Behavior asymptotic solution of reaction diffusion system with full matrix

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Abstract
The purpose of this paper is to prove that behavior asymptotic in the time of solutions for the strongly coupled reaction diffusion system:

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u - d_2 \Delta v &= f(u, v) \quad \text{in } \mathbb{R}^+ \times \Omega \\
\frac{\partial u}{\partial t} - d_3 \Delta u - d_4 \Delta v &= g(u, v) \quad \text{in } \mathbb{R}^+ \times \Omega \\
\frac{\partial u}{\partial \eta} &= \frac{\partial v}{\partial \eta} = 0 \quad \text{in } \mathbb{R}^+ \times \partial \Omega \\
u(., 0) &= u_0(.,) \quad v(., 0) = v_0(.) \quad \text{in } \Omega
\end{align*}
\]

(SRD)

with full matrix of diffusion coefficients.

We treat first second equations of SRD as a dynamical system in \(C(\Omega) \times C(\Omega)\) and apply Lyapunov type stability techniques. A key ingredient in this analysis is a result which establishes that the orbits of the dynamical system are precompact in \(C(\Omega) \times C(\Omega)\). As a consequence of Arzela-Ascoli theorem, this will be satisfied if the orbits are, for example, uniformly bounded in \(C^1(\Omega) \times C^1(\Omega)\) for \(t > 0\).

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1. Introduction

In this paper we study the following semilinear parabolic system

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u - d_2 \Delta v &= f(u, v), \quad \text{in } \mathbb{R}^+ \times \Omega \\
\frac{\partial v}{\partial t} - d_3 \Delta u - d_4 \Delta v &= g(u, v), \quad \text{in } \mathbb{R}^+ \times \Omega
\end{align*}
\]

(1)

Where \( \Omega \) is a regular and bounded domain of \( \mathbb{R}^n \), \( n \geq 1 \), \( u = u(t, x), v = v(t, x) \), \( x \in \Omega, t > 0 \) are real valued functions, \( \Delta \) denotes the Laplacian operator, and the constants of diffusion \( d_1, d_2, d_3, d_4 \) are assumed to be nonnegative. Also \( (d_1 + d_4)^2 < 4d_2d_3 \) which reflects the parabolicity of the system.

System (1) is subjected to the following boundary conditions

\[
\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \quad \text{in } \mathbb{R}^+ \times \partial \Omega
\]

(2)

and the initial data

\[
u(., 0) = u_0(.), \quad v(., 0) = v_0(.) \quad \text{in } \Omega
\]

(3)

which are assumed to be nonnegative.

Concerning the functions \( f \) and \( g \), we assume the following hypothesis:

\( \text{(H1)} \) \( f(r, s) \) and \( g(r, s) \) are continuously differentiable on \( \mathbb{R}^+ \times \mathbb{R}^+ \), such that \( f(0, s) \geq 0, g(r, 0) \geq 0 \forall r, s \geq 0 \)

We know that the problem (1)–(3) has a unique global solution see [17]. The main question we want to address is asymptotic behavior the solutions for system (1)–(3). In fact the subject of the asymptotic behavior of reaction diffusion systems has received a lot of attention in the last decades and several outstanding results have been proved by some of the major experts in the field.

This question has been investigated by many authors by considering special forms of the nonlinear terms \( f \) and \( g \).

In the case where diagonal \( d_1 \neq d_4 \) and \( d_3 = d_2 = 0 \), \( \Psi(v) \in C^1 \left( \mathbb{R}^+ \right) \)

\[
\lambda_1 \frac{\partial u}{\partial \eta} + (1 - \lambda_1)u = 0, \quad \text{in } \mathbb{R}^+ \times \partial \Omega
\]

\[
\lambda_2 \frac{\partial v}{\partial \eta} + (1 - \lambda_2)v = 0, \quad \text{in } \mathbb{R}^+ \times \partial \Omega
\]

(p)
Note that, the behavior of non-negative total solutions $p$ is treated in the papers [12, 16]. In particular, it is proved that there are two constants nonegative $u^*$, $v^*$ such that
\[ \| u(t) - u^* \|_\infty \to 0 \quad \text{and} \quad \| v(t) - v^* \|_\infty \to 0 \]
and $u^* \Psi(v^*) = 0$ it is obvious that $u^* = v^*$ where $u$ and $v$ have not Neumann boundary conditions. A. Barabanova [7] generalize the method of Haraux and youkana where $\Psi(v) = e^{av}$ (method of estimating energy) and then Rebiai Benachour [18] treated the same case with conditions more strong

\[
\begin{aligned}
\left\{ \begin{array}{l}
f(0, \eta) = g(0, \eta) = 0 \\
g(\zeta, 0) \geq 0, g(\zeta, \eta) \leq \Psi(\eta)f(\zeta, \eta) \\
g(\zeta, \eta) \leq C\Phi(\zeta)e^{a\eta^\beta} \quad C > 0 \text{ and } \alpha > 0, \Phi \in C^1(\mathbb{R}^+) \text{ and } \Phi(0) = 0
\end{array} \right.
\]

where $\Psi$ is a continuously differentiable function such that there is a constant $\beta \geq 1$ \lim_{\eta \to +\infty} \eta^{\beta-1}\Psi(\eta) = l \text{ where } l > 0$. Collet and Xin [10] have studied the same system (p) with the boundary conditions on $\mathbb{R}^n$ with a diagonal diffusion matrix and $\Psi(v) = v^m$, where $m \in \mathbb{N}^*$. They proved the existence of global solutions and showed that the norm $L^\infty$ of $v$ can not grow faster than $O(ln t)$. In addition, the system has been studied by Avrin [2] if $d_3 = d_2 = 0, v = \exp\{-E/v\}, E > 0$ and the space is variable $R$.

In the case of a triangular system according to:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - d_1 \Delta u = f(u, v) = -u\Psi(v) & \quad \text{in } \mathbb{R}^+ \times \Omega \\
\frac{\partial u}{\partial t} - d_3 \Delta u - d_4 \Delta v = g(u, v) = u\Psi(v) & \quad \text{in } \mathbb{R}^+ \times \Omega
\end{aligned}
\]

in the case of unbounded domain and $\Psi(v) = v^m$ is studied by Badraoui in [4,5]. In [5], he showed the existence of comprehensive solution if $v_0(x) \geq \frac{d_3}{d_1 - d_4}u_0(x)$ and $d_1 > d_4, d_3 > 0, d_1 < 0, d_3 < 0$. In [5], he proved that the norm $L^\infty$ of $v$ can not grow faster than $O(ln t)$ in the case of bounded domain, Kirane [13] studied the asymptotic behavior using the method of the invariant region and semi group theory.

Kouachi [15] obtained a result on uniform boundedness of solutions for a system like (SRD) with an overall complete matrix of diffusion coefficients satisfy the law of equilibrium. This result is generalized by Kouachi after [14] who used the notion of invariant regions and Lyapunov functional. Curiously, less attention was paid to the behavior of the solutions when the spatial variable $x$ approaches infinity despite the usefulness of this type of train for the digital processing of these problems. We are not aware of the article Gladnov [11], which generalizes a result of behavior $x$ tends to infinity with a semi-linear equation posed in $\mathbb{R}^+$ studied by Beberns and Fulks [8].

The remainder of this document is divided into two sections. Section 2 presents some material that we will use in this paper and in Section 3, we state and prove the main result.
2. Notations and preliminary

We put in what follows \( G(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \) we note that \( \{G(t), t \geq 0\} \) is a semi group nonlinear in the compact metric space \( X \)

\[ O(u_0, v_0) = \{G(t)(u_0, v_0), t \geq 0\} \]

is the orbit \((u_0, v_0)\).

First recall the following definition

**Definition 2.1.** All \( w\)-limite is defined by

\[ w(u_0, v_0) = \{(u, v) \in X : \exists t_n \to \infty : G(t_n) \to (u, v)\}. \]

**Inequalite**

We know that

\[ (\sqrt{2\varepsilon}a + \frac{b}{\sqrt{2\varepsilon}})^2 \geq 0 \]

so

\[ 2\varepsilon a^2 + \frac{b^2}{4\varepsilon} \geq -2ab \]

thus

\[ -ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon} \]

we have

\[ \nabla u = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right), \nabla v = \left( \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \ldots, \frac{\partial v}{\partial x_n} \right) \]

\[ |\nabla u| = \sqrt{\sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \]

then

\[ \int_{Q_T} |\nabla u| |\nabla v| \, dx dt = \sum_{i=1}^{n} \int_{Q_T} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx dt \]

replacing \( a = \frac{\partial u}{\partial x_i} b = \frac{\partial v}{\partial x_i} \) in (4) is obtained

\[ \sum_{i=1}^{n} \int_{Q_T} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx dt \geq \sum_{i=1}^{n} \int_{Q_T} -\varepsilon \left( \frac{\partial u}{\partial x_i} \right)^2 - \frac{1}{4\varepsilon} \left( \frac{\partial v}{\partial x_i} \right)^2 \, dx dt = -\varepsilon \int_{Q_T} |\nabla u|^2 \, dx dt - \frac{1}{4\varepsilon} \sum_{i=1}^{n} \int_{Q_T} |\nabla v|^2 \, dx dt \]
so
\[
\int_{Q_T} |\nabla u| |\nabla v| \, dx \, dt \geq -\varepsilon \int_{Q_T} |\nabla u|^2 \, dx \, dt - \frac{1}{4\varepsilon} \int_{Q_T} |\nabla v|^2 \, dx \, dt
\]  
(5)

3. Main result

**Theorem 3.1.** The solution \( w = (u, v) \) of the system (1)-(3) converges a constant vector of the form \( K = (K_1, K_2) \) as \( t \to \infty \), uniformly in \( \Omega \) i.e.

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}
\overset{t \to \infty}{\to}
\begin{pmatrix}
  K_1 \\
  K_2
\end{pmatrix}
\]

Furthermore, we have: \( K_1 \geq 0, K_2 \geq 0, f (K_1, K_2) = 0 \) and

\[
K_1 + K_2 = \frac{1}{\Omega} \int_{\Omega} (u_0(x) + v_0(x)) \, dx.
\]

The following lemma is a useful tool in the proof of the Theorem.

**Lemma 3.2.** Let \( (u, v) \) be a solution of (1)–(3). We have

(i) \( \int_{Q_T} |\nabla u|^2 \, dx \, dt < \infty \)

(ii) \( \int_{Q_T} |\nabla v|^2 \, dx \, dt < \infty \) here \( Q_T = \Omega \times [0, T] \) and \( 0 < T < \infty \).

**Proof.** We have

\[
\frac{\partial u}{\partial t} - d_1 \Delta u - d_2 \Delta v = f (u, v),
\]

by integrating over \((0, T)\) is obtained

\[
\int_0^T \frac{\partial u}{\partial t}(x, t) \, dt = d_1 \int_0^T \Delta u \, dt + d_2 \int_0^T \Delta v \, dt + \int_0^T f(u(x, t), v(x, t)) \, dt
\]

\[
u(x, T) - u(x, 0) = d_1 \int_0^T \Delta u \, dt + d_2 \int_0^T \Delta v \, dt + \int_0^T f(u(x, t), v(x, t)) \, dt
\]

and integrating a second time is obtained over \( \Omega \)

\[
\int_\Omega u(x, T) \, dx - \int_\Omega u(x, 0) \, dx = d_1 \int_\Omega \int_0^T \Delta u \, dt \, dx + d_2 \int_\Omega \int_0^T \Delta v \, dt \, dx + \int_\Omega \int_0^T f(u(x, t), v(x, t)) \, dt \, dx.
\]
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Green’s formula is applied to \( \int_{\Omega} \Delta u dx \) and \( \int_{\Omega} \Delta v dx \), we have

\[
\int_{\Omega} \Delta u dx = \int_{\partial \Omega} \frac{\partial u}{\partial \eta} d\sigma - \int_{\Omega} \nabla u \nabla 1 dx
\]
therefore \( \int_{\Omega} \Delta u dx = 0 \)

\[
\int_{\Omega} \Delta v dx = \int_{\partial \Omega} \frac{\partial v}{\partial \eta} d\sigma - \int_{\Omega} \nabla v \nabla 1 dx
\]
therefore \( \int_{\Omega} \Delta v dx = 0 \)

thus

\[
\int_{\Omega} \int_{0}^{T} f(u(x, t), v(x, t)) dt dx = \int_{\Omega} u_{0}(x) dx - \int_{\Omega} u(x, T) dx < \infty
\]
because \( u(T) \in C(\bar{\Omega}) \) i.e.

\[
\int_{Q_T} f(u(x, t), v(x, t)) dt dx < \infty
\]

In the same way \( \frac{\partial v}{\partial t} - d_{3} \Delta u - d_{4} \Delta v = g(u, v) \) we get that

\[
\int_{Q_T} g(u(x, t), v(x, t)) dt dx < \infty
\]

Multiply now the equation \( \frac{\partial u}{\partial t} - d_{1} \Delta u - d_{2} \Delta v = f(u, v) \) by \( u \) and integrating over \( Q_{T} \) we have

\[
\int_{\Omega} \int_{0}^{T} u \frac{\partial u}{\partial t} (x, t) dt dx = d_{1} \int_{\Omega} \int_{0}^{T} u \Delta u dt dx + d_{2} \int_{\Omega} \int_{0}^{T} u \Delta v dt dx + \int_{\Omega} \int_{0}^{T} u f(u(x, t), v(x, t)) dt dx
\]

by using the Green formula

\[
\int_{\Omega} u \Delta u dx = \int_{\partial \Omega} \frac{\partial u}{\partial \eta} d\sigma - \int_{\Omega} |\nabla u|^2 dx, \text{ therefore } \int_{\Omega} u \Delta u dx = -\int_{\Omega} |\nabla u|^2 dx
\]
and a simple calculation, it becomes

\[
\frac{1}{2} \int_{\Omega} [u^2(x, t)]_{0}^{T} dx = -d_{1} \int_{0}^{T} \int_{\Omega} |\nabla u|^2 x dt - d_{2} \int_{0}^{T} \int_{\Omega} |\nabla u| |\nabla v| x dt + \int_{0}^{T} \int_{\Omega} u(x, t) f(u(x, t), v(x, t)) x dt
\]
i.e.
\[
\int_{\Omega} u^2(x, T) + 2d_1 \int_{Q_T} |\nabla u|^2 \, dx \, dt + 2d_2 \int_{Q_T} |\nabla u| |\nabla v| \, dx \, dt
\]
\[
= \int_{\Omega} u_0^2(x) \, dx + 2 \int_{Q_T} u(x, t) f(u(x, t), v(x, t)) \, dx \, dt
\]
consequently
\[
2d_1 \int_{Q_T} |\nabla u|^2 \, dx \, dt + 2d_2 \int_{Q_T} |\nabla u| |\nabla v| \, dx \, dt \leq \int_{\Omega} u_0^2(x) \, dx \quad (6)
\]
\[
+ 2 \int_{Q_T} u(x, t) f(u(x, t), v(x, t)) \, dx \, dt
\]
Multiply the equation \( \frac{\partial v}{\partial t} - d_3 \Delta u - d_4 \Delta v = g(u, v) \) by \( v \) and integrating over \( Q_T \) is obtained
\[
\int_{\Omega} \int_0^T v \frac{\partial v}{\partial t} (x, t) \, dt \, dx = d_3 \int_{\Omega} \int_0^T v \Delta u \, dt \, dx + d_4 \int_{\Omega} \int_0^T v \Delta v \, dt \, dx
\]
\[
+ \int_{\Omega} \int_0^T v g(u(x, t), v(x, t)) \, dt \, dx
\]
by using the Green formula
\[
\int_{\Omega} v \Delta v \, dx = \int_{\partial \Omega} \frac{\partial v}{\partial \eta} \, d\sigma - \int_{\Omega} |\nabla v|^2 \, dx,
\]
therefore
\[
\int_{\Omega} v \Delta v \, dx = - \int_{\Omega} |\nabla v|^2 \, dx
\]
therefore
\[
\frac{1}{2} \int_{\Omega} [v^2(x, t)]_0^T \, dx = -d_3 \int_0^T \int_{\Omega} |\nabla u| |\nabla v| \, dx \, dt - d_4 \int_0^T \int_{\Omega} |\nabla v|^2 \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega} v(x, t) g(u(x, t), v(x, t)) \, dx \, dt
\]
\[
\int_{\Omega} v^2(x, T) + 2d_3 \int_{Q_T} |\nabla u| |\nabla v| \, dx \, dt + 2d_4 \int_{Q_T} |\nabla v|^2 \, dx \, dt
\]
\[
= \int_{\Omega} v_0^2(x) \, dx + 2 \int_{Q_T} v(x, t) g(u(x, t), v(x, t)) \, dx \, dt
\]
consequently
\[
2d_3 \int_{Q_T} |\nabla u| |\nabla v| \, dx \, dt + 2d_4 \int_{Q_T} |\nabla v|^2 \, dx \, dt \leq \int_{\Omega} v_0^2(x) \, dx \quad (7)
\]
\[
+ 2 \int_{Q_T} v(x, t) g(u(x, t), v(x, t)) \, dx \, dt
\]
Replacing (5) in the inequality (4) is found

$$
2d_1 \int_{Q_T} |\nabla u|^2 \, dx \, dt + 2d_2 \left( -\varepsilon \int_{Q_T} |\nabla u|^2 \, dx \, dt - \frac{1}{4 \varepsilon} \int_{Q_T} |\nabla v|^2 \, dx \, dt \right) 
\leq 2d_1 \int_{Q_T} |\nabla u|^2 \, dx \, dt + 2d_2 \int_{Q_T} |\nabla u| \, |\nabla v| \, dx \, dt
$$

$$
\leq \int_{\Omega} u_0^2(x) \, dx + 2 \int_{Q_T} u(x, t) f(u(x, t), v(x, t)) \, dx \, dt
$$

because \((d_2 \geq 0)\)

thus

$$
(2d_1 - 2\varepsilon d_2) \int_{Q_T} |\nabla u|^2 \, dx \, dt \leq \frac{d_2}{2\varepsilon} \int_{Q_T} |\nabla v|^2 \, dx \, dt
$$

$$
+ \int_{\Omega} u_0^2(x) \, dx + 2 \int_{Q_T} u(x, t) f(u(x, t), v(x, t)) \, dx \, dt
$$

Replacing (5) in the inequality (7) is found

$$
2d_4 \int_{Q_T} |\nabla v|^2 \, dx \, dt + 2d_5 \left( -\varepsilon \int_{Q_T} |\nabla u|^2 \, dx \, dt - \frac{1}{4 \varepsilon} \int_{Q_T} |\nabla v|^2 \, dx \, dt \right)
\leq 2d_4 \int_{Q_T} |\nabla v|^2 \, dx \, dt + 2d_5 \int_{Q_T} |\nabla u| \, |\nabla v| \, dx \, dt
$$

$$
\leq \int_{\Omega} v_0^2(x) \, dx + 2 \int_{Q_T} v(x, t) g(u(x, t), v(x, t)) \, dx \, dt
$$

because \((d_3 \geq 0)\)

so

$$
\left( 2d_4 - \frac{d_3}{2\varepsilon} \right) \int_{Q_T} |\nabla v|^2 \, dx \, dt \leq 2d_3 \varepsilon \int_{Q_T} |\nabla u|^2 \, dx \, dt + \int_{\Omega} v_0^2(x) \, dx
$$

$$
+ 2 \int_{Q_T} v(x, t) g(u(x, t), v(x, t)) \, dx \, dt
$$

We replace (9) in (8):

$$
\left( 2d_4 - \frac{d_3}{2\varepsilon} - \frac{d_2 d_3}{2d_1 - 2\varepsilon d_2} \right) \int_{Q_T} |\nabla u|^2 \, dx \, dt \leq \frac{1}{2d_1 - 2\varepsilon d_2} \times A
$$

where

$$
A = \int_{\Omega} u_0^2(x) \, dx + 2 \int_{Q_T} u(x, t) f(u(x, t), v(x, t)) \, dx \, dt
$$

$$
+ \int_{\Omega} v_0^2(x) \, dx + 2 \int_{Q_T} v(x, t) g(u(x, t), v(x, t)) \, dx \, dt
$$
since
\[ \int_{\Omega} u_0^2(x)dx < \infty, \int_{\Omega} v_0^2(x)dx < \infty \]
and
\[ \int_{Q_T} v(x, t)g(u(x, t), v(x, t))dxdt \leq \|v\|_{L^\infty(Q_T)} \int_{Q_T} g(u(x, t), v(x, t))dxdt < \infty \]
and
\[ \int_{Q_T} u(x, t)f(u(x, t), v(x, t))dxdt \leq \|u\|_{L^\infty(Q_T)} \int_{Q_T} f(u(x, t), v(x, t))dxdt < \infty \]
hence (ii)
\[ \int_{Q_T} |\nabla v|^2 dxdt < \infty \]
we have (8), from (ii) has
\[ (2d_1 - 2\varepsilon d_2) \int_{Q_T} |\nabla u|^2 dxdt < \infty, \]
\varepsilon for sufficiently large so that
\[ (2d_1 - 2\varepsilon d_2) > 0 \Rightarrow \varepsilon > \frac{d_1}{d_2} \]
consequently
\[ \int_{Q_T} |\nabla u|^2 dxdt < \infty; \forall T > 0 \]
Then \( u, v \) is globally bounded.

**Proof of theorem.** First we note that
\[ \int_{\Omega} \frac{\partial u}{\partial t}(x, t)dx = d_1 \int_{\Omega} \Delta u dx + d_2 \int_{\Omega} \Delta v dx + \int_{\Omega} f(u(x, t), v(x, t))dx \]
thus
\[ \int_{\Omega} \frac{\partial u}{\partial t}(x, t)dx = \int_{\Omega} f(u(x, t), v(x, t))dx \]
since
\[ \int_{\Omega} \Delta u \, dx = 0 \quad \text{and} \quad \int_{\Omega} \Delta v \, dx = 0 \]
and
\[ \int_{\Omega} \frac{\partial v}{\partial t}(x, t) \, dx = d_3 \int_{\Omega} \Delta u \, dx + d_4 \int_{\Omega} \Delta v \, dx + \int_{\Omega} g(u(x, t), v(x, t)) \, dx \]
then
\[ \int_{\Omega} \frac{\partial v}{\partial t}(x, t) \, dx = \int_{\Omega} g(u(x, t), v(x, t)) \, dx \]
since
\[ \int_{\Omega} \Delta u \, dx = 0 \quad \text{and} \quad \int_{\Omega} \Delta v \, dx = 0 \]
implies
\[ \int_{\Omega} \left( \frac{\partial u}{\partial t}(x, t) + \frac{\partial v}{\partial t}(x, t) \right) \, dx = 0 \quad \text{if} \quad g = -f \]
as
\[ \int_{0}^{t} \int_{\Omega} \left( \frac{\partial u}{\partial t}(x, t) + \frac{\partial v}{\partial t}(x, t) \right) \, dx \, dt = \int_{0}^{t} \int_{\Omega} \left( \frac{\partial u}{\partial t}(x, t) + \frac{\partial v}{\partial t}(x, t) \right) \, dx \, dt \]
\[ = \int_{\Omega} [u(x, t) + v(x, t)]_{0}^{t} \, dx \]
\[ = \int_{\Omega} [u(x, t) + v(x, t)] \, dx \]
\[ - \int_{\Omega} [u_{0}(x) + v_{0}(x)] \, dx = 0 \]
we deduce that
\[ \int_{\Omega} [u(x, t) + v(x, t)] \, dx = \int_{\Omega} [u_{0}(x) + v_{0}(x)] \, dx = 0 \quad (10) \]

Integrating the equation (1) in \( \Omega \) we have
\[ \int_{\Omega} \left( \frac{\partial u}{\partial t}(x, t) \right) \, dx = \int_{\Omega} f(u(x, t), v(x, t)) \, dx > 0 \]
this means that \( \frac{d}{dt} \int_{\Omega} u(x, t) \, dx > 0 \), i.e., the function \( t \to \int_{\Omega} u(x, t) \, dx \) is increasing and \( \Omega \) is bounded then \( t \to \frac{1}{|\Omega|} \int_{\Omega} u(x, t) \, dx \) is increasing and according to the positivity of \( u \) was
\[ \frac{1}{|\Omega|} \int_{\Omega} u(x, t) \, dx \geq 0 \]
therefore
\[ \frac{1}{|\Omega|} \int_{\Omega} u(x, t) \, dx \]
is bounded below and increasing then
\[ \exists \lim_{t \to \infty} \frac{1}{|\Omega|} \int_{\Omega} u(x, t) \, dx = l_1 \]
and the same thing for equation (2) was
\[ \int_{\Omega} \left( \frac{\partial v}{\partial t} (x, t) \right) \, dx = - \int_{\Omega} f(u(x, t), v(x, t)) \, dx < 0 \]
this means that \( \frac{d}{dt} \int_{\Omega} u(x, t) \, dx < 0 \), i.e. \( t \to \int_{\Omega} v(x, t) \, dx \) is decreasing and \( \Omega \) is bounded then
\[ t \to \frac{1}{|\Omega|} \int_{\Omega} v(x, t) \, dx \]
is decreasing.

We know that the solution \( v \) is bounded so
\[ \frac{1}{|\Omega|} \int_{\Omega} v(x, t) \, dx < \infty \]
thus \( \exists \mathcal{M} > 0 \) such as
\[ \frac{1}{|\Omega|} \int_{\Omega} v(x, t) \, dx \leq \mathcal{M} \]
therefore
\[ \frac{1}{|\Omega|} \int_{\Omega} v(x, t) \, dx \]
is bounded above and decreasing then
\[ \exists \lim_{t \to \infty} \frac{1}{|\Omega|} \int_{\Omega} v(x, t) \, dx = l_2. \]

On the other hand, since sets \( \{ u(t), t \geq 0 \} \) and \( \{ v(t), t \geq 0 \} \) are precompacts in \( C(\overline{\Omega}) \) there exists a sequence \( (t_n)_{n=0}, t_n \to \infty \) such that
\[ \lim_{n \to \infty} u(t_n) = u_s \quad \text{in} \quad C(\overline{\Omega}) \]
\[ \lim_{n \to \infty} v(t_n) = v_s \quad \text{in} \quad C(\overline{\Omega}) \]
Now, denote by \( w(u_0, v_0) \) the \( w \)-limite set for \( (u_0, v_0) \) and \( \Phi \) the set of the solution of the elliptic system
\[ \begin{cases} 
-d_1 \Delta u_s - d_2 \Delta v_s = f(u_s, v_s) & \text{in} \quad \mathbb{R}^+ \times \Omega \\
-d_3 \Delta u_s - d_4 \Delta v_s = -f(u_s, v_s) & \text{in} \quad \mathbb{R}^+ \times \Omega \\
\frac{\partial u_s}{\partial \eta} = \frac{\partial v_s}{\partial \eta} = 0 & \text{in} \quad \mathbb{R}^+ \times \partial \Omega 
\end{cases} \]
and prove $\Phi = \{(K_1, K_2)\}$ where $K_1, K_2$ are two constants; in fact.

Adding the equation (11) $d_2 \Delta v_s$ we get

$$-d_1 \Delta u_s = f (u_s, v_s) + d_2 \Delta v_s.$$  

Multiplying this equation by $u_s$ and integrating over $\Omega$ yields

$$-d_1 \int_{\Omega} u_s \Delta u_s dx = \int_{\Omega} u_s f (u_s, v_s) dx + d_2 \int_{\Omega} u_s \Delta v_s dx$$

Apply Green formula:

$$d_1 \int_{\Omega} |\nabla u_s|^2 dx = \int_{\Omega} u_s f (u_s, v_s) dx + d_2 \int_{\Omega} |\nabla u_s| |\nabla v_s| dx.$$  

Adding the equation (12) $d_4 \Delta v$ we get

$$-d_3 \Delta u_s = -f (u_s, v_s) + d_4 \Delta v_s.$$  

Multiplying this equation by $u_s$ and integrating over $\Omega$ yields

$$-d_3 \int_{\Omega} u_s \Delta u_s dx = -\int_{\Omega} u_s f (u_s, v_s) dx + d_4 \int_{\Omega} u_s \Delta v_s dx$$

Apply Green formula:

$$d_3 \int_{\Omega} |\nabla u_s|^2 dx = -\int_{\Omega} u_s f (u_s, v_s) dx + d_4 \int_{\Omega} |\nabla u_s| |\nabla v_s| dx$$

Supposing $d_4 = -d_2$ therefore

$$(d_1 + d_3) \int_{\Omega} |\nabla u_s|^2 dx = 0$$

we deduce that

$$\int_{\Omega} |\nabla u_s|^2 dx = 0 \Rightarrow \nabla u_s = 0 \Rightarrow u_s = K_1 \quad (13)$$

Adding the two equations of (s), multiplying the result by $v_s$ and integrating over $\Omega$ yields

$$-d_1 \int_{\Omega} v_s \Delta u_s dx - d_2 \int_{\Omega} v_s \Delta v_s dx - d_3 \int_{\Omega} v_s \Delta u_s dx - d_4 \int_{\Omega} v_s \Delta v_s dx = 0$$

$$(d_1 - d_3) \int_{\Omega} |\nabla u_s| |\nabla v_s| dx + (-d_2 - d_4) \int_{\Omega} |\nabla v_s|^2 dx = 0$$

therefore

$$(-d_2 - d_4) \int_{\Omega} |\nabla v_s|^2 dx = 0$$
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i.e.

\[ \int_{\Omega} |\nabla v_s|^2 \, dx = 0 \Rightarrow \nabla v_s = 0 \Rightarrow v_s = K_2 \]  

(14)

replacing \( u = K_1, v = K_2 \) in equation (11). It is clear that \( f(K_1, K_2) = 0 \).

Hereafter, we are going to show that

\[ w(u_0, v_0) = \Phi = \{(K_1, K_2)\}. \]

We observe that \( w(u_0, v_0) \neq \emptyset \), because \((u_s, v_s) \in w(u_0, v_0)\).

Now, \( \forall x \in \Omega, \sigma \in ]-1, 1[ \) and let

\[ p_n(x, \sigma) = u(x, t_n + \sigma), q_n(x, \sigma) = v(x, t_n + \sigma) \]

multiply the first equation (1) by \( \frac{\partial u}{\partial t} \)

\[ \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} - d_1 \frac{\partial u}{\partial t} \Delta u - d_2 \frac{\partial u}{\partial t} \Delta v = \frac{\partial u}{\partial t} f(u, v) \]

and integrate over \( \Omega \) we get

\[ \int_{\Omega} \left( \frac{\partial u}{\partial t} \right)^2 \, dx - d_1 \int_{\Omega} \frac{\partial u}{\partial t} \Delta u \, dx - d_2 \int_{\Omega} \frac{\partial u}{\partial t} \Delta v \, dx = \int_{\Omega} \frac{\partial u}{\partial t} f(u, v) \, dx \]

as

\[ \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 = d_1 \int_{\Omega} \frac{\partial u}{\partial t} \Delta u \, dx + d_2 \int_{\Omega} \frac{\partial u}{\partial t} \Delta v \, dx + \int_{\Omega} \frac{\partial u}{\partial t} f(u, v) \, dx \]

integrating result over \((t_0, +\infty)\), we have

\[ \int_{t_0}^{+\infty} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \, dt = d_1 \int_{t_0}^{+\infty} \int_{\Omega} \frac{\partial u}{\partial t} \Delta u \, dx \, dt + d_2 \int_{t_0}^{+\infty} \int_{\Omega} \frac{\partial u}{\partial t} \Delta v \, dx \, dt + \int_{t_0}^{+\infty} \int_{\Omega} \frac{\partial u}{\partial t} f(u, v) \, dx \, dt < \infty \]

thus

\[ \frac{\partial u}{\partial t} \in L^2(t_0, +\infty, L^2(\Omega)) \]
∀σ ∈ ]−1, 1[ we have

\[ p_n(x, \sigma) - u(x, t_n) = u(x, t_n + \sigma) - u(x, t_n) \]

\[ = \int_{t_n}^{t_n+\sigma} \frac{\partial u}{\partial t}(x, t)dt \leq \int_{t_{n-1}}^{t_{n+1}} \frac{\partial u}{\partial t}(x, t)dt \]

(because \( t_{n-1} < t_n \), \( \sigma < 1 \), \( t_n + \sigma < t_n + 1 \))

\[ \leq \left( \int_{t_{n-1}}^{t_{n+1}} (1)^2 dt \right)^{\frac{1}{2}} \left( \int_{t_{n-1}}^{t_{n+1}} \left( \frac{\partial u}{\partial t}(x, t) \right)^2 dt \right)^{\frac{1}{2}} \]

(by inequality of Cauchy Schwartz)

\[ \leq \sqrt{2} \left( \int_{t_{n-1}}^{t_{n+1}} \left( \frac{\partial u}{\partial t}(x, t) \right)^2 dt \right)^{\frac{1}{2}} \]

thus

\[ |p_n(x, \sigma) - u(x, t_n)|^2 = 2 \int_{t_{n-1}}^{t_{n+1}} \left( \frac{\partial u}{\partial t}(x, t) \right)^2 dt \]

integrating the latter inequality \( \Omega \) yields

\[ \int_{\Omega} |p_n(x, \sigma) - u(x, t_n)|^2 dx \leq 2 \int_{\Omega} \int_{t_{n-1}}^{t_{n+1}} \left( \frac{\partial u}{\partial t}(x, t) \right)^2 dt dx \]

we pass to the limit as \( n \to \infty \), we have

\[ \| p_n(x, \sigma) - u_s \|^2_{L^2(\Omega)} \leq 2 \lim_{n\to\infty} \int_{\Omega} \int_{t_{n-1}}^{t_{n+1}} \left( \frac{\partial u}{\partial t}(x, t) \right)^2 dt dx = 0 \]

so

\[ \| p_n(x, \sigma) - u_s \|^2_{L^2(\Omega)} \to 0 \quad n \to \infty \]

As a result we will all \( \sigma \in ]−1, 1[ \),

\[ \| p_n(x, \sigma) - u_s \|^2_{L^2(\Omega)} \to 0, \quad \text{hence} \quad \sup_{-1 < \sigma < 1} \| p_n(x, \sigma) - u_s \|^2_{L^2(\Omega)} \to 0 \]

and by the same method are obtained

\[ \sup_{-1 < \sigma < 1} \| q_n(x, \sigma) - v_s \|^2_{L^2(\Omega)} \to 0 \quad n \to \infty \]

Also, we can have

\[ \sup_{-1 < \sigma < 1} \| \nabla p_n(x, \sigma) - \nabla u_s \|^2_{L^2(\Omega)} \to 0 \quad \text{and} \quad \sup_{-1 < \sigma < 1} \| \nabla q_n(x, \sigma) - \nabla v_s \|^2_{L^2(\Omega)} \to 0 \quad n \to \infty \]
through positivity and boundedness of the solution was

\[ 0 \leq u(x, t_n + \sigma) \leq M \]
\[ 0 \leq v(x, t_n + \sigma) \leq N \]

as \( f \in C^\infty(\mathbb{R}_+) \), we can conclude, using Lebesgue’s theorem, that

\[ f(p_n(x, \sigma), q_n(x, \sigma)) \rightarrow f(u_s, v_s) \]

in \( L^2(\Omega \times (-1, 1)) \) weak.

Now, let \( \xi_i \in C^1(\overline{\Omega}) \) such that \( \xi_i = 0 \) on \( \partial\Omega \) where \( i = 1, 2 \) and let \( \gamma \in C^1(\overline{\Omega}) \) such that \( \text{supp} \gamma \subset [-1, 1], \int_{-1}^{1} \gamma(s)ds = 1 \) and \( \gamma(-1) = \gamma(1) \).

We multiply the first equation (1.1) by \( \gamma(t - t_n)\xi_1 \) and integrate over \((t_n - 1, t_n + 1) \times \Omega\), we obtain

\[
\int_{t_n-1}^{t_n+1} \int_{\Omega} \gamma(t - t_n)\xi_1 \frac{\partial u}{\partial t} dx dt - d_1 \int_{t_n-1}^{t_n+1} \int_{\Omega} \gamma(t - t_n)\xi_1 \Delta u dx dt
\]
\[
- d_2 \int_{t_n-1}^{t_n+1} \int_{\Omega} \gamma(t - t_n)\xi_1 \Delta v dx dt
\]
\[
= \int_{t_n-1}^{t_n+1} \int_{\Omega} \gamma(t - t_n)\xi_1 \gamma(u, v) dx dt \tag{15}
\]

Calculate the integral

\[
\int_{t_n-1}^{t_n+1} \gamma(t - t_n)\xi_1 \frac{\partial u}{\partial t} dt
\]

by part, we find

\[
\int_{t_n-1}^{t_n+1} \gamma(t - t_n)\xi_1 \frac{\partial u}{\partial t} dt = - \int_{t_n-1}^{t_n+1} \gamma(t - t_n)\xi_1 u(x, t) dt \tag{16}
\]

to calculate \( \int_{\Omega} \gamma(t - t_n)\xi_1 \Delta u dx \) using Green’s formula

\[
\int_{\Omega} \gamma(t - t_n)\xi_1 \Delta u dx = \int_{\partial\Omega} \gamma(t - t_n)\xi_1 \frac{\partial u}{\partial \eta} d\sigma - \int_{\Omega} \nabla \gamma(t - t_n)\xi_1 \nabla u dx
\]
\[
= - \int_{\Omega} \nabla \gamma(t - t_n)\xi_1 \nabla u dx
\]

so

\[
\int_{\Omega} \gamma(t - t_n)\xi_1 \Delta u dx = - \int_{\Omega} \nabla \gamma(t - t_n)\xi_1 \nabla u dx \tag{17}
\]
to calculate $\int_{\Omega} \gamma(t - t_n)\xi_1 \Delta vdx$ using Green’s formula

\[
\int_{\Omega} \gamma(t - t_n)\xi_1 \Delta vdx = \int_{\partial\Omega} \gamma(t - t_n)\xi_1 \frac{\partial v}{\partial \eta} d\sigma - \int_{\Omega} \nabla \gamma(t - t_n)\xi_1 \nabla vdx
\]

so

\[
\int_{\Omega} \gamma(t - t_n)\xi_1 \Delta vdx = -\int_{\Omega} \nabla \gamma(t - t_n)\xi_1 \nabla vdx
\] (18)

injected (16) and (17) and (18) in (15) is obtained

\[
- \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \gamma'(t - t_n)\xi_1 u(x,t) dx dt + d_1 \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \nabla \gamma(t - t_n)\xi_1 \nabla u dx dt
\]

\[
+ d_2 \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \nabla \gamma(t - t_n)\xi_1 \nabla v dx dt - \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \gamma(t - t_n)\xi_1 f(u,v) dx dt
\]

\[
= 0
\] (19)

We multiply equation (2) by $\gamma(t - t_n)\xi_2$ and integrating over $(t_n - 1, t_n + 1) \times \Omega$, we have

\[
\int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \gamma(t - t_n)\xi_2 \frac{\partial v}{\partial t} dx dt - d_3 \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \gamma(t - t_n)\xi_2 \Delta u dx dt
\]

\[
- d_4 \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \gamma(t - t_n)\xi_2 \Delta v dx dt
\]

\[
= - \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \gamma(t - t_n)\xi_2 f(u,v) dx dt
\] (20)

Calculate the integral

\[
\int_{t_{n-1}}^{t_{n+1}} \gamma(t - t_n)\xi_2 \frac{\partial v}{\partial t} dt
\]

by part, we find

\[
\int_{t_{n-1}}^{t_{n+1}} \gamma(t - t_n)\xi_2 \frac{\partial v}{\partial t} dt = - \int_{t_{n-1}}^{t_{n+1}} \gamma'(t - t_n)\xi_2 v(x,t) dt
\] (21)

to calculate $\int_{\Omega} \gamma(t - t_n)\xi_2 \Delta u dx$ using Green’s formula

\[
\int_{\Omega} \gamma(t - t_n)\xi_2 \Delta u dx = \int_{\partial\Omega} \gamma(t - t_n)\xi_2 \frac{\partial u}{\partial \eta} d\sigma - \int_{\Omega} \nabla \gamma(t - t_n)\xi_2 \nabla u dx
\]

\[
= - \int_{\Omega} \nabla \gamma(t - t_n)\xi_2 \nabla u dx
\]
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\[
\int_{\Omega} \gamma(t - t_n) \xi_2 \Delta u \, dx = - \int_{\Omega} \nabla \gamma(t - t_n) \xi_2 \nabla u \, dx \tag{22}
\]

to calculate \( \int_{\Omega} \gamma(t - t_n) \xi_2 \Delta v \, dx \) using Green's formula

\[
\int_{\Omega} \gamma(t - t_n) \xi_2 \Delta v \, dx = \int_{\partial \Omega} \gamma(t - t_n) \xi_2 \frac{\partial v}{\partial \eta} \, d\sigma - \int_{\Omega} \nabla \gamma(t - t_n) \xi_2 \nabla v \, dx
\]

so

\[
\int_{\Omega} \gamma(t - t_n) \xi_2 \Delta v \, dx = - \int_{\Omega} \nabla \gamma(t - t_n) \xi_2 \nabla v \, dx
\]

so

\[
\int_{\Omega} \gamma(t - t_n) \xi_1 \Delta v \, dx = - \int_{\Omega} \nabla \gamma(t - t_n) \xi_1 \nabla v \, dx. \tag{23}
\]

Injected (21) and (22) and (23) in (20) is obtained

\[
\begin{align*}
&- \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \gamma'(t - t_n) \xi_2 v(x, t) \, dx \, dt + d_3 \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \nabla \gamma(t - t_n) \xi_2 \nabla u \, dx \, dt \\
&+ d_4 \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \nabla \gamma(t - t_n) \xi_2 \nabla v \, dx \, dt + \int_{t_{n-1}}^{t_{n+1}} \int_{\Omega} \gamma(t - t_n) \xi_2 f(u, v) \, dx \, dt \\
&= 0 \tag{24}
\end{align*}
\]

By making the following change of variable

\[
\begin{pmatrix}
\sigma = t - t_n \\
\sigma = t_n - 1 \\
\sigma = t_n + 1
\end{pmatrix} 
\begin{pmatrix}
d\sigma = dt \\
\sigma = -1 \\
\sigma = 1
\end{pmatrix}
\]

therefore the integral (19) becomes

\[
\begin{align*}
&\int_{-1}^{+1} \int_{\Omega} \gamma'(\sigma) \xi_1 p_n(x, \sigma) \, dx \, d\sigma - d_1 \int_{-1}^{+1} \int_{\Omega} \nabla \gamma(\sigma) \xi_1 \nabla p_n(x, \sigma) \, dx \, d\sigma \\
&- d_2 \int_{-1}^{+1} \int_{\Omega} \nabla \gamma(\sigma) \xi_1 \nabla q_n(x, \sigma) \, dx \, dt \\
&+ \int_{-1}^{+1} \int_{\Omega} \gamma(\sigma) \xi_1 f(p_n(x, \sigma), q_n(x, \sigma)) \, dx \, dt \\
&= 0 \tag{25}
\end{align*}
\]
The same applies to the integral (24)

\[
\int_{-1}^{+1} \int_{\Omega} \gamma'(\sigma) \xi_2 q_n(x, \sigma) dx d\sigma - d_3 \int_{-1}^{+1} \int_{\Omega} \nabla \gamma(\sigma) \xi_2 \nabla p_n(x, \sigma) dx d\sigma \\
- d_4 \int_{-1}^{+1} \int_{\Omega} \nabla \gamma(\sigma) \xi_2 \nabla q_n(x, \sigma) dx dt \\
- \int_{-1}^{+1} \int_{\Omega} \gamma(\sigma) \xi_2 f(p_n(x, \sigma), q_n(x, \sigma)) dx dt \\
= 0 \quad (26)
\]

Using Lesbegue’s theorem we have

\[
\lim_{n \to \infty} \int_{-1}^{+1} \int_{\Omega} \gamma'(\sigma) \xi_1 p_n(x, \sigma) dx d\sigma \\
= \int_{-1}^{+1} \int_{\Omega} \gamma'(\sigma) \xi_1 u_s dx d\sigma = \int_{-1}^{+1} \gamma'(\sigma) d\sigma \int_{\Omega} \xi_1 u_s dx \\
= [\gamma(\sigma)]_{-1}^{+1} \int_{\Omega} \xi_1 u_s dx = 0 \text{ because } \gamma(1) = \gamma(-1)
\]

and same method for

\[
\lim_{n \to \infty} \int_{-1}^{+1} \int_{\Omega} \gamma'(\sigma) \xi_2 q_n(x, \sigma) dx dt \\
= \int_{-1}^{+1} \int_{\Omega} \gamma'(\sigma) \xi_2 v_s dx d\sigma = \int_{-1}^{+1} \gamma'(\sigma) d\sigma \int_{\Omega} \xi_2 v_s dx \\
= [\gamma(\sigma)]_{-1}^{+1} \int_{\Omega} \xi_2 v_s dx = 0 \text{ because } \gamma(1) = \gamma(-1)
\]

from inequality (25), we have

\[
-d_1 \int_{\Omega} \nabla \xi_1 \nabla u_s - d_2 \int_{\Omega} \nabla \xi_1 \nabla v_s + \int_{\Omega} \xi_1 f(u_s, v_s) dx = 0
\]

and inequality (26) yields

\[
-d_3 \int_{\Omega} \nabla \xi_2 \nabla u_s - d_4 \int_{\Omega} \nabla \xi_2 \nabla v_s - \int_{\Omega} \xi_2 f(u_s, v_s) dx = 0
\]

But this form it is the same when we multiply (8) by \( \xi_1 \) and (9) by \( \xi_2 \) and integrating over \( \Omega \) hence \( w = \Phi \). Finally, combining (13) and (14) with (10) yields

\[
\int_{\Omega} (K_1 + K_2) dx = \int_{\Omega} (u_0 + v_0) dx \\
(K_1 + K_2) |\Omega| = \int_{\Omega} (u_0 + v_0) dx
\]
i.e.

\[ K_1 + K_2 = \frac{1}{|\Omega|} \int_{\Omega} (u_0 + v_0) \, dx \]

the proof of the theorem is complete.

References


