Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 3 (2016), pp. 2175–2194

http://www.ripublication.com/gjpam.htm

Behavior asymptotic solution of reaction diffusion system with full matrix

Moumeni Abdelkader and Mebarki Maroua

Departement of mathematics,
University Badji Mokhtar Annaba - Algeria,
Laboratoire de Mathématiques,
Dynamique et Modélisation B.P. 12,
Annaba 23000.

Abstract

The purpose of this paper is to prove that behavior asymptotic in the time of solutions for the strongly coupled reaction diffusion system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u - d_2 \Delta v = f(u, v) & in\ \mathbb{R}^+ \times \Omega \\ \dfrac{\partial u}{\partial t} - d_3 \Delta u - d_4 \Delta v = g(u, v) & in \mathbb{R}^+ \times \Omega \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0 & in \mathbb{R}^+ \times \partial\Omega \\ u(.,0) = u_0(.), v(.,0) = v_0(.) & in \Omega \end{cases} \qquad \text{(SRD)}$$

with full matrix of diffusion coefficients.

We treat first second equations of SRD as a dynamical system in $C(\overline{\Omega}) \times C(\overline{\Omega})$ and apply Lyapunov type stability techniques. A key ingredient in this analysis is a result which establishes that the orbits of the dynamical system are precompact in $C(\overline{\Omega}) \times C(\overline{\Omega})$. As a consequence of Arzela-Ascoli theorem, this will be satisfied if the orbits are, for example, uniformly bounded in $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ for $t > 0$.

AMS subject classification: 35K57, 35B40, 35B45.
Keywords: Reaction diffusion system, semi group, global existence, asymptotic behavior.

1. Introduction

In this paper we study the following semilinear parabolic system

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u - d_2 \Delta v = f(u, v) & in \mathbb{R}^+ \times \Omega \\ \dfrac{\partial u}{\partial t} - d_3 \Delta u - d_4 \Delta v = g(u, v) & in \mathbb{R}^+ \times \Omega \end{cases} \tag{1}$$

Where Ω is a regular and bounded domain of R^n, $(n \geq 1)$, $u = u(t, x)$, $v = v(t, x)$, $x \in \Omega$, $t > 0$ are real valued functions, Δ denotes the Laplacian operator, and the constants of diffusion d_1, d_2, d_3, d_4 are assumed to be nonnegative. Also $(d_1 + d_4)^2 \prec 4d_2d_3$ which reflects the parabolicity of the system.

System (1) is subjected to the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \qquad in \mathbb{R}^+ \times \partial\Omega \tag{2}$$

and the initial data

$$u(., 0) = u_0(.), v(., 0) = v_0(.) \qquad in \Omega \tag{3}$$

which are assumed to be nonnegative.

Concerning the functions f and g, we assume the following hypothesis:

(**H1**) $f(r, s)$ and $g(r, s)$ are continuously differentiable on $R^+ \times R^+$, such that $f(0, s) \geq 0$, $g(r, 0) \geq 0 \forall r, s \geq 0$

We know that the problem (1)–(3) has a unique global solution see [17]. The main question we want to address is asymptotic behavior the solutions for system (1)–(3). In fact the subject of the asymptotic behavior of reaction diffusion systems has received a lot of attention in the last decades and several outstanding results have been proved by some of the major experts in the field.

This question has been investigated by many authors by considering special forms of the nonlinear terms f and g.

In the case where diagonal $d_1 \neq d_4$ and $d_3 = d_2 = 0$, $\Psi(v) \in C^1\left(R^+\right)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = f(u, v) = -u\Psi(v) & \text{in } \mathbb{R}^+ \times \Omega \\ \dfrac{\partial v}{\partial t} - d_4 \Delta v = g(u, v) = u\Psi(v) & \text{in } \mathbb{R}^+ \times \Omega \\ \lambda_1 \dfrac{\partial u}{\partial \eta} + (1 - \lambda_1)u = 0 & \text{in } \mathbb{R}^+ \times \partial\Omega \\ \lambda_2 \dfrac{\partial v}{\partial \eta} + (1 - \lambda_2)v = 0 & \text{in } \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{p}$$

Note that, the behavior of non-negative total solutions (p) is treated in the papers [12, 16]. In particular, it is proved that there are two constants nonegative u^*, v^* such that $\left\|u(t)-u^*\right\|_\infty \to 0$ $\left\|v(t)-v^*\right\|_\infty \to 0$ and $u^*\Psi(v^*) = 0$ it is obvious that $u^* = v^*$ where u and v have not Neumann boundary conditions. A. Barabanova [7] generalize the method of Haraux and youkana where $\Psi(v) = e^{\alpha v}$ (method of estimating energy) and then Rebiai Benachour [18] treated the same case with conditions more strong

$$\begin{cases} f(0,\eta) = g(0,\eta) = 0 \\ g(\zeta,0) \succeq 0, g(\zeta,\eta) \preceq \Psi(\eta) f(\zeta,\eta) \\ g(\zeta,\eta) \preceq C\Phi(\zeta)e^{\alpha\eta^\beta} \quad C \succ 0 \text{ and } \alpha \succ 0, \Phi \in C^1\left(R^+\right) \text{ and } \Phi(0) = 0 \end{cases}$$

where Ψ is a continuously differentiable function such that there is a constant $\beta \succeq 1$ $\lim\limits_{\eta\to+\infty} \eta^{\beta-1}\Psi(\eta) = l$ where $l \succ 0$. Collet and Xin [10] have studied the same system (p) with the boundary conditions on R^n with a diagonal diffusion matrix and $\Psi(v) = v^m$, where $m \in N^*$. They proved the existence of global solutions and showed that the norm L^∞ of v can not grow faster than $O(lnt)$. In addition, the system has been studied by Avrin [2] if $d_3 = d_2 = 0$, $v = exp\{-E/v\}$, $E > 0$ and the space is variable R.

In the case of a triangular system according to:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = f(u,v) = -u\Psi(v) & \text{in } \mathbb{R}^+ \times \Omega \\ \dfrac{\partial u}{\partial t} - d_3\Delta u - d_4\Delta v = g(u,v) = u\Psi(v) & \text{in } \mathbb{R}^+ \times \Omega \end{cases}$$

in the case of unbounded domain and $\Psi(v) = v^m$ is studied by Badraoui in [4,5]. In [5], he showed the existence of comprehensive solution if $v_0(x) \succeq \dfrac{d_3}{d_1 - d_4}u_0(x)$ and $d_1 > d_4$, $d_3 > 0$, where $d_1 < 0$, $d_3 < 0$. In [5], he proved that the norm L^∞ of v can not grow faster than $O(lnt)$ in the case of bounded domain, Kirane [13] studied the asymptotic behavior using the method of the invariant region and semi group theory

Kouachi [15] obtained a result on uniform boundedness of solutions for a system like (SRD) with an overall complete matrix of diffusion coefficients satisfy the law of equilibrium. This result is generalized by Kouachi after [14] who used the notion of invariant regions and Lyapunov functional. Curiously, less attention was paid to the behavior of the solutions when the spatial variable x approaches infinity despite the usefulness of this type of train for the digital processing of these problems. We are not aware of the article Gladnov [11], which generalizes a result of behavior x tends to infinity with a semi-linear equation posed in R^+ studied by Beberns and Fulks [8].

The remainder of this document is divided into two sections. Section 2 presents some material that we will use in this paper and in Section 3, we state and prove the main result.

2. Notations and preliminary

We put in what follows $G(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ we note that $\{G(t), t \succeq 0\}$ is a semi group nonlinear in the compact metric space X

$$O(u_0, v_0) = \{G(t)(u_0, v_0), t \succeq 0\}$$

is the orbit (u_0, v_0).

First recall the following definition

Definition 2.1. All **w-limite** is defined by

$$w(u_0, v_0) = \{(u, v) \in X : \exists t_n \to \infty : G(t_n) \to (u, v)\}.$$

Inequalite

We know that

$$\left(\sqrt{2\varepsilon}a + \frac{b}{\sqrt{2\varepsilon}}\right)^2 \succeq 0$$

so

$$2\varepsilon a^2 + \frac{b^2}{2\varepsilon} \succeq -2ab$$

thus

$$-ab \preceq \varepsilon a^2 + \frac{b^2}{4\varepsilon} \tag{4}$$

we have

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right), \nabla v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \ldots, \frac{\partial v}{\partial x_n}\right)$$

$$|\nabla u| = \sqrt{\sum_{i=1}^{n}(\frac{\partial u}{\partial x_i})^2 et \nabla u \nabla v} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$$

then

$$\int_{Q_T} |\nabla u| \, |\nabla v| \, dxdt = \sum_{i=1}^{n} \int_{Q_T} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dxdt$$

replacing $a = \dfrac{\partial u}{\partial x_i}$ $b = \dfrac{\partial v}{\partial x_i}$ in (4) is obtained

$$\begin{aligned} \sum_{i=1}^{n} \int_{Q_T} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dxdt &\succeq \sum_{i=1}^{n} \int_{Q_T} \left(-\varepsilon\left(\frac{\partial u}{\partial x_i}\right)^2 - \frac{1}{4\varepsilon}\left(\frac{\partial v}{\partial x_i}\right)^2\right)\left(\frac{\partial v}{\partial x_i}\right)^2 dxdt \\ &= -\varepsilon \sum_{i=1}^{n} \int_{Q_T} \left(\frac{\partial u}{\partial x_i}\right)^2 dxdt - \frac{1}{4\varepsilon} \sum_{i=1}^{n} \int_{Q_T} \left(\frac{\partial v}{\partial x_i}\right)^2 dxdt \\ &= -\varepsilon \int_{Q_T} |\nabla u|^2 \, dxdt - \frac{1}{4\varepsilon} \int_{Q_T} |\nabla v|^2 \, dxdt \end{aligned}$$

so

$$\int_{Q_T} |\nabla u| \, |\nabla v| \, dxdt \succeq -\varepsilon \int_{Q_T} |\nabla u|^2 \, dxdt - \frac{1}{4\varepsilon} \int_{Q_T} |\nabla v|^2 \, dxdt \tag{5}$$

3. Main result

Theorem 3.1. The solution $w = (u, v)$ of the system (1)-(3) converges a constant vector of the form $K = (K_1, K_2)$ as $t \to \infty$, uniformly in $\overline{\Omega}$ i.e.

$$\begin{pmatrix} u \underset{t\to\infty}{\to} K_1 \\ v \underset{t\to\infty}{\to} K_2 \end{pmatrix}$$

Furthermore, we have: $K_1 \succeq 0$, $K_2 \succeq 0$, $f\,(K_1, K_2) = 0$ and

$$K_1 + K_2 = \frac{1}{\overline{\Omega}} \int_{\Omega} (u_0(x) + v_0(x)) \, dx.$$

The following lemma is a useful tool in the proof of the Theorem.

Lemma 3.2. Let (u, v) be a solution of (1)–(3). We have

(i) $\displaystyle\int_{Q_T} |\nabla u|^2 \, dxdt \prec \infty$

(ii) $\displaystyle\int_{Q_T} |\nabla v|^2 \, dxdt \prec \infty$ here $Q_T = \Omega \times [0, T]$ and $0 \prec T \prec \infty$.

Proof. We have

$$\frac{\partial u}{\partial t} - d_1 \Delta u - d_2 \Delta v = f\,(u, v)\,,$$

by integrating over $(0, T)$ is obtained

$$\int_0^T \frac{\partial u}{\partial t}(x, t)dt = d_1 \int_0^T \Delta u dt + d_2 \int_0^T \Delta v dt + \int_0^T f(u(x, t), v(x, t))dt$$

$$u(x, T) - u(x, 0) = d_1 \int_0^T \Delta u dt + d_2 \int_0^T \Delta v dt + \int_0^T f(u(x, t), v(x, t))dt$$

and integrating a second time is obtained over Ω

$$\int_{\Omega} u(x, T)dx - \int_{\Omega} u(x, 0)dx = d_1 \int_{\Omega}\int_0^T \Delta u dt dx + d_2 \int_{\Omega}\int_0^T \Delta v dt dx + \int_{\Omega}\int_0^T f(u(x, t), v(x, t))dt dx.$$

Green's formula is applied to $\int_{\Omega} \Delta u dx$ and $\int_{\Omega} \Delta v dx$, we have

$$\int_{\Omega} \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} d\sigma - \int_{\Omega} \nabla u \nabla 1 dx \qquad \text{therefore } \int_{\Omega} \Delta u dx = 0$$

$$\int_{\Omega} \Delta v dx = \int_{\partial\Omega} \frac{\partial v}{\partial \eta} d\sigma - \int_{\Omega} \nabla v \nabla 1 dx \qquad \text{therefore } \int_{\Omega} \Delta v dx = 0$$

thus

$$\int_{\Omega} \int_{0}^{T} f(u(x,t), v(x,t)) dt dx = \int_{\Omega} u_0(x) dx - \int_{\Omega} u(x,T) dx \prec \infty$$

because $u(T) \in C(\overline{\Omega})$ i.e.

$$\int_{Q_T} f(u(x,t), v(x,t)) dt dx \prec \infty$$

In the same way $\frac{\partial v}{\partial t} - d_3 \Delta u - d_4 \Delta v = g\,(u, v)$ we get that

$$\int_{Q_T} g(u(x,t), v(x,t)) dt dx \prec \infty$$

Multiply now the equation $\frac{\partial u}{\partial t} - d_1 \Delta u - d_2 \Delta v = f\,(u, v)$ by u and integrating over Q_T we have

$$\int_{\Omega} \int_{0}^{T} u \frac{\partial u}{\partial t}(x,t) dt dx = d_1 \int_{\Omega} \int_{0}^{T} u \Delta u dt dx + d_2 \int_{\Omega} \int_{0}^{T} u \Delta v dt dx + \int_{\Omega} \int_{0}^{T} u f(u(x,t), v(x,t)) dt dx$$

by using the Green formula

$$\int_{\Omega} u \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} d\sigma - \int_{\Omega} |\nabla u|^2 dx, \quad \text{therefore } \int_{\Omega} u \Delta u dx = - \int_{\Omega} |\nabla u|^2 dx$$

and a simple calculation, it becomes

$$\frac{1}{2} \int_{\Omega} [u^2(x,t)]_0^T dx = -d_1 \int_{0}^{T} \int_{\Omega} |\nabla u|^2 dx dt - d_2 \int_{0}^{T} \int_{\Omega} |\nabla u| \, |\nabla v| \, dx dt + \int_{0}^{T} \int_{\Omega} u(x,t) f(u(x,t), v(x,t)) dx dt$$

i.e.

$$\int_\Omega u^2(x,T) + 2d_1 \int_{Q_T} |\nabla u|^2 \, dxdt + 2d_2 \int_{Q_T} |\nabla u| \, |\nabla v| \, dxdt$$
$$= \int_\Omega u_0^2(x)dx + 2 \int_{Q_T} u(x,t) f(u(x,t), v(x,t))dxdt$$

consequently

$$2d_1 \int_{Q_T} |\nabla u|^2 \, dxdt + 2d_2 \int_{Q_T} |\nabla u| \, |\nabla v| \, dxdt \preceq \int_\Omega u_0^2(x)dx \tag{6}$$
$$+2 \int_{Q_T} u(x,t) f(u(x,t), v(x,t))dxdt$$

Multiply the equation $\frac{\partial v}{\partial t} - d_3 \Delta u - d_4 \Delta v = g\,(u, v)$ by v and integrating over Q_T is obtained

$$\int_\Omega \int_0^T v \frac{\partial v}{\partial t}(x,t)dtdx = d_3 \int_\Omega \int_0^T v \Delta u dtdx + d_4 \int_\Omega \int_0^T v \Delta v dtdx$$
$$+ \int_\Omega \int_0^T v g(u(x,t), v(x,t))dtdx$$

by using the Green formula

$$\int_\Omega v \Delta v dx = \int_{\partial\Omega} \frac{\partial v}{\partial \eta} d\sigma - \int_\Omega |\nabla v|^2 \, dx, \quad \text{therfore} \int_\Omega v \Delta v dx = - \int_\Omega |\nabla v|^2 \, dx$$

therefore

$$\frac{1}{2} \int_\Omega [v^2(x,t)]_0^T dx = -d_3 \int_0^T \int_\Omega |\nabla u| \, |\nabla v| \, dxdt - d_4 \int_0^T \int_\Omega |\nabla v|^2 \, dxdt$$
$$+ \int_0^T \int_\Omega v(x,t) g(u(x,t), v(x,t))dxdt$$

$$\int_\Omega v^2(x,T) + 2d_3 \int_{Q_T} |\nabla u| \, |\nabla v| \, dxdt + 2d_4 \int_{Q_T} |\nabla v|^2 \, dxdt$$
$$= \int_\Omega v_0^2(x)dx + 2 \int_{Q_T} v(x,t) g(u(x,t), v(x,t))dxdt$$

consequently

$$2d_3 \int_{Q_T} |\nabla u| \, |\nabla v| \, dxdt + 2d_4 \int_{Q_T} |\nabla v|^2 \, dxdt \preceq \int_\Omega v_0^2(x)dx \tag{7}$$
$$+2 \int_{Q_T} v(x,t) g(u(x,t), v(x,t))dxdt$$

Replacing (5) in the inequality (4) is found

$$\begin{aligned} & 2d_1 \int_{Q_T} |\nabla u|^2 \, dxdt + 2d_2 \left(-\varepsilon \int_{Q_T} |\nabla u|^2 \, dxdt - \frac{1}{4\varepsilon} \int_{Q_T} |\nabla v|^2 \, dxdt \right) \\ \preceq \; & 2d_1 \int_{Q_T} |\nabla u|^2 \, dxdt + 2d_2 \int_{Q_T} |\nabla u| \, |\nabla v| \, dxdt \\ \preceq \; & \int_{\Omega} u_0^2(x) dx + 2 \int_{Q_T} u(x,t) f(u(x,t), v(x,t)) dxdt \qquad \text{because } (d_2 \succeq 0) \end{aligned}$$

thus

$$\begin{aligned} & (2d_1 - 2\varepsilon d_2) \int_{Q_T} |\nabla u|^2 \, dxdt \preceq \frac{d_2}{2\varepsilon} \int_{Q_T} |\nabla v|^2 \, dxdt \\ & + \int_{\Omega} u_0^2(x) dx + 2 \int_{Q_T} u(x,t) f(u(x,t), v(x,t)) dxdt \end{aligned} \tag{8}$$

Replacing (5) in the inequality (7) is found

$$\begin{aligned} & 2d_4 \int_{Q_T} |\nabla v|^2 \, dxdt + 2d_3 \left(-\varepsilon \int_{Q_T} |\nabla u|^2 \, dxdt - \frac{1}{4\varepsilon} \int_{Q_T} |\nabla v|^2 \, dxdt \right) \\ \preceq \; & 2d_4 \int_{Q_T} |\nabla v|^2 \, dxdt + 2d_3 \int_{Q_T} |\nabla u| \, |\nabla v| \, dxdt \\ \preceq \; & \int_{\Omega} v_0^2(x) dx + 2 \int_{Q_T} v(x,t) g(u(x,t), v(x,t)) dxdt \qquad \text{because } (d_3 \succeq 0) \end{aligned}$$

so

$$\begin{aligned} \left(2d_4 - \frac{d_3}{2\varepsilon} \right) \int_{Q_T} |\nabla v|^2 \, dxdt \; \preceq \; & 2d_3 \varepsilon \int_{Q_T} |\nabla u|^2 \, dxdt + \int_{\Omega} v_0^2(x) dx \\ & + 2 \int_{Q_T} v(x,t) g(u(x,t), v(x,t)) dxdt \end{aligned} \tag{9}$$

We replace (9) in (8):

$$\left(2d_4 - \frac{d_3}{2\varepsilon} - \frac{d_2 d_3}{2d_1 - 2\varepsilon d_2} \right) \int_{Q_T} |\nabla u|^2 \, dxdt \preceq \frac{1}{2d_1 - 2\varepsilon d_2} \times A$$

where

$$\begin{aligned} A \; = \; & \int_{\Omega} u_0^2(x) dx + 2 \int_{Q_T} u(x,t) f(u(x,t), v(x,t)) dxdt \\ & + \int_{\Omega} v_0^2(x) dx + 2 \int_{Q_T} v(x,t) g(u(x,t), v(x,t)) dxdt \end{aligned}$$

since

$$\int_\Omega u_0^2(x)dx \prec \infty, \int_\Omega v_0^2(x)dx \prec \infty$$

and

$$\int_{Q_T} v(x,t)g(u(x,t), v(x,t))dxdt$$
$$\preceq \|v\|_{L^\infty(Q_T)} \int_{Q_T} g(u(x,t), v(x,t))dxdt \prec \infty$$

and

$$\int_{Q_T} u(x,t)f(u(x,t), v(x,t))dxdt$$
$$\preceq \|u\|_{L^\infty(Q_T)} \int_{Q_T} f(u(x,t), v(x,t))dxdt \prec \infty$$

hence (ii)

$$\int_{Q_T} |\nabla v|^2 \, dxdt \prec \infty$$

we have (8), from (ii) has

$$(2d_1 - 2\varepsilon d_2) \int_{Q_T} |\nabla u|^2 \, dxdt \prec \infty,$$

ε for sufficiently large so that

$$(2d_1 - 2\varepsilon d_2) \succ 0 \Rightarrow \varepsilon \succ \frac{d_1}{d_2}$$

consequently

$$\int_{Q_T} |\nabla u|^2 \, dxdt \prec \infty; \forall T \succ 0$$

Then u, v is globally bounded.

Proof of theorem. First we note that

$$\int_\Omega \frac{\partial u}{\partial t}(x,t)dx = d_1 \int_\Omega \Delta u dx + d_2 \int_\Omega \Delta v dx + \int_\Omega f(u(x,t), v(x,t))dx$$

thus

$$\int_\Omega \frac{\partial u}{\partial t}(x,t)dx = \int_\Omega f(u(x,t), v(x,t))dx$$

since

$$\int_\Omega \Delta u dx = 0 \text{ and } \int_\Omega \Delta v dx = 0$$

and

$$\int_\Omega \frac{\partial v}{\partial t}(x,t)dx = d_3 \int_\Omega \Delta u dx + d_4 \int_\Omega \Delta v dx + \int_\Omega g(u(x,t), v(x,t))dx$$

then

$$\int_\Omega \frac{\partial v}{\partial t}(x,t)dx = \int_\Omega g(u(x,t), v(x,t))dx$$

since

$$\int_\Omega \Delta u dx = 0 \text{ and } \int_\Omega \Delta v dx = 0$$

impliying

$$\int_\Omega \left(\frac{\partial u}{\partial t}(x,t) + \frac{\partial v}{\partial t}(x,t) \right) dx = 0 \quad \text{if } g = -f$$

as

$$\begin{aligned} \int_0^t \int_\Omega \left(\frac{\partial u}{\partial t}(x,t) + \frac{\partial v}{\partial t}(x,t) \right) dx &= \int_\Omega \int_0^t \left(\frac{\partial u}{\partial t}(x,t) + \frac{\partial v}{\partial t}(x,t) \right) dt dx \\ &= \int_\Omega [u(x,t) + v(x,t)]_0^t dx \\ &= \int_\Omega [u(x,t) + v(x,t)] dx \\ &\quad - \int_\Omega [u_0(x) + v_0(x)] dx = 0 \end{aligned}$$

we deduce that

$$\int_\Omega [u(x,t) + v(x,t)]dx = \int_\Omega [u_0(x) + v_0(x)]dx = 0 \tag{10}$$

Integrating the equation (1) in Ω we have

$$\int_\Omega (\frac{\partial u}{\partial t}(x,t)dx = \int_\Omega f(u(x,t), v(x,t))dx \succ 0$$

this means that $\frac{d}{dt} \int_\Omega u(x,t)dx \succ 0$, ie the fonction $t \to \int_\Omega u(x,t)dx$ is increasing and Ω is bounded then $t \to \frac{1}{|\Omega|} \int_\Omega u(x,t)dx$ is increasing and according to the positivity of u was

$$\frac{1}{|\Omega|} \int_\Omega u(x,t)dx \geq 0$$

therefore

$$\frac{1}{|\Omega|}\int_{\Omega} u(x,t)dx$$

is bounded below and increasing then

$$\exists \lim_{t\to\infty}\frac{1}{|\Omega|}\int_{\Omega} u(x,t)dx = l_1$$

and the same thing for equation (2) was

$$\int_{\Omega}\left(\frac{\partial v}{\partial t}(x,t)dx = -\int_{\Omega} f(u(x,t),v(x,t)\right)dx \prec 0$$

this means that $\frac{d}{dt}\int_{\Omega} u(x,t)dx \prec 0$, i.e. $t \to \int_{\Omega} v(x,t)dx$ is decreasing and Ω is bounded then

$$t \to \frac{1}{|\Omega|}\int_{\Omega} v(x,t)dx$$

is decreasing.

We know that the solution v is bounded so

$$\frac{1}{|\Omega|}\int_{\Omega} v(x,t)dx \prec \infty$$

thus $\exists M \succ 0$ such as

$$\frac{1}{|\Omega|}\int_{\Omega} v(x,t)dx \preceq M$$

therefore

$$\frac{1}{|\Omega|}\int_{\Omega} v(x,t)dx$$

is bounded above and decreasing then

$$\exists \lim_{t\to\infty}\frac{1}{|\Omega|}\int_{\Omega} v(x,t)dx = l_2.$$

On the other hand, since sets $\{u(t), t \succeq 0\}$ and $\{v(t), t \succeq 0\}$ are precompacts in $C(\overline{\Omega})$ there exists a sequence $(t_n)_{n\geq 0}$, $t_n \to \infty$ such that

$$\begin{array}{rcll} \lim\limits_{n\to\infty} u(t_n) &=& u_s & \text{in} \quad C(\overline{\Omega}) \\ \lim\limits_{n\to\infty} v(t_n) &=& v_s & \text{in} \quad C(\overline{\Omega}) \end{array}$$

Now, denote by $w(u_0, v_0)$ the w-limite set for (u_0, v_0) and Φ the set of the solution of the elliptic system

$$\left\{\begin{array}{lll} -d_1\Delta u_s - d_2\Delta v_s = f\,(u_s, v_s) & in\ \mathbb{R}^+\times\Omega & (11) \\ -d_3\Delta u_s - d_4\Delta v_s = -f\,(u_s, v_s) & in\ \mathbb{R}^+\times\Omega & (12) \\ \dfrac{\partial u_s}{\partial \eta} = \dfrac{\partial v_s}{\partial \eta} = 0 & in\ \mathbb{R}^+\times\partial\Omega & \end{array}\right. \qquad \text{(s)}$$

and prove $\Phi = \{(K_1, K_2)\}$ where K_1, K_2 are two constants; in fact.

Adding the equation (11) $d_2 \Delta v_s$ we get

$$-d_1 \Delta u_s = f(u_s, v_s) + d_2 \Delta v_s.$$

Multipliying this equation by u_s and integrating over Ω yields

$$-d_1 \int_\Omega u_s \Delta u_s dx = \int_\Omega u_s f(u_s, v_s)\, dx + d_2 \int_\Omega u_s \Delta v_s dx$$

Apply Green formular:

$$d_1 \int_\Omega |\nabla u_s|^2\, dx = \int_\Omega u_s f(u_s, v_s)\, dx + d_2 \int_\Omega |\nabla u_s| \, |\nabla v_s|\, dx.$$

Adding the equation (12) $d_4 \Delta v$ we get

$$-d_3 \Delta u_s = -f(u_s, v_s) + d_4 \Delta v_s.$$

Multipliying this equation by u_s and integrating over Ω yields

$$-d_3 \int_\Omega u_s \Delta u_s dx = -\int_\Omega u_s f(u_s, v_s)\, dx + d_4 \int_\Omega u_s \Delta v_s dx$$

Apply Green formular:

$$d_3 \int_\Omega |\nabla u_s|^2\, dx = -\int_\Omega u_s f(u_s, v_s)\, dx + d_4 \int_\Omega |\nabla u_s| \, |\nabla v_s|\, dx$$

Supposing $d_4 = -d_2$ therefore

$$(d_1 + d_3) \int_\Omega |\nabla u_s|^2\, dx = 0$$

we deduce that

$$\int_\Omega |\nabla u_s|^2\, dx = 0 \Rightarrow \nabla u_s = 0 \Rightarrow u_s = K_1 \tag{13}$$

Adding the two equations of (s), multiplying the result by v_s and integrating over Ω yields

$$-d_1 \int_\Omega v_s \Delta u_s dx - d_2 \int_\Omega v_s \Delta v_s dx - d_3 \int_\Omega v_s \Delta u_s dx - d_4 \int_\Omega v_s \Delta v_s dx = 0$$

$$(-d_1 - d_3) \int_\Omega |\nabla u_s| \, |\nabla v_s|\, dx + (-d_2 - d_4) \int_\Omega |\nabla v_s|^2\, dx = 0$$

therefore

$$(-d_2 - d_4) \int_\Omega |\nabla v_s|^2\, dx = 0$$

i.e.

$$\int_{\Omega} |\nabla v_s|^2 \, dx = 0 \Rightarrow \nabla v_s = 0 \Rightarrow v_s = K_2 \tag{14}$$

replacing $u = K_1$, $v = K_2$ in equation (11). It is clear that $f\,(K_1, K_2) = 0$.

Hereafter, we are going to show that

$$w\,(u_0, v_0) = \Phi = \{(K_1, K_2)\}.$$

We observe that $w\,(u_0, v_0) \neq \varnothing$, because $(u_s, v_s) \in w\,(u_0, v_0)$.

Now, $\forall x \in \Omega$, $\sigma \in \,]-1, 1[$ and let

$$p_n(x,\sigma) = u(x, t_n + \sigma), q_n(x,\sigma) = v(x, t_n + \sigma)$$

multiply the first equation (1) by $\dfrac{\partial u}{\partial t}$

$$\frac{\partial u}{\partial t}\frac{\partial u}{\partial t} - d_1 \frac{\partial u}{\partial t}\Delta u - d_2 \frac{\partial u}{\partial t}\Delta v = \frac{\partial u}{\partial t} f\,(u, v)$$

and integrate over Ω we get

$$\int_{\Omega} \left(\frac{\partial u}{\partial t}\right)^2 dx - d_1 \int_{\Omega} \frac{\partial u}{\partial t}\Delta u dx - d_2 \int_{\Omega} \frac{\partial u}{\partial t}\Delta v dx = \int_{\Omega} \frac{\partial u}{\partial t} f\,(u, v)\, dx$$

as

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} = d_1 \int_{\Omega} \frac{\partial u}{\partial t}\Delta u dx + d_2 \int_{\Omega} \frac{\partial u}{\partial t}\Delta v dx + \int_{\Omega} \frac{\partial u}{\partial t} f\,(u, v)\, dx$$

intégrating result over $(t_0, +\infty)$, we have

$$\begin{aligned} \int_{t_0}^{+\infty} \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} dt = d_1 \int_{t_0}^{+\infty} \int_{\Omega} \frac{\partial u}{\partial t}\Delta u dx dt + d_2 \int_{t_0}^{+\infty} \int_{\Omega} \frac{\partial u}{\partial t}\Delta v dx dt \\ + \int_{t_0}^{+\infty} \int_{\Omega} \frac{\partial u}{\partial t} f\,(u, v)\, dx dt \prec \infty \end{aligned}$$

thus

$$\frac{\partial u}{\partial t} \in L^2\left(t_0, +\infty, L^2\,(\Omega)\right)$$

$\forall \sigma \in]-1, 1[$ we have

$$\begin{aligned} p_n(x,\sigma) - u(x,t_n) &= u(x, t_n+\sigma) - u(x,t_n) \\ &= \int_{t_n}^{t_n+\sigma} \frac{\partial u}{\partial t}(x,t)dt \leq \int_{t_{n-1}}^{t_n+1} \frac{\partial u}{\partial t}(x,t)dt \\ &\quad \text{(because } t_{n-1} \prec t_n \text{ , } \sigma \prec 1 \ t_n+\sigma \prec t_n+1) \\ &\leq \left(\int_{t_{n-1}}^{t_n+1} (1)^2 dt\right)^{\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n+1} \left(\frac{\partial u}{\partial t}(x,t)\right)^2 dt\right)^{\frac{1}{2}} \\ &\quad \text{(by inequality of Cauchy Schwartz)} \\ &\preceq \sqrt{2}\left(\int_{t_{n-1}}^{t_n+1} \left(\frac{\partial u}{\partial t}(x,t)\right)^2 dt\right)^{\frac{1}{2}} \end{aligned}$$

thus

$$|p_n(x,\sigma) - u(x,t_n)|^2 = 2\int_{t_{n-1}}^{t_n+1} \left(\frac{\partial u}{\partial t}(x,t)\right)^2 dt$$

integrating the latter inequality Ω yields

$$\int_\Omega |p_n(x,\sigma) - u(x,t_n)|^2 \, dx \leq 2\int_\Omega \int_{t_{n-1}}^{t_n+1} \left(\frac{\partial u}{\partial t}(x,t)\right)^2 dt dx$$

we pass to the limit as $n \to \infty$, we have

$$\|p_n(x,\sigma) - u_s\|^2_{L^2(\Omega)} \leq 2 \lim_{n\to\infty} \int_\Omega \int_{t_{n-1}}^{t_n+1} \left(\frac{\partial u}{\partial t}(x,t)\right)^2 dt dx = 0$$

so

$$\|p_n(x,\sigma) - u_s\|^2_{L^2(\Omega)} \underset{n\to\infty}{\to} 0 \, .$$

As a result we will all $\sigma \in]-1, 1[$,

$$\|p_n(x,\sigma) - u_s\|^2_{L^2(\Omega)} \underset{n\to\infty}{\to} 0, \text{ hence } \sup_{-1\prec\sigma\prec 1} \|p_n(x,\sigma) - u_s\|^2_{L^2(\Omega)} \underset{n\to\infty}{\to} 0$$

and by the same method are obtained

$$\sup_{-1\prec\sigma\prec 1} \|q_n(x,\sigma) - v_s\|^2_{L^2(\Omega)} \underset{n\to\infty}{\to} 0$$

Also, we can have

$$\sup_{-1\prec\sigma\prec 1} \|\nabla p_n(x,\sigma) - \nabla u_s\|^2_{L^2(\Omega)} \underset{n\to\infty}{\to} 0 \text{ and } \sup_{-1\prec\sigma\prec 1} \|\nabla q_n(x,\sigma) - \nabla v_s\|^2_{L^2(\Omega)} \underset{n\to\infty}{\to} 0$$

through positivity and boundedness of the solution was

$$\begin{array}{rcl} 0 & \leq & u(x, t_n + \sigma) \preceq M \\ 0 & \leq & v(x, t_n + \sigma) \preceq N \end{array}$$

as $f \in C^{\infty}(\mathbb{R}^{+})$, we can conclude, using Lebesgue's theorem, that

$$f(p_n(x,\sigma), q_n(x,\sigma)) \rightarrow f(u_s, v_s) \text{ in } L^2(\Omega \times (-1,1)) \text{ weak}$$

Now, let $\xi_i \in C^1\left(\overline{\Omega}\right)$ such that $\xi_i = 0$ on $\partial\Omega$ where $i = 1, 2$ and let $\gamma \in C^1\left(\overline{\Omega}\right)$ such that supp$\gamma \subset [-1, 1]$, $\int_{-1}^{1} \gamma(s)ds = 1$ and $\gamma(-1) = \gamma(1)$.

We multiply the first equation (1.1) by $\gamma(t-t_n)\xi_1$ and integrate over $(t_n - 1, t_n + 1) \times \Omega$, we obtain

$$\begin{array}{rl} & \int_{t_n-1}^{t_n+1} \int_{\Omega} \gamma(t-t_n)\xi_1 \frac{\partial u}{\partial t} dxdt - d_1 \int_{t_n-1}^{t_n+1} \int_{\Omega} \gamma(t-t_n)\xi_1 \Delta u dxdt \\ & -d_2 \int_{t_n-1}^{t_n+1} \int_{\Omega} \gamma(t-t_n)\xi_1 \Delta v dxdt \\ = & \int_{t_n-1}^{t_n+1} \int_{\Omega} \gamma(t-t_n)\xi_1 f(u,v)\, dxdt \end{array} \tag{15}$$

Calculate the integral

$$\int_{t_n-1}^{t_n+1} \gamma(t-t_n)\xi_1 \frac{\partial u}{\partial t} dt$$

by part, we find

$$\int_{t_n-1}^{t_n+1} \gamma(t-t_n)\xi_1 \frac{\partial u}{\partial t} dt = -\int_{t_n-1}^{t_n+1} \gamma'(t-t_n)\xi_1 u(x,t) dt \tag{16}$$

to calculate $\int_{\Omega} \gamma(t-t_n)\xi_1 \Delta u dx$ using Green's formula

$$\begin{array}{rcl} \int_{\Omega} \gamma(t-t_n)\xi_1 \Delta u dx & = & \int_{\partial\Omega} \gamma(t-t_n)\xi_1 \frac{\partial u}{\partial \eta} d\sigma - \int_{\Omega} \nabla\gamma(t-t_n)\xi_1 \nabla u dx \\ & = & -\int_{\Omega} \nabla\gamma(t-t_n)\xi_1 \nabla u dx \end{array}$$

so

$$\int_{\Omega} \gamma(t-t_n)\xi_1 \Delta u dx = -\int_{\Omega} \nabla\gamma(t-t_n)\xi_1 \nabla u dx \tag{17}$$

to calculate $\int_\Omega \gamma(t-t_n)\xi_1 \Delta v dx$ using Green's formula

$$\begin{aligned}\int_\Omega \gamma(t-t_n)\xi_1 \Delta v dx &= \int_{\partial\Omega} \gamma(t-t_n)\xi_1 \frac{\partial v}{\partial \eta} d\sigma - \int_\Omega \nabla\gamma(t-t_n)\xi_1 \nabla v dx \\ &= -\int_\Omega \nabla\gamma(t-t_n)\xi_1 \nabla v dx\end{aligned}$$

so

$$\int_\Omega \gamma(t-t_n)\xi_1 \Delta v dx = -\int_\Omega \nabla\gamma(t-t_n)\xi_1 \nabla v dx \tag{18}$$

injected (16) and (17) and (18) in (15) is obtained

$$\begin{aligned}&-\int_{t_n-1}^{t_n+1}\int_\Omega \gamma'(t-t_n)\xi_1 u(x,t) dxdt + d_1 \int_{t_n-1}^{t_n+1}\int_\Omega \nabla\gamma(t-t_n)\xi_1 \nabla u dxdt \\ &+d_2 \int_{t_n-1}^{t_n+1}\int_\Omega \nabla\gamma(t-t_n)\xi_1 \nabla v dxdt - \int_{t_n-1}^{t_n+1}\int_\Omega \gamma(t-t_n)\xi_1 f(u,v)\, dxdt \\ =\ & 0\end{aligned} \tag{19}$$

We multiply equation (2) by $\gamma(t-t_n)\xi_2$ and integrating over $(t_n-1, t_n+1)\times\Omega$, we have

$$\begin{aligned}&\int_{t_n-1}^{t_n+1}\int_\Omega \gamma(t-t_n)\xi_2 \frac{\partial v}{\partial t} dxdt - d_3 \int_{t_n-1}^{t_n+1}\int_\Omega \gamma(t-t_n)\xi_2 \Delta u dxdt \\ &-d_4 \int_{t_n-1}^{t_n+1}\int_\Omega \gamma(t-t_n)\xi_2 \Delta v dxdt \\ =\ & -\int_{t_n-1}^{t_n+1}\int_\Omega \gamma(t-t_n)\xi_2 f(u,v)\, dxdt\end{aligned} \tag{20}$$

Calculate the integral

$$\int_{t_n-1}^{t_n+1} \gamma(t-t_n)\xi_2 \frac{\partial v}{\partial t} dt$$

by part, we find

$$\int_{t_n-1}^{t_n+1} \gamma(t-t_n)\xi_2 \frac{\partial v}{\partial t} dt = -\int_{t_n-1}^{t_n+1} \gamma'(t-t_n)\xi_2 v(x,t) dt \tag{21}$$

to calculate $\int_\Omega \gamma(t-t_n)\xi_2 \Delta u dx$ using Green's formula

$$\begin{aligned}\int_\Omega \gamma(t-t_n)\xi_2 \Delta u dx &= \int_{\partial\Omega} \gamma(t-t_n)\xi_2 \frac{\partial u}{\partial \eta} d\sigma - \int_\Omega \nabla\gamma(t-t_n)\xi_2 \nabla u dx \\ &= -\int_\Omega \nabla\gamma(t-t_n)\xi_2 \nabla u dx\end{aligned}$$

so

$$\int_{\Omega} \gamma(t-t_n)\xi_2 \Delta u dx = -\int_{\Omega} \nabla\gamma(t-t_n)\xi_2 \nabla u dx \tag{22}$$

to calculate $\int_{\Omega} \gamma(t-t_n)\xi_2 \Delta v dx$ using Green's formula

$$\begin{aligned}\int_{\Omega} \gamma(t-t_n)\xi_2 \Delta v dx &= \int_{\partial\Omega} \gamma(t-t_n)\xi_2 \frac{\partial v}{\partial \eta} d\sigma - \int_{\Omega} \nabla\gamma(t-t_n)\xi_2 \nabla v dx \\ &= -\int_{\Omega} \nabla\gamma(t-t_n)\xi_2 \nabla v dx\end{aligned}$$

so

$$\int_{\Omega} \gamma(t-t_n)\xi_1 \Delta v dx = -\int_{\Omega} \nabla\gamma(t-t_n)\xi_1 \nabla v dx. \tag{23}$$

Injected (21) and (22) and (23) in (20) is obtained

$$\begin{aligned}&-\int_{t_n-1}^{t_n+1}\int_{\Omega} \gamma'(t-t_n)\xi_2 v(x,t) dxdt + d_3 \int_{t_n-1}^{t_n+1}\int_{\Omega} \nabla\gamma(t-t_n)\xi_2 \nabla u dxdt \\ &+d_4 \int_{t_n-1}^{t_n+1}\int_{\Omega} \nabla\gamma(t-t_n)\xi_2 \nabla v dxdt + \int_{t_n-1}^{t_n+1}\int_{\Omega} \gamma(t-t_n)\xi_2 f(u,v)\, dxdt \\ = \;& 0\end{aligned} \tag{24}$$

By making the following change of variable

$$\left(\begin{array}{c} \sigma = t - t_n \to d\sigma = dt \\ si \left\{ \begin{array}{l} t = t_n - 1 \\ t = t_n + 1 \end{array} \right. \to \left\{ \begin{array}{l} \sigma = -1 \\ \sigma = 1 \end{array} \right. \end{array} \right)$$

therefore the integral (19) becomes

$$\begin{aligned}&\int_{-1}^{+1}\int_{\Omega} \gamma'(\sigma)\xi_1 p_n(x,\sigma) dxd\sigma - d_1 \int_{-1}^{+1}\int_{\Omega} \nabla\gamma(\sigma)\xi_1 \nabla p_n(x,\sigma) dxd\sigma \\ &-d_2 \int_{-1}^{+1}\int_{\Omega} \nabla\gamma(\sigma)\xi_1 \nabla q_n(x,\sigma) dxdt \\ &+\int_{-1}^{+1}\int_{\Omega} \gamma(\sigma)\xi_1 f\left(p_n(x,\sigma), q_n(x,\sigma)\right) dxdt \\ = \;& 0\end{aligned} \tag{25}$$

The same applies to the integral (24)

$$\begin{aligned}
&\int_{-1}^{+1}\int_{\Omega}\gamma'(\sigma)\xi_2 q_n(x,\sigma)dxd\sigma - d_3\int_{-1}^{+1}\int_{\Omega}\nabla\gamma(\sigma)\xi_2\nabla p_n(x,\sigma)dxd\sigma\\
&-d_4\int_{-1}^{+1}\int_{\Omega}\nabla\gamma(\sigma)\xi_2\nabla q_n(x,\sigma)dxdt\\
&-\int_{-1}^{+1}\int_{\Omega}\gamma(\sigma)\xi_2 f\left(p_n(x,\sigma),q_n(x,\sigma)\right)dxdt\\
=\;&0
\end{aligned} \tag{26}$$

Using Lesbegue's theorem we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{-1}^{+1}\int_{\Omega}\gamma'(\sigma)\xi_1 p_n(x,\sigma)dxd\sigma\\
&=\int_{-1}^{+1}\int_{\Omega}\gamma'(\sigma)\xi_1 u_s dxd\sigma = \int_{-1}^{+1}\gamma'(\sigma)d\sigma\int_{\Omega}\xi_1 u_s dx\\
&=[\gamma\,(\sigma)]_{-1}^{+1}\int_{\Omega}\xi_1 u_s dx = 0 \text{ because } \gamma(1)=\gamma(-1)
\end{aligned}$$

and same method for

$$\begin{aligned}
&\lim_{n\to\infty}\int_{-1}^{+1}\int_{\Omega}\gamma'(\sigma)\xi_2 q_n(x,\sigma)dxdt\\
&=\int_{-1}^{+1}\int_{\Omega}\gamma'(\sigma)\xi_2 v_s dxd\sigma = \int_{-1}^{+1}\gamma'(\sigma)d\sigma\int_{\Omega}\xi_2 v_s dx\\
&=[\gamma\,(\sigma)]_{-1}^{+1}\int_{\Omega}\xi_2 v_s dx = 0 \text{ because } \gamma(1)=\gamma(-1)
\end{aligned}$$

from inequality (25), we have

$$-d_1\int_{\Omega}\nabla\xi_1\nabla u_s - d_2\int_{\Omega}\nabla\xi_1\nabla v_s + \int_{\Omega}\xi_1 f\left(u_s,v_s\right)dx = 0$$

and inequality (26) yields

$$-d_3\int_{\Omega}\nabla\xi_2\nabla u_s - d_4\int_{\Omega}\nabla\xi_2\nabla v_s - \int_{\Omega}\xi_2 f\left(u_s,v_s\right)dx = 0$$

But this form it is the same when we multiply (8) by ζ_1 and (9) by ζ_2 and integrating over Ω hence $w=\Phi$ Finally, combining (13) and (14) with (10) yields

$$\begin{aligned}
\int_{\Omega}(K_1+K_2)\,dx &= \int_{\Omega}(u_0+v_0)\,dx\\
(K_1+K_2)\,|\Omega| &= \int_{\Omega}(u_0+v_0)\,dx
\end{aligned}$$

i.e.

$$K_1 + K_2 = \frac{1}{|\Omega|} \int_\Omega (u_0 + v_0)\, dx$$

the proof of the theorem is complete. ■

References

[1] N. D. Alikakos, Lp-bounds of solutions of reaction-diffusion equations, Comm. Differential Equations 4, pp. 827–868, 1979.

[2] J. D. Avrin, Qualitative Theory for a Model of Laminar Flames with Arbitrary Nonnegative Initial Data, J. Differential Equations, 84, pp. 290–308, 1990.

[3] H. Amann, Global existence for semilinear parabolic systems, J. Reine Angew. Math. 360, 47-83, 1985.

[4] S. Badraoui, Existence of Global Solutions for Systems of Reaction-Diffusion Equations on Unbounded Domains, Electron. J. Diff. Eqns., Vol 2002, No. 74, pp. 1–10, 2002.

[5] S. Badraoui, Large Time Asymptotic Bounds of L^1 Solutions for Some Reaction-Diffusion Equations, Arab J. Math. Sc. Volume 8, Number 2, pp. 27–39, 2002.

[6] M. Ball, Stongly continuous semi-group, weak solutions and variation of constants mula, Proc. Amer. Math. Soc. 63, pp. 370–373, 1977.

[7] A. Barabanova, On the global existence of solutions of a reaction-di usion equation with exponential nonlinearity, Proc. Amer. Math. Soc., 122, 827–831, 1994.

[8] J. W. Beberns & W. Fulks; The small Heat-Loss Problem, J. Diff. Equat., 57, pp. 324–332, 1985.

[9] Chen Caisheng; Global Existence and Asymptotic Behavior of Solutions for a Strongly Coupled Parabolic System, Journal of Mathematical Research & Exposition, Vol. 14, No. 2, 1994.

[10] P. Collet & J. Xin; Global Existence and Large Time Asymptotic Bounds of L^1 Solutions of Thermal Diffusive Combustion Systems on Rn, Ann. Sc. Norm. Sup. Pisa, Cl. Sc., IV. Ser., 23, pp. 625–642, 1996.

[11] A. L. Gladnov; Behavior of Solutions of Semilinear Parabolic Equations as $x!1$, Mathematics Notes, Vol 51, No 2, pp. 124–128, 1990.

[12] A. Haraux, A. Youkana, On a result of K. Masuda concerning reaction-diffusion equations, Tohoku. Math. J., 40, pp. 159–163, 1988.

[13] M. Kirane, Global Bounds and Asymptotics for a System of Reaction-Diffusion Equations, Journal of Mathematical Analysis and Applications, 138, pp. 328–342, 1989.

[14] S. Kouachi, Global Existence of Solutions in Invariant Regions for Reaction-Diffusion Systems with a Balance Law and a Full Matrix of Diffusion Coefficients, EJQTDE, No. 4, pp. 1–10, 2003.

[15] S. Kouachi, Uniform Boundedness and Global Existence of Solutions for Reaction-Diffusion Systems with a Balance Law and a Full matrix of Diffusion Coefficients, EJQTDE, No. 7, pp. 1–9, 2001.

[16] K. Masuda, On the global existence and asymptotic behaviour of solution of reaction-diffusion equations, Hokkaido Math. J., 12, 360–370, 1983.

[17] M. Mebarki, A. Moumeni, Global solution of system reaction diffusion with full matrix, Global Journal of Mathematical Analysis, pp. 04–25, 2015.

[18] B. Rebiai, S. Benachour, Global classical solutions for reaction diffusion systems with nonlinearities of exponentiel growth, hal-00400191, version 1, 2009.

[19] A. Pazy, Semi-groups of Linear Operators and Applications to Partial Differential Equations, Springer, New York, 1983.

[20] M. Pierre, Weak solutions and supersolutions in L1 for reaction-diffusion systems, J. Evol. Equ., 3, pp. 153–168, 2003.

[21] J. Smoller, Shock waves and reaction-diffusion equations, Springer-Verlag, New York 1983.