Average Run Length of Control Chart for ARX(1) Process with Exponential White Noise

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Abstract

The objective of Statistical Process Control (SPC) is to monitor the operation of in-control processes. One of efficient tools of SPC is the Cumulative Sum control chart which widely used in several of application such as pharmaceutics, engineering, industries and in other areas. For many processes of interested in observations which are closely spaced in time will be correlated. The measure of operation is the Average Run Length(ARL). The main goal of this paper is to derive analytical solution for ARL of the CUSUM control chart for Autoregressive which has one Explanatory variable (ARX(1)) with exponential white noise. Checking the accuracy of results, obtained from exact expressions with Numerical Integral Equation by Gauss-Legendre rule was compared. An excellent agreement between the exact expressions and numerical solutions were found. This fact is an additional indication that the exact expressions are sufficiently high accuracy.

Keywords: Autoregressive with one Explanatory variable, Cumulative Sum control chart (CUSUM), Average Run Length (ARL)

1. Introduction

The SPC is a collection of statistical technique that used to improve the performance of a process, and to reduce variation of product parameter. The methods of SPC are applied to production process within the industrial setting. Theoretically, such methods are not limited to industrial applications, however. Physicians are often interested in monitoring a patient's test readings over time. Traffic engineers may be interested in monitoring traffic levels at particular intersections. The main tools in SPC is to monitor process are control charts. The control charts are statistical tools that can help in establishing process capabilities, identifying problems
that cause out of control conditions and maintain control of product and process quality. The Shewhart charts introduced by Shewhart [1] was simple to understand and used very extensively for on-line process monitoring, but have the disadvantage that they are not effective for the quick detection of small changes in process parameters. This disadvantage arised because the statistic plotted at each sampling point is a function of data taken only that sampling point. The ability to detect small parameter changes can be significantly improved by using control chart on which a statistic is plotted that incorporates information across past samples, in addition to the information in the current sample. A good alternative to the Shewhart control chart when interested in detecting small shifts is the Exponential Weighted Moving Average (EWMA) introduced by Roberts [2] and the Cumulative sum (CUSUM) chart purposed by Page[3].Traditional control charts are based on the assumption that process outputs are independent and identically distributed (IID).

However, there has been increasing awareness that the observations from many processes are autocorrelated and that autocorrelation can have an adverse effect on the performance of control charts (AlwanandRoberts [4]). As violation has a significant impact on the performance of the classical SPC procedure (see, e.g., Johnson and Bagshaw [5], Bagshaw and Johnson [6], Harris and Ross [7]). For example, positive autocorrelation can produce severe negative bias in traditional estimators of the process standard deviation, and this bias produces control limits that are much tighter than desired. Tight control limits, combined with autocorrelation in the observations being plotted, can result in an average false alarm rate much higher than expected or desired. A very high false alarm rate will cause process personnel to waste effort in unproductive searching for special causes. This can lead to a loss of confidence in control chart.

The CUSUM chart is primarily used for maintain (rather than improve) current control of a process (Duncan [8]). The primary advantage of the CUSUM chart is that it will identified a sudden or persistent change in the process average more rapidly than a Shewhart control chart incorporating the initial Shewhart interpretation rule. Furthermore, it is often possible to pinpoint the exact sample where the change in the process occurred (Wetherill and Brown [9]). The performance of CUSUM control charts in the presence of autocorrelation has been studied in a number of contexts. See, for example, Johnson and Bagshaw [5], Bagshaw and Johnson [6], Harris and Ross [7], Yashchin [10], Superville and Adams [11], Tseng and Adams [12], Runger, Willemain, and Prabhu [13], Schmid [14], Vander Weil [15], VanBrackle and Reynolds [16], and Timmer, Pignatiello, and Longnecker [17]. Therefore, the focus of this paper is to study the Fredholm type integral equations method to derive a closed-form solution of Average Run Length (ARL) for Autoregressive with one Explanatory variable (ARX(1)) with exponential white noise.

The key measure of the control charts methodologies will be average run length (ARL), for in-control (ARL0) and out of control (ARL1). One method of comparing and contrasting control chart methods is to compare their shift detection capabilities through their ARL. Harris and Ross [7] discussed the three ways that ARLs can be calculated: using Markov Chain Approach (MCA), solving a set of Integral Equation (IE), or using Monte Carlo simulation (MC). In 1987, Crowder [18] used a numerical
technique to determine ARL of control chart. He used an IE (known as a Fredholm integral equation of the second kind) for the run length and its variance and approximated the solution using a system of linear equations thus giving exact expressions for the mean and variance. Later, Lucus and Saccucci [19] studied the run length distribution of the control chart by using MCA. Vanbrackle and Reynold [16] were estimated the ARL by using an IE technique and MCA to evaluate EWMA and CUSUM control charts in case of the first order Autoregressive, AR (1) process with additional random error.

The purpose of this paper is to derive analytical formulas and use numerical methods to find ARL of CUSUM control chart for ARX (1) processes with exponential white noise for detecting of a change in process mean. We used integral equation to derive exact expression for ARL.

The operations of the paper are the following: in section 2 introduced the characteristics of CUSUM control chart for ARX (1) processes. The derivation of closed-form expression of ARL is expressed in section 3, the numerical method for solving integral equation to obtain approximation of ARL is presented in section 4, the comparison of the results is addressed in Section 5 and conclusions are presented in section 6.

2. The ARX(1) Processes for CUSUM Control Chart

The CUSUM control chart is a chronological plot of the cumulative sum of deviations of process means. It is well known that the CUSUM chart identifies small shifts in a process much faster than the standard x control chart (Juran and Gryna [20]). Given \( Y_t \) be a sequence of the auto-regressivewith Explanatory variable: ARX (1) random processes. The CUSUM processes regress the current value \( Y_t \) on the past values of itself \( Y_{t-1}, Y_{t-2}, \ldots, Y_{t-r}, Y_{t-r-1} \) and past random errors that occurred in past time periods \( \epsilon_{t-1}, \epsilon_{t-2}, \epsilon_{t-3}, \ldots, \epsilon_{t-r} \). Thus, the current value is a white noise error term.

The definition of CUSUM statistics based on ARX (1) process is the following recursion:

\[
C_t = \max(C_{t-1} + \epsilon_t - a, 0) \quad ; \quad t = 1, 2, \ldots
\]

where \( C_t \) is the CUSUM statistics, \( \epsilon_t \) is a sequence of independent and identically distribution random variables. The value of \( C_0 \) is an initial value of CUSUM statistics, \( C_0 = u \) and \( a \) is non-zero constant.

The general Autoregressive with Explanatory variable: ARX (1) processes can be written as:

\[
Y_t = \phi_1 Y_{t-1} + \epsilon_t + (\delta_1 Y_{t-1} - \phi_1 \delta Y_{t-1} - \delta \epsilon_{t-1})
\]

\[
+ (\delta_2 Y_{t-2} - \phi_1 \delta Y_{t-2} - \delta \epsilon_{t-2})
\]

\[
+ \ldots + (\delta r Y_{t-r} - \phi_1 \delta Y_{t-r} - \delta \epsilon_{t-r})
\]

\[
+ \omega_0 X_{t-b} - \omega_1 X_{t-(b+1)} - \ldots - \omega_s X_{t-(b+s)}
\]

\[
- (\omega_0 \phi_1 X_{t-b-1} - \omega_1 \phi_1 X_{t-(b+1)-1} - \ldots - \omega_s \phi_1 X_{t-(b+s)-1})
\]

(2)
where $\epsilon_t$ is to be a white noise processes assumed with exponential distribution. The initial value is normally to be the process mean, an autoregressive coefficient $-1 < \phi < 1$. The parameter $-1 < \delta < 1$ and $-1 < \omega < 1$ is coefficient of transfer function. It is assumed that the initial value of ARX(1) processes $Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p}, \ldots, Y_{t-r-1} = 1$ and $X_{t-b}, X_{t-b-1}, \ldots, X_{t-(b+1)-1} = 1$.

In this paper, the case of positive change in distribution which crossing the upper control limit raises alarm is mainly discussed. Given $\epsilon_t, t=1,2,\ldots$ is a sequence of independent identically distribution random variables with exponential parameter ($\alpha$). It is normally assumed that under in control state, the parameter has known in-control value $(\alpha = \alpha_0)$. The parameter $\alpha$ could be changed to out-of-control value $(\alpha = \alpha_1)$, when $(\theta = \infty)$, is the change-point time.

The first passage times for the CUSUM can be written as:

$$\tau_h = \inf(t > 0 : C_t > h), \quad h > u$$

where $\tau_h$ is a stopping time $h$ is a constant parameter known as Upper control Limit (UCL).

The ARL is the expectation value of $\tau_h$. Most two characteristics are used the performances of control chart are $ARL_0$ and $ARL_1$ as following:

$$ARL_0 = E_\infty(\tau_h)$$

(4)

$$ARL_1 = E_\theta(\tau_h - \theta + 1 | \tau_h \geq \theta)$$

(5)

Where $E_\infty(.)$ is the expectation corresponding to the target value and is assumed to be large enough. $E_\theta(.)$ is the expectation under the assumption that change-point occurs at time $\theta = 1$.

### 3. Exact expression of CUSUM control chart for ARX(1) Processes

The notations $P_e$ denote the probability measure and $E_e$ denote the expression corresponding chart after it is reset at $u \in [0,h]$. The solution of integral equation is as following

$$H(u) = 1+E_e[\{0 < C_i < b\} \{H(C_i)\}] + P_e\{C_i = 0\} H(0)$$

(6)

Therefore, the integral equation of CUSUM control chart is

$$Y_t = \phi Y_{t-1} + \epsilon_t + (\delta Y_{t-1} - \phi \delta Y_{t-1} - \delta \epsilon_{t-1}) + (\delta Y_{t-2} - \phi \delta Y_{t-2} - \delta \epsilon_{t-2}) + \ldots$$

$$+ (\delta Y_{t-r} - \phi \delta Y_{t-r} - \delta \epsilon_{t-r}) + \ldots$$

$$+ \omega_0 X_{t-b} - \omega X_{t-(b+1)} - \ldots - \omega X_{t-(b+1)} - \ldots$$

$$- (\omega_0 \phi X_{t-b-1} - \omega \phi X_{t-(b+1)-1} - \ldots - \omega \phi X_{t-(b+1)-1})$$

So,

$$H(u) = 1 + \left[ (\alpha \phi^u + \omega \phi^u) \left( \delta Y_{t-r} - \phi \delta Y_{t-r} - \delta \epsilon_{t-r} \right) \right] \times$$

$$e^{-\alpha(\delta Y_{t-r} - \phi \delta Y_{t-r} - \delta \epsilon_{t-r}) - \ldots - \alpha(\delta Y_{t-b} - \phi \delta Y_{t-b} - \delta \epsilon_{t-b})} \times$$

$$e^{-\omega X_{t-b} - \omega X_{t-(b+1)} - \ldots - \omega X_{t-(b+1)}}$$

$$\times \left( \alpha \phi^u X_{t-(b+1)-1} + \omega \phi^u X_{t-(b+1)-1} - \ldots - \omega \phi^u X_{t-(b+1)-1} \right)$$
Average Run Length of Control Chart for Arx(1) Process

\[ k \int_0^H(y) e^{-\alpha y} dy + (1 - e^{-\alpha y} \phi Y_{t-2} - \delta Y_{t-3} + \delta \varepsilon_{t-1}) \times \]

\[ e^{\alpha (\delta Y_{t-2} - \phi Y_{t-3} - \delta \varepsilon_{t-1})} \times \]

\[ - \alpha X_{i_{(t+1)}} - \alpha Y_{i_{(t+1)}} - \alpha X_{i_{(t+1)}} \times \]

\[ \times H(0) \]

(7).

Let \( k = \int_0^H(y) e^{-\alpha y} dy \). Consequently, \( H(u) \) can be rewritten as

\[ H(u) = 1 + \left[ e^{\alpha (u - \phi Y_{t-2} - \delta Y_{t-3} + \delta \varepsilon_{t-1})} \times \]

\[ e^{\alpha (\delta Y_{t-2} - \phi Y_{t-3} - \delta \varepsilon_{t-1})} \times \]

\[ - \alpha X_{i_{(t+1)}} - \alpha Y_{i_{(t+1)}} - \alpha X_{i_{(t+1)}} \times \]

\[ \times H(0) \]

In particular at \( u = 0 \), we obtain \( H(0) \) as following form

\[ H(0) = 1 + \left[ e^{\alpha (u - \phi Y_{t-2} - \delta Y_{t-3} + \delta \varepsilon_{t-1})} \times \]

\[ e^{\alpha (\delta Y_{t-2} - \phi Y_{t-3} - \delta \varepsilon_{t-1})} \times \]

\[ - \alpha X_{i_{(t+1)}} - \alpha Y_{i_{(t+1)}} - \alpha X_{i_{(t+1)}} \times \]

\[ \times H(0) \]

Substituting \( H(0) \) into Equation , then \( H(u) \) as following form

\[ H(u) = 1 + \left[ e^{\alpha (u - \phi Y_{t-2} - \delta Y_{t-3} + \delta \varepsilon_{t-1})} \times \]

\[ e^{\alpha (\delta Y_{t-2} - \phi Y_{t-3} - \delta \varepsilon_{t-1})} \times \]

\[ - \alpha X_{i_{(t+1)}} - \alpha Y_{i_{(t+1)}} - \alpha X_{i_{(t+1)}} \times \]

\[ \times H(0) \]

Substituting \( H(0) \) into Equation , then \( H(u) \) as following form

\[ H(u) = 1 + \left[ e^{\alpha (u - \phi Y_{t-2} - \delta Y_{t-3} + \delta \varepsilon_{t-1})} \times \]

\[ e^{\alpha (\delta Y_{t-2} - \phi Y_{t-3} - \delta \varepsilon_{t-1})} \times \]

\[ - \alpha X_{i_{(t+1)}} - \alpha Y_{i_{(t+1)}} - \alpha X_{i_{(t+1)}} \times \]

\[ \times H(0) \]

Consequently,

\[ H(u) = 1 + k \alpha + e^{\alpha (u - \phi Y_{t-2} - \delta Y_{t-3} + \delta \varepsilon_{t-1})} \times \]

\[ e^{\alpha (\delta Y_{t-2} - \phi Y_{t-3} - \delta \varepsilon_{t-1})} \times \]

\[ - \alpha X_{i_{(t+1)}} - \alpha Y_{i_{(t+1)}} - \alpha X_{i_{(t+1)}} \times \]

\[ \times H(0) \]

To find a constant \( k \) as following form

\[ k = \int_0^H(y) e^{-\alpha y} dy \]
Substituting a constant $k$ into Equation (9) as follows

$$H(u) = e^{ah} \left( 1 + e^{a(\alpha Y_1 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \right) \times$$

$$e^{a(\delta_1 Y_2 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times \ldots \times e^{a(\delta_1 Y_m - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t+h} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t+(h-1)} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t+1} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times \alpha^h e^{\alpha Y_1 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2}$$

Thus, we get the explicit formulas for $ARL$ of CUSUM control chart as follows

$$ARL_0 = e^{a^h} \left( 1 + e^{a(\alpha Y_1 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\delta_1 Y_2 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times \ldots \times e^{a(\delta_1 Y_m - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t+h} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t+(h-1)} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times \alpha^h e^{\alpha Y_1 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2} \right)$$

Since the process is in-control state with exponential parameter $\alpha=\alpha_0$, we obtain the explicit formula for $ARL_0$ as follows

$$ARL_0 = e^{a^h} \left( 1 + e^{a(\alpha Y_1 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\delta_1 Y_2 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times \ldots \times e^{a(\delta_1 Y_m - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t+h} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t+(h-1)} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times \alpha^h e^{\alpha Y_1 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2} \right)$$

Since the process is out-of-control state with exponential parameter $\alpha=\alpha_1$, The explicit formula for $ARL_1$ can be written as follows

$$ARL_1 = e^{a^h} \left( 1 + e^{a(\alpha Y_1 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\delta_1 Y_2 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times \ldots \times e^{a(\delta_1 Y_m - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t+h} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t+(h-1)} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times e^{a(\alpha X_{t} - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2)} \times \alpha^h e^{\alpha Y_1 - \delta Y_1 - \delta_1 e_1 - \delta_2 e_2} \right)$$

where $-1 < \phi_1 < 1$ is Autoregressive coefficient, $-1 < \delta_i < 1$ and $-1 < \omega_i < 1$ is parameter of transfer function, $X_{t+h}, X_{t+(h-1)}, \ldots, X_{t+(h+n-1)}$ is Explanatory variable, $0 < \lambda < 1, Y_{t-1}, Y_{t-2}, \ldots, Y_{t-m}$ and $\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots, \varepsilon_{t-r}$ are smoothing parameter and initial values respectively.

4. Numerical Integral Equation

Generally, the Integral Equation could not be analytically solved $H(u)$ and it is necessary to use numerical methods to solve them. Kantorovich and krylov [21]; Atkinson and Han [22] have been developed numerical schemes for solving integral equation. We shall use a quadrature rule to approximate the integral by finite sum of area of rectangles with based on $h/m$ beginning at zero. Particularly, once the choice of a quadrature rule is made, the interval $[0,h]$ is divided into a partition $0 \leq a_1 \leq a_2 \leq \ldots \leq a_m \leq m$ and set of constant weighted $w_j = (h/m) \geq 0$.
\[
\int_{0}^{b} W(x) f(x) dx \approx \sum_{j=1}^{m} w_j f(\alpha_j)
\]
where \( \alpha_j = \frac{h}{m} \left( \frac{2j-1}{2} \right) \) and \( w_j = \frac{h}{m} ; j = 1, 2, \ldots, m \).

Let \( \tilde{H}(u) \) denote to the numerical approximation to integral equation \( \bar{H}(u) \), which can be found as the solution of linear equation as follows:

\[
\begin{align*}
\tilde{H}(a_i) &= 1 + \tilde{H}(a_i) F((a_i - a_j - \phi Y_{t-1}) - (\delta Y_{t-1} - \phi \delta Y_{t-1} - \delta \varepsilon_{t-1}) \\
- (\delta Y_{t-1} - \phi \delta Y_{t-2} - \delta \varepsilon_{t-1}) - (\delta Y_{t-2} - \phi \delta Y_{t-3} - \delta \varepsilon_{t-2}) - \cdots - (\delta Y_{t-r} - \phi \delta Y_{t-r+1} - \delta \varepsilon_{t-r}) \\
&- \omega_0 X_{t-b} + \omega_1 X_{t-(b+1)} + \cdots + \omega_{m} X_{t-(b+m)} + (\omega_0 \phi X_{t-b-1} - \omega_1 \phi X_{t-(b+1)-1} - \cdots - \omega_{m} \phi X_{t-(b+m)-1}) \\
+ \sum_{j=1}^{m} w_j \tilde{H}(a_j) f((a_j + a_j - a_j - \phi Y_{t-1}) - (\delta Y_{t-1} - \phi \delta Y_{t-1} - \delta \varepsilon_{t-1}) \\
- (\delta Y_{t-1} - \phi \delta Y_{t-2} - \delta \varepsilon_{t-1}) - (\delta Y_{t-2} - \phi \delta Y_{t-3} - \delta \varepsilon_{t-2}) - \cdots - (\delta Y_{t-r} - \phi \delta Y_{t-r+1} - \delta \varepsilon_{t-r}) \\
&- \omega_0 X_{t-b} + \omega_1 X_{t-(b+1)} + \cdots + \omega_{m} X_{t-(b+m)} + (\omega_0 \phi X_{t-b-1} - \omega_1 \phi X_{t-(b+1)-1} - \cdots - \omega_{m} \phi X_{t-(b+m)-1}) \\
+ \sum_{j=1}^{m} w_j \tilde{H}(a_j) f((a_j + a_j - a_j - \phi Y_{t-1}) - (\delta Y_{t-1} - \phi \delta Y_{t-1} - \delta \varepsilon_{t-1}) \\
- (\delta Y_{t-1} - \phi \delta Y_{t-2} - \delta \varepsilon_{t-1}) - (\delta Y_{t-2} - \phi \delta Y_{t-3} - \delta \varepsilon_{t-2}) - \cdots - (\delta Y_{t-r} - \phi \delta Y_{t-r+1} - \delta \varepsilon_{t-r}) \\
&- \omega_0 X_{t-b} + \omega_1 X_{t-(b+1)} + \cdots + \omega_{m} X_{t-(b+m)} + (\omega_0 \phi X_{t-b-1} - \omega_1 \phi X_{t-(b+1)-1} - \cdots - \omega_{m} \phi X_{t-(b+m)-1}) \\
&= \ldots \\
\tilde{H}(a_m) &= 1 + \tilde{H}(a_m) F((a_m - a_m - \phi Y_{t-1}) - (\delta Y_{t-1} - \phi \delta Y_{t-1} - \delta \varepsilon_{t-1}) \\
- (\delta Y_{t-1} - \phi \delta Y_{t-2} - \delta \varepsilon_{t-1}) - (\delta Y_{t-2} - \phi \delta Y_{t-3} - \delta \varepsilon_{t-2}) - \cdots - (\delta Y_{t-r} - \phi \delta Y_{t-r+1} - \delta \varepsilon_{t-r}) \\
&- \omega_0 X_{t-b} + \omega_1 X_{t-(b+1)} + \cdots + \omega_{m} X_{t-(b+m)} + (\omega_0 \phi X_{t-b-1} - \omega_1 \phi X_{t-(b+1)-1} - \cdots - \omega_{m} \phi X_{t-(b+m)-1}) \\
+ \sum_{j=2}^{m} w_j \tilde{H}(a_j) f((a_j + a_j - a_j - \phi Y_{t-1}) - (\delta Y_{t-1} - \phi \delta Y_{t-1} - \delta \varepsilon_{t-1}) \\
- (\delta Y_{t-1} - \phi \delta Y_{t-2} - \delta \varepsilon_{t-1}) - (\delta Y_{t-2} - \phi \delta Y_{t-3} - \delta \varepsilon_{t-2}) - \cdots - (\delta Y_{t-r} - \phi \delta Y_{t-r+1} - \delta \varepsilon_{t-r}) \\
&- \omega_0 X_{t-b} + \omega_1 X_{t-(b+1)} + \cdots + \omega_{m} X_{t-(b+m)} + (\omega_0 \phi X_{t-b-1} - \omega_1 \phi X_{t-(b+1)-1} - \cdots - \omega_{m} \phi X_{t-(b+m)-1}) \\
+ \sum_{j=2}^{m} w_j \tilde{H}(a_j) f((a_j + a_j - a_j - \phi Y_{t-1}) - (\delta Y_{t-1} - \phi \delta Y_{t-1} - \delta \varepsilon_{t-1}) \\
- (\delta Y_{t-1} - \phi \delta Y_{t-2} - \delta \varepsilon_{t-1}) - (\delta Y_{t-2} - \phi \delta Y_{t-3} - \delta \varepsilon_{t-2}) - \cdots - (\delta Y_{t-r} - \phi \delta Y_{t-r+1} - \delta \varepsilon_{t-r}) \\
&- \omega_0 X_{t-b} + \omega_1 X_{t-(b+1)} + \cdots + \omega_{m} X_{t-(b+m)} + (\omega_0 \phi X_{t-b-1} - \omega_1 \phi X_{t-(b+1)-1} - \cdots - \omega_{m} \phi X_{t-(b+m)-1}) \\
&= \cdots
\end{align*}
\]
\[ -\alpha_0 X_{t-b} + \alpha_1 X_{t-(b+1)} + \ldots + \alpha_s X_{t-(s+x)} + (\alpha_1 \phi X_{t-b-1} - \alpha_2 \phi X_{t-(b+1)-1} - \ldots - \alpha_s \phi X_{t-(s+x)-1}) \].

Solving set of equations for the approximate values \( \tilde{H}(a_1), \tilde{H}(a_2), \ldots, \tilde{H}(a_n) \), the numerical integration for function \( H(u) \) is

\[
\tilde{H}(u) = 1 + \tilde{H}(a_1)F((a - u - \phi Y_{t-1} - (\delta Y_{t-1} - \phi \delta Y_{t-1} - \delta \epsilon_{t-1})) \\
- (\delta Y_{t-1} - \phi \delta Y_{t-2} - \delta \epsilon_{t-1}) - (\delta Y_{t-2} - \phi \delta Y_{t-3} - \delta \epsilon_{t-2}) - \ldots - (\delta Y_{t-r} - \phi \delta Y_{t-r-1} - \delta \epsilon_{t-r}) \\
- \alpha_0 X_{t-b} + \alpha_1 X_{t-(b+1)} + \ldots + \alpha_s X_{t-(s+x)} + (\alpha_1 \phi X_{t-b-1} - \alpha_2 \phi X_{t-(b+1)-1} - \ldots - \alpha_s \phi X_{t-(s+x)-1}) \\
+ \sum_{j=1}^{m} w_j \tilde{H}(a_j) \left( (a_j + a - u - \phi Y_{t-1} - (\delta Y_{t-1} - \phi \delta Y_{t-1} - \delta \epsilon_{t-1})) \\
- (\delta Y_{t-1} - \phi \delta Y_{t-2} - \delta \epsilon_{t-1}) - (\delta Y_{t-2} - \phi \delta Y_{t-3} - \delta \epsilon_{t-2}) - \ldots - (\delta Y_{t-r} - \phi \delta Y_{t-r-1} - \delta \epsilon_{t-r}) \\
- \alpha_0 X_{t-b} + \alpha_1 X_{t-(b+1)} + \ldots + \alpha_s X_{t-(s+x)} + (\alpha_1 \phi X_{t-b-1} - \alpha_2 \phi X_{t-(b+1)-1} - \ldots - \alpha_s \phi X_{t-(s+x)-1}) \right),
\]

with \( a_j = \frac{h}{m} \left( \frac{2j-1}{2} \right) \) and \( w_j = \frac{h}{m} ; j = 1, 2, \ldots, m \).

5. Comparison results of CUSUM Control Charts by Exact Expression and Numerical Integral Equation methods

In this section, the results of ARL_0 and ARL_1 for ARX (p) processes, which are obtained from the exact expression with numerical solution of integral equation method are compared. The results of ARL are expressed in Table 1 to Table 3. The parameter value for in-control parameter \( \beta_0 = 1 \) and parameter for out-of-control \( \beta_1 = 1.01, 1.02, 1.03, 1.04, 1.05, 1.06, 1.07, 1.08, 1.09, 1.10, 1.3, 1.5, 3, \) and 5 respectively. The performance of the purposed exact expression is considered by the computational times and the absolute percentage difference.

\[
\text{Diff} \% = \left( \frac{\bar{L}(u) - L(u)}{L(u)} \right) \times 100\%.
\]

The results from Table 1 and Table 2 present that these methods are in good agreement. The analytical results agree with numerical approximation with an absolute percentage difference less than 0.05% for \( m = 1,500 \) iterations and for computational times of approximately 50 second. The computational times for the proposed analytical exact expressions are less than 1 second.
Table 1: Comparison of ARL₀ and ARL₁ of CUSUM control chart by exact expression with numerical integral equation for ARX(1) process with \( r=0, s=0, b=3 \) \( \omega_0 = 0.1 \) and \( \phi_1 = 0.1 \).

<table>
<thead>
<tr>
<th>Parameter values of EWMA chart ( a = 2.5, u = 1 ) and ( h = 3.959 )</th>
<th>Exact expression</th>
<th>Numerical IE (Time used)</th>
<th>Diff (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>1.00</td>
<td>370.165</td>
<td>370.034(57.46)</td>
</tr>
<tr>
<td>1.01</td>
<td>346.354</td>
<td>346.243(56.11)</td>
<td>0.032048</td>
</tr>
<tr>
<td>1.02</td>
<td>324.498</td>
<td>324.397(57.35)</td>
<td>0.031125</td>
</tr>
<tr>
<td>1.03</td>
<td>304.405</td>
<td>304.313(55.47)</td>
<td>0.030223</td>
</tr>
<tr>
<td>1.04</td>
<td>285.909</td>
<td>285.827(56.38)</td>
<td>0.028680</td>
</tr>
<tr>
<td>1.05</td>
<td>268.86</td>
<td>268.785(55.22)</td>
<td>0.027896</td>
</tr>
<tr>
<td>1.06</td>
<td>253.122</td>
<td>253.054(56.18)</td>
<td>0.026865</td>
</tr>
<tr>
<td>1.07</td>
<td>238.577</td>
<td>238.519(57.28)</td>
<td>0.024311</td>
</tr>
<tr>
<td>1.08</td>
<td>225.117</td>
<td>225.063(57.53)</td>
<td>0.023988</td>
</tr>
<tr>
<td>1.09</td>
<td>212.647</td>
<td>212.598(56.07)</td>
<td>0.023043</td>
</tr>
<tr>
<td>1.10</td>
<td>201.078</td>
<td>201.032(55.43)</td>
<td>0.022877</td>
</tr>
<tr>
<td>1.30</td>
<td>79.1033</td>
<td>79.0860(55.34)</td>
<td>0.021870</td>
</tr>
<tr>
<td>1.50</td>
<td>40.4176</td>
<td>40.4094(57.08)</td>
<td>0.020288</td>
</tr>
<tr>
<td>3.00</td>
<td>5.49036</td>
<td>5.48934(56.58)</td>
<td>0.018578</td>
</tr>
<tr>
<td>5.00</td>
<td>2.74180</td>
<td>2.74134(56.44)</td>
<td>0.016777</td>
</tr>
</tbody>
</table>

Table 2: Comparison of ARL₀ and ARL₁ of CUSUM control chart by exact expression with numerical integral equation for ARX(1) process with \( r=0, s=1, b=3 \) \( \omega_0 = 0.1, \omega_1 = 0.2 \) and \( \phi_1 = 0.01 \).

<table>
<thead>
<tr>
<th>Parameter values of EWMA chart ( a = 2.5, u = 1 ) and ( h = 3.2984 )</th>
<th>Exact expression</th>
<th>Numerical IE (Time used)</th>
<th>Diff (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>1.00</td>
<td>370.133</td>
<td>369.989(57.03)</td>
</tr>
<tr>
<td>1.01</td>
<td>347.615</td>
<td>347.482(56.45)</td>
<td>0.038261</td>
</tr>
<tr>
<td>1.02</td>
<td>326.863</td>
<td>326.742(56.32)</td>
<td>0.037019</td>
</tr>
<tr>
<td>1.03</td>
<td>307.711</td>
<td>307.598(57.18)</td>
<td>0.036723</td>
</tr>
<tr>
<td>1.04</td>
<td>290.014</td>
<td>289.908(56.54)</td>
<td>0.036550</td>
</tr>
<tr>
<td>1.05</td>
<td>273.639</td>
<td>273.541(55.43)</td>
<td>0.035814</td>
</tr>
<tr>
<td>1.06</td>
<td>258.468</td>
<td>258.378(56.33)</td>
<td>0.034821</td>
</tr>
<tr>
<td>1.07</td>
<td>244.396</td>
<td>244.314(55.41)</td>
<td>0.033552</td>
</tr>
<tr>
<td>1.08</td>
<td>231.327</td>
<td>231.254(56.15)</td>
<td>0.031557</td>
</tr>
<tr>
<td>1.09</td>
<td>219.176</td>
<td>219.108(55.47)</td>
<td>0.031025</td>
</tr>
<tr>
<td>1.10</td>
<td>207.865</td>
<td>207.801(56.48)</td>
<td>0.030789</td>
</tr>
<tr>
<td>1.30</td>
<td>85.4719</td>
<td>85.4462(56.39)</td>
<td>0.030068</td>
</tr>
<tr>
<td>1.50</td>
<td>44.6600</td>
<td>44.6742(57.11)</td>
<td>0.028661</td>
</tr>
<tr>
<td>3.00</td>
<td>5.88310</td>
<td>5.88160(56.31)</td>
<td>0.025497</td>
</tr>
<tr>
<td>5.00</td>
<td>2.80747</td>
<td>2.80692(56.44)</td>
<td>0.019591</td>
</tr>
</tbody>
</table>
6. Conclusion

Exact expression for the ARL of CUSUM in the case of ARX(1) process with exponential white noise are derived. These formulas are very accurate, and easy to calculate and program. More specifically, the exact expression take computational time much less than numerical integral equation.

References


