau) \right) + O(\lambda^{-2}). \end{aligned} \tag{36}$$

Comparing (21), (31) with (35) and (36), we note that for $m \geq 0$,

$$\frac{dZ_k^{(m,s)}(t)}{dt} = V_k^{(m,s)}(t), \qquad \frac{d\hat{Z}_k^{(0)}(\tau))}{d\tau} = 0, \tag{37}$$

$$\frac{d\hat{Z}_k^{(m+1)}(\tau)}{d\tau} = \hat{V}_k^{(m)}(\tau), \qquad \frac{dZ_k^{(m+2,f)}(\tau)}{d\tau} = V_k^{(m+1,f)}(\tau) \tag{38}$$

by use of the identity $V_k = \dot{Z}_k$. In particular, substitution of $\tau = \lambda t$ into the fast variables gives the relation

$$\frac{dZ_k^{(m+2,f)}(t,\lambda)}{d\tau} = V_k^{(m+1,f)}(t,\lambda). \tag{39}$$

From (39) it is reasonable to assume that the fast variables can appear in the second or higher order of the asymptotic expansion (17). If the expansion (17) has a fast variable

$Z_k^{(1,f)}$ in the first order, it implies that a fast variable $V_k^{(0,f)}$ exists in the zero-th order of the asymptotic expansion (34). This contradicts the fact that the zero-th order fast variable $v_k^{(0,f)}$ is suppressed by the initialization (49), as discussed in [9].

We assume that the constitutive laws $d_k^v(t)$ and $d_k^p(t)$ have a formally uniformly valid asymptotic expansion as follows

$$d_k^q(t,\lambda) = d_k^{q(0,s)}(t) + \lambda^{-1}\left(d_k^{q(1,s)}(t) + d_k^{q(1,f)}(t,\lambda)\right) + O(\lambda^{-2}), q = v, p, \quad (40)$$

where $d_k^{q(m,s)}(t), d_k^{q(m,f)}(\tau) \in ([0,\infty))$, and we assume $d_k^{q(0,s)}(t) = d_k^{q\infty}(t)$. Similarly, we can show that the expansion (40) leads to the outer limit expansion

$$d_k^q(t) = d_k^{q(0,s)}(t) + \lambda^{-1} d_k^{q(1,s)}(t) + \lambda^{-2} d_k^{q(2,s)}(t) + O(\lambda^{-3}), q = v, p, \quad (41)$$

under the limit process $\lambda \to \infty$, t fixed $\neq 0$ and that it leads the inner limit asymptotic expansion of the constitutive law in the form

$$\begin{aligned} d_k^q(\tau) &= \hat{d}_k^{q(0)}(\tau) + \lambda^{-1}\hat{d}_k^{q(1)}(\tau) + \text{¥Ï}(\lambda^{-2}) \\ &= d_k^{q(0,s)}(0) + \lambda^{-1}\left(\tau\frac{dd_k^q(0,s)}{dt}(0) + d_k^{q(1,s)}(0) + d_k^{q(1,f)}(\tau)\right) + O(\lambda^{-2}) \end{aligned} \quad (42)$$

under the limit process $\lambda \to \infty$, τ fixed $\neq \infty$. The property $d_1^q d_2^q = 1$ implied from the $\mu_1^q + \mu_2^q = 1$ gives the relation

$$\sum_{j=0}^{n} d_1^{q(j,s)} d_2^{q(n-j,s)} = \delta_{n0}, \quad \sum_{j=0}^{n} \hat{d}_1^{q(j)} \hat{d}_2^{q(n-j)} = \delta_{n0}, \quad n = 0, 1, 2 \quad (43)$$

between the terms in the expansion (40), where δ_{n0} is the Kronecker symbol.

4. Summary of the Outer Limit Process

In this section, we summarize the outer limiting behavior of a compressible two-phase flow model as the Mach number tends to zero [4, 10]. Formal outer limit asymptotic expansions are derived for the solutions $U_k \equiv (\beta_k, v_k, \rho_k, p_k)^{tr}$ of compressible two-phase equations (4)-(6). The slow variables $U_k^{(m,s)} \equiv \left(\beta_k^{(m,s)}, v_k^{(m,s)}, \rho_k^{(m,s)}, p_k^{(m,s)}\right)^{tr}$, $m = 0, 1, 2$, in the outer limit asymptotic expansions have been evaluated through second order in closed form. This is the order the incompressible pressure first appears. The uniformly valid variables in space requires matching at the boundaries of the regions existing in each order of λ^{-1} since in higher order in λ^{-1}, there exist transition layers in the intermediate region of the mixing zone edges $Z_k(t)$ and $Z_k^{\infty}(t)$ for the compressible and incompressible flow.

The first order expansion is defined by five regions, $\mathcal{E}_k^{(1)} \cup \mathcal{T}_k^{(1)} \cup \mathcal{M} \cup \mathcal{T}_{k'}^{(1)} \cup \mathcal{E}_{k'}^{(1)}$, including two transition-layers through $z = \overline{Z}_i^{(1,s)}$, $i = k, k'$. In second order, we have two additional transition-layers extending out to $z = \overline{Z}_i^{(2,s)}$, so the second order expansion is uniquely defined by seven regions, $\mathcal{E}_k^{(2)} \cup \mathcal{T}_k^{(2)} \cup \mathcal{T}_k^{(1)} \cup \mathcal{M} \cup \mathcal{T}_{k'}^{(1)} \cup \mathcal{T}_{k'}^{(2)} \cup \mathcal{E}_{k'}^{(2)}$. The regions are defined by

$$\begin{aligned} \mathcal{E}_{k'}^{(n)} &= \left\{(z,t) : (-1)^k \overline{Z}_k^{(n,s)} \le (-1)^k z\right\}, \\ \mathcal{T}_{k'}^{(n)} &= \left\{(z,t) : (-1)^k \overline{Z}_k^{(n-1,s)} \le (-1)^k z < (-1)^k \overline{Z}_k^{(n,s)}\right\}, \\ \mathcal{M} &= \left\{(z,t) : Z_1^{(0,s)} < z < Z_2^{(0,s)}\right\}, \end{aligned} \tag{44}$$

where for $n = 1, 2$,

$$\overline{Z}_k^{(n,s)}(t) \equiv \sum_{j=0}^{n} \lambda^{-j} Z_k^{(j,s)} \tag{45}$$

denotes the position the position of boundaries of the transition-layers.

[4] examined the closed form solution previously introduced for the incompressible limit, which has nonuniqueness parameterized by pressure solutions. This extra degree of freedom is resolved by the equations

$$d_1^{v(1,s)} = \left(\frac{|V_2|}{|V_1|}\right)^{(1,s)} = \frac{-V_1^\infty V_2^{(1,s)} + V_2^\infty V_1^{(1,s)}}{V_1^{\infty 2}}, \tag{46}$$

$$\begin{aligned} d_1^{v(2,s)} &= \left(\frac{V_2 - v_2(Z_1,t) + \int_{Z_1}^{Z_2} \frac{1}{\rho_2}\frac{D_2\rho_2}{Dt} dz}{-V_1 + v_1(Z_2,t) + \int_{Z_1}^{Z_2} \frac{1}{\rho_1}\frac{D_1\rho_1}{Dt} dz}\right)^{(2,s)} \\ &= -\frac{d_1^{v(1,s)} V_1^{(1,s)} + d_1^{v\infty} V_1^{(2,s)} + V_2^{(2,s)}}{V_1^\infty} \\ &\quad + \frac{1}{V_1^\infty}\left\{\int_{Z_1^\infty}^{z^{+\infty}} \frac{\beta_1^\infty + d_1^{v\infty}}{\rho_1^\infty a_1^2}\left(\frac{\partial p_2^\infty}{\partial t} + v_2^\infty \frac{\partial p_2^\infty}{\partial z}\right) dz \right. \\ &\quad \left. + \int_{z^{-\infty}}^{Z_2^\infty} \frac{\beta_2^\infty - 1}{\rho_2^\infty a_2^2}\left(\frac{\partial p_2^\infty}{\partial t} + v_2^\infty \frac{\partial p_2^\infty}{\partial z}\right) dz\right\} \end{aligned} \tag{47}$$

in the expansion of the constitutive quantity d_k^v introduced in [11]. Here the limiting squared sound speed $a_k^2 = c_k^2(\rho_k^\infty) = \partial p_k/\partial \rho_k(\rho_k^\infty)$ and $z = z^{+\infty}$ ($z = z^{-\infty}$) is defined as the position of the upper(lower) wall of $\mathcal{D}$. In the incompressible case [6, 7], the velocity constitutive law was interpreted as a ratio of the volumetric growth rates for the mixing zone for the two phases,

$$d_1^{v\infty} = \frac{|V_2^\infty|}{|V_1^\infty|}. \tag{48}$$

The identities (46) and (47) show an extension of the volumetric mixing rate constraint determining the v^* closure to the fully compressible case, including the relative compressibility of the two fluids. Specifically, (47) links the compressible variables parameterized by d_k^v and the incompressible pressures. This shows that the two-phase incompressible equations, in contrast to the single phase case, remember the compressible fluids from which they are derived. The identities (46) and (47) are required as necessary and sufficient conditions for the convergence of compressible pressures to the incompressible pressures through the second order of the expansion. These equations can be regarded as constitutive constraints on the expansion. This aspect of two-phase flow in the incompressible limit appears to be unique.

In the outer limit asymptotic process we assume:

H0. Formulas (13), (18) and (35) are valid, and the compressible equations (4)-(6) have initial conditions

$$\begin{aligned}
\beta_k(z,0) &= \beta_k^\infty(z,0) + \lambda^{-1}\beta_k^{(1,s)}(z,0) + \lambda^{-2}\beta_k^{(2,s)}(z,0)\,,\\
v_k(z,0) &= v_k^\infty(z,0) + \lambda^{-1}v_k^{(1,s)}(z,0) + \lambda^{-2}v_k^{(2,s)}(z,0)\,,\\
\rho_k(z,0) &= \rho_k^\infty + \lambda^{-2}a_k^{-2}p_k^\infty(z,0)\,,\\
p_k(z,0) &= p_k(\rho_k^\infty) + \lambda^{-2}p_k^\infty(z,0)\,,
\end{aligned} \tag{49}$$

where $p_k(\rho_k^\infty) = A_k\rho_k^{\infty\,\gamma_k}$, the limiting squared sound speed $a_k^2 = c_k^2(\rho_k^\infty) = \dfrac{dp_k}{d\rho_k}(\rho_k^\infty)$ and $\beta_k^{(1,s)}(z,0)$ and $\beta_k^{(2,s)}(z,0)$ are given functions smooth in $\mathcal{M}$ and vanishing outside of $\mathcal{M}$.

H1. The first order term $d_k^{v(1,s)}$ satisfies (46).

H2. The second order term $d_k^{v(2,s)}$ satisfies (47).

With the above conditions we have the following

Theorem 4.1. Assume H0-H2. Then the compressible solutions of (4)-(6) have uniformly valid outer expansions in z to $O(\lambda^{-2})$ of the form

$$\begin{aligned}
\beta_k &= \beta_k^\infty + +\lambda^{-1}\beta_k^{(1,s)} + \lambda^{-2}\beta_k^{(2,s)} + O(\lambda^{-3}),\\
v_k &= v_k^\infty + \lambda^{-1}v_k^{(1,s)} + \lambda^{-2}v_k^{(2,s)} + O(\lambda^{-3}),\\
\rho_k &= \rho_k^\infty + \lambda^{-2}a_k^{-2}p_k^\infty + (\lambda^{-3}),\\
p_k &= p^{(0)} + \lambda^{-2}p_k^\infty + O(\lambda^{-3}),
\end{aligned} \tag{50}$$

where the universal constant $p^{(0)} = p_k(p_k^\infty) = A_k\rho_k^{\infty\gamma_k}$ and $a_k^2 = c_k^2(\rho_k^\infty) = dp_k/d\rho_k(\rho_k^\infty)$. Here $\beta_k^{(m,s)}$ and $v_k^{(m,s)}$, $m = 1, 2$, are given in closed form as the following

$$v_k^{(1,s)}(z,t) = V_k^\infty\mu_{k'}^{v(1,s)} + V_k^{(1,s)}\mu_{k'}^{v\infty}, \tag{51}$$

$$\begin{aligned} v_k^{(2,s)}(z,t) &= \sum_{j=0}^{2} V_k^{(j,s)} \mu_{k'}^{v(2-j,s)} + \mu_k^{v\infty} \int_z^{z+\infty} \sum_{k=1}^{2} \frac{\beta_k^\infty}{\rho_k^\infty a_k^2} \frac{D_k p_k^\infty}{Dt} dz \\ &\quad - \mu_{k'}^{v\infty} d_{k'}^{v\infty} \int_{\overline{Z}_k^{(2,s)}}^{z} \frac{\beta_k^\infty + d_k^{v\infty}}{\rho_k^\infty a_k^2} \frac{D_k p_k^\infty}{Dt} + \frac{\beta_{k'}^\infty - 1}{\rho_{k'}^\infty a_{k'}^2} \frac{D_{k'} p_{k'}^\infty}{Dt} dz, \end{aligned} \tag{52}$$

$$\beta_k^{(1,s)}(z,t) = \begin{cases} 0 & \text{in } \mathcal{E}_1^{(1)},\ \mathcal{E}_2^{(1)} \\ -\beta_{k'}^{(1,s)}(z,t) & \text{in } \mathcal{T}_k^{(1)} \\ \left(\beta_k^{(1,s)}(z,0)\dfrac{dz_0(\beta_k^\infty)}{d\beta_k^\infty} + \displaystyle\int_0^t b^{(1,s)}(\beta_k^\infty, s)\, ds\right) \dfrac{\partial \beta_k^\infty}{\partial z} & \text{in } \mathcal{M} \\ \dfrac{\lambda\left(z - \overline{Z}_k^{(1,s)}\right)}{\xi_k(t)} & \text{in } \mathcal{T}_{k'}^{(1)} \end{cases}, \tag{53}$$

$$\beta_k^{(2,s)}(z,t) = \begin{cases} 0 & \text{in } \mathcal{E}_1^{(2)},\ \mathcal{E}_2^{(2)} \\ -\beta_{k'}^{(2,s)}(z,t) & \text{in } \mathcal{T}_k^{(2)},\ \mathcal{T}_k^{(1)} \\ \left(\beta_k^{(2,s)}(z,0)\dfrac{dz_0(\beta_k^\infty)}{d\beta_k^\infty} + \displaystyle\int_0^t b^{(2,s)}(\beta_k^\infty, s)\, ds\right) \dfrac{\partial \beta_k^\infty}{\partial z} & \text{in } \mathcal{M} \\ \dfrac{\beta_k^{(1,s)}}{\xi_k(t)} \left[3\beta_k^{(1,s)} \displaystyle\int_0^t V_k^\infty d_{k'}^{v\infty}(1 - d_{k'}^{v\infty})\, dt \right. & \\ \left. +2\displaystyle\int_0^t V_k^\infty d_{k'}^{v(1,s)} + V_k^{(1,s)} d_{k'}^{v\infty}\, dt - Z_k^{(2,s)}\right] & \text{in } \mathcal{T}_{k'}^{(1)} \\ \dfrac{\lambda^2\left(z - \overline{Z}_k^{(2,s)}\right)}{\xi_k(t)} & \text{in } \mathcal{T}_{k'}^{(2)} \end{cases}, \tag{54}$$

where $z_0(\beta_k^\infty)$ is the inverse of the initial volume fraction $\beta_k^\infty(z,0)$ and

$$\xi_k(t) = 2\int_0^t V_k^{\infty\,2} / V_{k'}^\infty\, dt, \tag{55}$$

$$b^{(1,s)}(\beta_k^\infty, t) = \sum_{k=1}^{2} \left(2\mu_k^{v\infty} - \beta_{k'}^\infty\right) \frac{\mu_{k'}^{v\infty\,2}}{\beta_{k'}^\infty} V_k^{(1,s)}. \tag{56}$$

$$b_2^{(2,s)}(\beta_k^\infty, t) = \frac{\partial v^{*\infty}}{\partial \beta_k^\infty} \beta_k^{(2,s)} - v^{*(2,s)} - v^{*(1,s)} \frac{\partial \beta_k^{(1,s)}}{\partial \beta_k^\infty}. \tag{57}$$

The volume fraction β_k^∞, velocity v_k^∞ and constant density ρ_k^∞ of incompressible flow are proved to be the outer limit $\beta_k^{(0,s)}$, $v_k^{(0,s)}$, $\rho_k^{(0,s)}$ describing this incompressible limit process as $\lambda \to \infty$. The pressures $p_k^{(0,s)}$, $p_k^{(1,s)}$, $p_k^{(2,\text{eff})} \equiv p_k^{(2,s)} - p_k^\infty$ are not affected by the gravity force $g = g(t)$ and satisfy the pressure mechanical equilibrium condition. In fact, it is shown that (4.1), (4.2) are necessary and sufficient conditions for the pressure equilibrium condition. The second order term in the expansion of p_k satisfies $p_k^{(2,s)} = p_k^\infty + O(\lambda^{-1})$ and therefore, it is fixed to be bounded, under the assumption that p_k^∞ satisfies (4.2). Actually, (4.2) is derived as a condition equivalent to a bounded second order term $p_k^{(2,s)}$. The transition layers $\mathcal{T}_i^{(1)}$, $\mathcal{T}_i^{(2)}$, $i = k, k'$, affect an $O(\lambda^{-1})$ term in the second order of the expansion.

5. Conclusion

The formally uniformly valid expansions which lead to the outer and inner expansion under the corresponding limit process have been discussed. We examined the reduced expansions and conditions required in the derivation of outer and inner expansions for compressible solutions.

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