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# On Clique Secure Domination in Graphs

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**Abstract**

Let $G$ be a connected simple graph. A nonempty subset $S$ of the vertex set V(G) is a clique in $G$ if the graph $\langle S \rangle$ induced by $S$ is complete. A clique $S$ in $G$ is a clique dominating set if it is a dominating set. A clique dominating set $S$ is a clique secure dominating set in $G$ if for every vertex $u \in V(G) \setminus S$, there exists a vertex $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup S$ is a dominating set in $G$. The clique secure domination number, denoted by $\gamma_{cls}(G)$, is the smallest cardinality of a clique secure dominating set in $G$. A clique secure dominating set having cardinality equal to $\gamma_{cls}(G)$ is called a $\gamma_{cls}$-$set$ of $G$. In this paper, we show that given positive integers $k$ and $n$ such that $n \geq 4$ and $1 \leq k \leq n$, there exists a connected graph $G$ with $|V(G)| = n$ and $\gamma_{cls}(G) = k$. Also, we show that for any positive integers $k$, $m$ and $n$ such that $1 \leq k \leq m$, there exists a connected graph $G$ with $|V(G)| = n$, $\gamma_{cls}(G) = m$, and $\gamma_{cl}(G) = k$. Further, we give the characterization of the clique secure dominating set resulting from the join of two graphs and give some important results.

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## 1. Introduction

The concept of domination in graphs introduced by Claude Berge in 1958 and Oystein Ore in 1962 [9] is currently receiving much attention in literature. Following the article of Ernie Cockayne and Stephen Hedetniemi [3], the domination in graphs became an area of study by many researchers. One type of domination parameter is the secure domination in graphs. This was studied and introduced by E.J. Cockayne et.al [1, 4, 5]. Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible. The clique domination in graphs can be read in the paper of Cozzen and Kelleher [8, 11]. For the general concepts not mentioned, the readers may be referred to [7].

A *graph* $G$ is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of $G$ and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply $uv$) of distinct elements from $V(G)$ called the *edge-set* of $G$. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of $G$. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of $G$. If $|V(G)| = 1$, then $G$ is called a trivial graph. If $E(G) = \varnothing$, then $G$ is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of $v$. The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of $X$ in $G$ is the set $N_G(X) = \bigcup_{v\in X} N_G(v)$. The *closed neighborhood* of $X$ in $G$ is the set $N_G[X] = \bigcup_{v\in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ [resp. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$].

Let $x, y \in V(G)$. Any $x$-$y$ path of length equal to $d_G(x, y)$ (the distance between the vertices $x$ and $y$ in $G$) is called an $x$-$y$ *geodesic*. The interval $I[x, y] = I_G[x, y]$ consists of $x$, $y$ and all vertices lying on any $x$-$y$ geodesic. If $S \subseteq V(G)$, then the *geodetic closure* of $S$ is the set $I[S] = I_G[S] = \bigcup_{x,y\in S} I[x, y]$. $S$ is *convex* if $I[x, y] \subseteq S$ for any $x, y \in S$, i.e., $I_G[S] = S$. A complete graph of order $n$, denoted by $K_n$, is the graph in which every pair of its distinct vertices are joined by an edge. A nonempty subset $S$ of $V(G)$ is a clique in $G$ if the graph $\langle S\rangle$ induced by $S$ is complete.

A subset $S$ of $V(G)$ is a *dominating set* of $G$ if for every $v \in V(G)\backslash S$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A clique $S$ in $G$ is a clique dominating set if it is a dominating set. The *clique domination number* of $G$, denoted by $\gamma_{cl}(G)$, is the smallest cardinality of a dominating set of $G$. A clique dominating set of cardinality $\gamma_{cl}(G)$ is called a $\gamma_{cl}$*-set* of $G$. A dominating set $S$ of $V(G)$ is a *secure dominating set* of $G$ if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G$. The minimum cardinality of a secure dominating set of $G$, denoted by $\gamma_s(G)$, is called the *secure domination number* of $G$. A secure dominating set of cardinality $\gamma_s(G)$ is called a $\gamma_s$*-set* of $G$.

In [6], Enriquez and Canoy, defined the secure clique dominating sets in a graph $G$ to characterized the secure convex dominating sets in the join of two graphs. This motivate the researchers to study the clique secure dominating set resulting from the join and the corona of two graphs. Accordingly, a clique dominating set $S$ in a graph $G$ is a *clique secure dominating set* of $G$ if for every $u$ $V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set in $G$. The *clique secure domination number* of $G$ denoted by $\gamma_{cls}(G)$ is the minimum cardinality of a clique secure dominating set of $G$. A clique secure dominating set of cardinality $\gamma_{cls}(G)$ is called $\gamma_{cls}$-*set*. Unless otherwise stated, all graphs in this paper are assumed to be simple and connected.

## 2. Results

Since $\gamma_{cl}(G)$ does not always exists in a connected nontrivial graph G, we denote by $\mathcal{CS}(G)$ be a family of all graphs with clique secure dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{CS}(G)$. Further, since $G = K_n$ $(n \geq 3)$ is a clique in $G$ and the $\gamma_{cls}(G) = 1$, it follows that $\gamma_{cls}(G) \neq n$ for any $G$ of order $n \geq 2$.

From the definitions, the following remarks are immediate.

**Remark 2.1.** A clique dominating set $S$ that is secure, is a clique secure dominating set of a graph $G$.

**Remark 2.2.** Let $G$ be a nontrivial connected graph. Then

$(i)$ $\gamma(G) \leq \gamma_s(G) \leq \gamma_{cls}(G)$; and

$(ii)$ $1 \leq \gamma_{cls}(G) \leq n - 1$.

**Theorem 2.3. [6]** Let $G$ be a connected graph of order $n \geq 1$. Then $\gamma_{scon}(G) = 1$ if and only if $G = K_n$.

**Lemma 2.4.** Every clique secure dominating set of a graph $G$ is a secure dominating set of $G$.

*Proof.* Suppose that $S$ is a clique secure dominating set of a graph $G$. Then $S$ is a clique dominating set of $G$ (and hence a dominating set of $G$). Since for every $u \in V(G) \setminus S$ there exists $v \in S \cap N(u)$ such that $S_u = (S \setminus \{v\}) \cup \{v\}$ is dominating set, it follows that $S$ is a secure dominating set of $G$. ■

The following result characterize the clique secure dominating set of cardinality equal to one.

**Theorem 2.5.** Let $G$ be a connected graph of order $n \geq 1$. Then $\gamma_{cls}(G) = 1$ if and only if $G = K_n$.

*Proof.* Suppose that $\gamma_{cls}(G) = 1$. Let $S = \{x\}$ be a clique secure dominating set in $G$. Then $S$ is a secure dominating set of $G$ by Lemma 2.4. Thus, $G = K_n$ by Theorem 2.3.

For the converse, suppose that $G = K_n$. Then $\gamma_s(G) = 1$ by Theorem 2.3. Let $S = \{x\}$ be a $\gamma_s$-$set$ of $G$. Since $S$ is a clique dominating set of $G$, it follows that $S$ is a clique secure dominating set of $G$. Therefore $\gamma_{cls}(G) = 1$. ■

The next result says that the value of the parameter $\gamma_{cls}$ ranges over all positive integers.

**Theorem 2.6.** Given positive integers $k$ and $n$ such that $n \geq 4$ and $1 \leq k \leq n$, there exists a connected graph $G$ with $|V(G)| = n$ and $\gamma_{cls}(G) = k$.

*Proof.* Consider the following cases:

*Case 1.* Suppose $1 \leq k < n - 1$.

Let $G = K_k \circ H$ where $H$ is a complete graph of order $r$ and let $n = k(r + 1)$. Then the set $V(K_k)$ is a minimum clique secure dominating set of $G$. Hence $\gamma_{cls}(G) = k$ and $|V(G)| = |V(K_k \circ H)| = k(r + 1) = n$.

*Case 2.* Suppose $k = n - 1$.
Let $G = P_3 = [v_1, v_2, v_3]$. Then the set $A = \{v_1, v_2\}$ is a minimum clique secure dominating set of $G$. Hence $|V(G)| = 3 = n$ and $\gamma_{cls}(G) = 2 = k = n - 1$.

This proves the assertion. ■

**Theorem 2.7.** Given positive integers $k$, $m$ and $n$ such that $1 \leq k \leq m$ there exists a connected graph $G$ with $|V(G)| = n$, $\gamma_{cls}(G) = m$, and $\gamma_{cl}(G) = k$.

*Proof.* Consider the following cases:

*Case 1.* Suppose that $1 = k < m$.
Let $r = 3m - 1$ and $n = r + 1$ and let $G = K_1 + H$ where $H$ is obtained from $P_r = [v_1, v_2, \ldots, v_r]$ by adding the edges $v_2v_5, v_2v_8, \ldots, v_2v_{r-3}, v_5v_8, v_5, v_11, \ldots, v_5v_{r-3}$, and so on (see Figure 1).

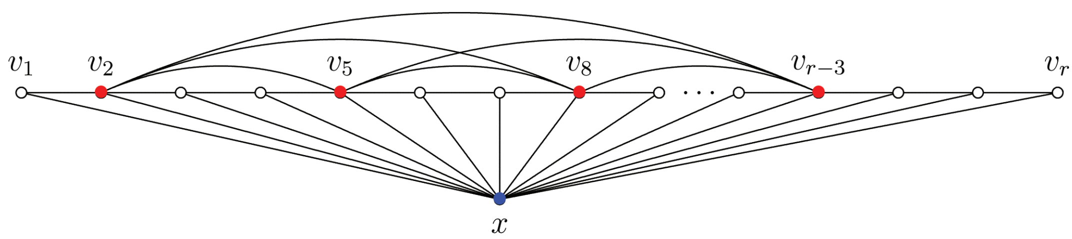


Figure 1:

The sets $A = \{x\}$ and $B = \{x, v_2, v_5, \ldots, v_{r-3}\}$ are, respectively, a minimum clique dominating set and a minimum clique secure dominating sets of $G$. Thus, $|V(G)| = r + 1 = n$, $\gamma_{cl}(G) = |A| = 1 = k$ and $\gamma_{clr}(G) = |B| = 1+[(r-3)+1]/3 = (r+1)/3 = m$.

*Case 2.* Suppose that $1 \leq k = m$.
Let $G = K_k \circ H$ where $H$ is a complete graph of order $r$ and let $n = k(r+1)$. Then the set $V(K_k)$ is a minimum clique dominating set and minimum clique secure dominating set of $G$. Hence $|V(G)| = |V(K_k \circ H)| = k(r+1) = n$, $\gamma_{cl}(G) = k$ and $\gamma_{cls}(G) = k = m$.

*Case 3.* Suppose that $1 < k < m$.
Let $P_r = [x_1, x_2, \ldots, x_r]$ and $P_{n-r} = [y_1, y_2, \ldots, y_{n-r}]$ and let $G = P_r + P_{n-r}$ ($r \geq 6$ and $n \geq 8$). If $n = 8$, then the sets $A_1 = \{x_1, y_1\}$ is a clique dominating set and $B_1 = \{x_2, x_3, y_1\}$ is a minimum secure clique dominating set of $G$. Thus, $\gamma_{cl}(G) = 2 = k$ and $\gamma_{cls}(G) = 3 = m$. If $n > 9$, then $\gamma_{cl}(G) = 2 = k$ and $\gamma_{cls}(G) = 4 = m$. Further, $|V(G)| = r + (n - r) = n$.

This proves the assertion. ■

**Corollary 2.8.** The difference $\gamma_{cls} - \gamma_{cl}$ can be made arbitrarily large.

We need the following theorem for our next result.

**Theorem 2.9. [2]** Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_s(G) = 2$ if and only if $G$ is non-complete and there exists distinct vertices $x$ and $y$ that dominate $G$ and satisfy one of the following conditions:

$(i)$ $N(x) \setminus \{y\} = N(y) \setminus \{x\} = V(G) \setminus \{x, y\}$.

$(ii)$ $\langle N(x) \setminus N[y] \rangle$ and $\langle N(y) \setminus N[x] \rangle$ are complete and for each $u \in N(x) \cap N(y)$ either $\langle (N(x) \setminus N[y]) \cup \{u\} \rangle$ or $\langle (N(y) \setminus N[x]) \cup \{u\} \rangle$ is complete.

$(iii)$ $N(x) \setminus \{y\} = V(G) \setminus \{x, y\}$, $N(x) \setminus N[y] \neq \varnothing$ and $\langle N(x) \setminus N[y] \rangle$ is complete.

Next, we characterize all connected graphs having secure clique domination number equal to two.

**Theorem 2.10.** Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_{cls}(G) = 2$ if and only if $G$ is non-complete and there exists distinct and adjacent vertices $x$ and $y$ that dominate $G$ and satisfy one of the following conditions:

$(i)$ $N(x) \setminus \{y\} = N(y) \setminus \{x\} = V(G) \setminus \{x, y\}$.

$(ii)$ $\langle N(x) \setminus N[y] \rangle$ and $\langle N(y) \setminus N[x] \rangle$ are complete and for each $u \in N(x) \cap N(y)$ either $\langle (N(x) \setminus N[y]) \cup \{u\} \rangle$ or $\langle (N(y) \setminus N[x]) \cup \{u\} \rangle$ is complete.

$(iii)$ $N(x) \setminus \{y\} = V(G) \setminus \{x, y\}$, $N(x) \setminus N[y] \neq \varnothing$ and $\langle N(x) \setminus N[y] \rangle$ is complete.

*Proof.* Suppose that $\gamma_{cls}(G) = 2$. Let $S = \{x, y\}$ be a $\gamma_{cls}$-*set* of $G$. Then $x$ and $y$ are distinct and adjacent vertices of $G$. Further, $S$ is a secure dominating set of $G$ by Lemma 2.4. Since $\gamma_{cls}(G) \neq 1$, $G$ is non-complete by Theorem 2.5 and hence $\gamma_s(G) \neq 1$ by Theorem 2.3. This implies that $\gamma_s(G) = 2$. Thus, conditions $(i)$, $(ii)$, and $(iii)$ holds by Theorem 2.9.

For the converse, suppose that $G$ is non-complete and there exists distinct and adjacent vertices $x$ and $y$ that dominate $G$ and satisfy one of the following conditions $(i)$, $(ii)$, and $(iii)$. Suppose first that condition $(i)$ holds. Then $\gamma_s(G) = 2$ by Theorem 2.9. Let $S = \{x, y\}$ be a $\gamma_s$-*set* of $G$. Since $x$ and $y$ are adjacent, $S$ is a clique dominating set of $G$. Since $S$ is a secure dominating set, it follows that $S$ is a clique secure dominating set of $G$ by Remark 2.1. Since $G$ is non-complete, $\gamma_{cls}(G) \neq 1$ by Theorem 2.5. Thus, $\gamma_{cls}(G) = 2$. Similarly, if conditions $(ii)$ or $(iii)$ holds, $\gamma_{cls}(G) = 2$. ■

The next result characterized the clique secure dominating sets in the join of two graphs.

**Theorem 2.11.** Let $G$ and $H$ be connected non-complete graphs. Then a proper subset $S$ of $V(G + H)$ is a clique secure dominating set in $G + H$ if and only if one of the following statements holds:

$(i)$ $S$ is a clique secure dominating set of $G$ and $|S| \geq 2$.

$(ii)$ $S$ is a clique secure dominating set of $H$ and $|S| \geq 2$.

$(iii)$ $S = S_G \cup S_H$ where $S_G = \{v\} \subset V(G)$ and $S_H = \{w\} \subset V(H)$ and

- $(a)$ $S_G$ is a dominating set of $G$ and $S_H$ is a dominating set of $H$; or
- $(b)$ $S_G$ is dominating set of $G$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in $H$; or
- $(c)$ $S_H$ is dominating set of $H$ and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in $G$; or
- $(d)$ $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in $G$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in $H$.

$(iv)$ $S = S_G \cup S_H$ where $S_G$ is a clique in $G$ ($|S_G| \geq 2$) and $S_H = \{w\} \subset V(H)$ and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in $G$.

$(v)$ $S = S_G \cup S_H$ where $S_G = \{v\} \subset V(G)$ and $S_H$ is a clique in $H$ ($|S_H| \geq 2$) and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in $H$.

$(vi)$ $S = S_G \cup S_H$ where $S_G$ is a clique in $G$ ($|S_G| \geq 2$) and $S_H$ is a clique in $H$ ($|S_H| \geq 2$).

*Proof.* Suppose that $S$ is a clique secure dominating set of $G+H$. Consider the following cases:

*Case 1.* Suppose that $S \subseteq V(G)$ or $S \subseteq V(H)$.
If $S \subseteq V(G)$, then $S$ is a clique secure dominating set of $G$. Now suppose that $|S| = 1$, say $S = \{a\}$. Since $S$ is a clique secure dominating set of $G + H$, $\{z\}$ is a dominating set of $G + H$ (and hence in $H$) for every $z \in V(H)$. This implies that $H$ is a complete graph, contrary to our assumption. Thus, $|S| \geq 2$. This shows that statement $(i)$ holds. Similarly, statement $(ii)$ holds if $S \subseteq V(H)$.

*Case 2.* Suppose that $S_G = S \cap V(G) \neq \varnothing$ and $S_H = S \cap V(H) \neq \varnothing$. Then $S = S_G \cup S_H$. Consider the following subcases.

*Subcase 1.* Suppose that $S_G = \{v\} \subset V(G)$ and $S_H = \{w\} \subset V(H)$. If $S_G$ is a dominating set of $G$ and $S_H$ is a dominating set of $H$, then we are done with $(iiia)$. Suppose that $S_G$ is a dominating set of $G$ and $S_H$ is not a dominating set of $H$. Let $x \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since $S$ is a clique secure dominating set of $G + H$, $\{w, x\}$ is a dominating set in $G + H$ (and hence in $H$). Since $wx \notin E(H)$, $xy \in E(H)$ for every $y \notin N_H(w)$. This implies that $y \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since $x$ was arbitrarily chosen, it follows that the subgraph $\langle (V(H) \setminus S_H) \setminus N_H(S_H) \rangle$ induced by $(V(H) \setminus S_H) \setminus N_H(S_H)$ is complete. Hence, $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in $H$. This proves $(iiib)$. Similarly, if $S_H$ is dominating set of $H$ and $S_G$ is not a dominating set of $G$, then $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in $G$. This proves $(iiic)$. If $S_G$ is not a dominating set of $G$ and $S_H$ is not a dominating set of $H$, then $(iiid)$ holds by following similar arguments in $(iiib)$ and $(iiic)$.

*Subcase 2.* Suppose that $S_G$ is a clique in $G$ ($|S_G| \geq 2$) and $S_H = \{w\} \subset V(H)$. If $S_G$ is a dominating set of $G$, then $(i)$ holds. Suppose that $S_G$ is not a dominating set of $G$. Let $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since $S$ is a clique secure dominating set of $G + H$, $S_x = (S \setminus \{w\}) \cup \{x\}$ is a dominating set of $G + H$ (and hence of $G$). Since $vx \notin E(G)$ for every $v \in S_G$, $xy \in E(G)$ for every $y \notin N_G(S_G)$ (otherwise, $S_x$ is not dominating set of $G + H$). This implies that $y \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since $x$ was arbitrarily chosen, it follows that the subgraph $\langle (V(G) \setminus S_G) \setminus N_G(S_G) \rangle$ induced by $(V(G) \setminus S_G) \setminus N_G(S_G)$ is complete. Hence $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in $G$. This proves $(iv)$. Similarly, $(v)$ holds, if $S_G = \{v\} \subset V(G)$ and $S_H \subseteq V(H)$ ($|S_H| \geq 2$).

*Subcase 3.* Suppose that $S_G$ is a clique in $G$ and $S_H$ is a clique in $H$. Let $|S_G| \geq 2$. If $S_G$ is a dominating set of $G$, then $(i)$ holds. Suppose that $S_G$ is not a dominating set of $G$. If $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in $G$, then $(iv)$ holds. Suppose that $(V(G) \setminus S_G) \setminus N_G(S_G)$ is not a clique in $G$. If $|S_H| = 1$, say $S_H = \{w\}$, then there exists $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$ such that $S_x = (S \setminus \{w\}) \cup \{x\}$ is not a dominating set of $G$(and hence of $G + H$). This contradict to our assumption that $S$ is a clique secure dominating set of $G + H$. Thus, $|S_H| \geq 2$. Similarly, if $|S_H| \geq 2$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is not a clique in $H$, then $|S_G| \geq 2$. This proves $(vi)$.

For the converse, suppose first that statement $(i)$ holds. Since $S$ is a clique secure dominating set of $G$ and $|S| \geq 2$, $S$ is a clique dominating set of $G + H$. Let $u \in V(G + H) \setminus S$. If $u \in V(G) \setminus S$, $S_u = (S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G$ and hence in $G + H$ for some $v \in S$. Similarly, if $u \in V(H) \setminus S$, then $S_u$ is a dominating set of $G + H$. Thus, $S$ is a clique secure dominating set of $G + H$. Similarly, if statement $(ii)$ holds then $S$ is a clique secure dominating set of $G + H$.

Next, suppose that $S = S_G \cup S_H$ where $S_G = \{v\} \subset V(G)$ and $S_H = \{w\} \subset V(H)$ and $(iiia)$ to $(iiid)$ holds. Suppose first that $(iiia)$ holds. Then $S = \{v, w\}$ is a dominating set of $G + H$ and $vw \in E(G + H)$, that is, $S$ is a clique dominating set of $G + H$. Let $x \in V(G + H) \setminus S$. Then $S_x = \{v, x\}$. Since $S_G = \{v\}$ is a dominating set of $G$ and hence in $G + H$, it follows that $S_x$ is a dominating set in $G + H$. Thus, $S$ is a clique secure dominating set of $G+H$.

Suppose that $(iiib)$ holds. Then $S = S_G \cup S_H = \{v, w\}$ is a clique dominating set of $G+H$. Let $x \in V(G+H) \setminus S$. If $x \in V(G) \setminus S$, then $S_x = \{v, x\}$ is a dominating set of $G$ and hence of $G+H$, that is, $S$ is a clique secure dominating set of $G+H$. Now, let $x \in V(H) \setminus S$. If $xw \in E(H)$, then $S_x = \{v, x\}$ is a dominating set of $G+H$, that is, $S$ is a clique secure dominating set of $G+H$. If $xw \notin E(H)$, then $x \in [(V(H) \setminus S_H) \setminus N_H(S_H)]$ and $S_x = \{w, x\}$. Thus,

$$\begin{aligned}
N_H[S_x] &= S_x \cup N_H(S_x) \\
&= \{w, x\} \cup N_H(\{w, x\}) \\
&= \{w\} \cup \{x\} \cup N_H(w) \cup N_H(x) \\
&= \{w\} \cup N_H(w) \cup \{x\} \cup N_H(x) \\
&= S_H \cup N_H(S_H) \cup N_H[x] \\
&= S_H \cup N_H(S_H) \cup [(V(H) \setminus S_H) \setminus N_H(S_H)] \\
&= V(H).
\end{aligned}$$

This implies that $S_x$ is a a dominating set of $H$ and hence of $G+H$. Again, $S$ is a clique secure dominating set of $G+H$. Similarly, $S$ is a clique secure dominating set of $G+H$ if $(iiic)$ holds.

Suppose that $(iiid)$ holds. Then $S = S_G \cup S_H = \{v, w\}$ is a clique dominating set of $G+H$. Let $x \in V(G+H) \setminus S$. Consider $x \in V(G) \setminus S$. If $xv \in E(G)$, then $S_x = \{w, x\}$ is a dominating set of $G+H$, that is, $S$ is a clique secure dominating set of $G+H$. If $xv \notin E(G)$, then $x \in [(V(G) \setminus S_G) \setminus N_G(S_G)]$ and $S_x = \{v, x\}$. Thus,

$$\begin{aligned}
N_G[S_x] &= S_x \cup N_G(S_x) \\
&= \{v, x\} \cup N_G(\{v, x\}) \\
&= \{v\} \cup \{x\} \cup N_G(v) \cup N_G(x) \\
&= \{v\} \cup N_G(v) \cup \{x\} \cup N_G(x) \\
&= S_G \cup N_G(S_G) \cup N_G[x] \\
&= S_G \cup N_G(S_G) \cup [(V(G) \setminus S_G) \setminus N_G(S_G)] \\
&= V(G).
\end{aligned}$$

This implies that $S_x$ is a a dominating set of $G$ and hence of $G+H$, that is, $S$ is a clique secure dominating set of $G+H$. Similarly, if $x \in V(H) \setminus S$, then $S$ is a clique secure dominating set of $G+H$.

Suppose that $(iv)$ holds. Then $S = S_G \cup S_H$ is a clique dominating set of $G+H$. Let $x \in V(G+H) \setminus S$. Consider $x \in V(G) \setminus S$. If $xv \in E(G)$ for some $v \in S$, then $S_x = (S \setminus \{v\}) \cup \{x\}$ is a dominating set of $G+H$, that is, $S$ is a clique secure dominating set of $G+H$. If $xv \notin E(G)$, then $x \in [(V(G) \setminus S_G) \setminus N_G(S_G)]$ and $S_x = (S \setminus \{w\}) \cup \{x\}$.

Thus,

$$\begin{aligned} N_G[S_x] &= S_x \cup N_G(S_x) \\ &= [(S \setminus \{w\}) \cup \{x\}] \cup N_G((S \setminus \{w\}) \cup \{x\}) \\ &= S \setminus \{w\} \cup \{x\} \cup N_G(S \setminus \{w\}) \cup N_G(x) \\ &= S \setminus \{w\} \cup N_G(S \setminus \{w\}) \cup \{x\} \cup N_G(x) \\ &= S_G \cup N_G(S_G) \cup N_G[x] \\ &= S_G \cup N_G(S_G) \cup [(V(G) \setminus S_G) \setminus N_G(S_G)] \\ &= V(G). \end{aligned}$$

This implies that $S_x$ is a a dominating set of $G$ and hence of $G + H$, that is, $S$ is a clique secure dominating set of $G + H$. Similarly, if $x \in V(H) \setminus S$, then $S$ is a clique secure dominating set of $G + H$.

By the same arguments utilized above, $S$ is a clique secure dominating set of $G + H$ if any of the following statements $(v)$ or $(vi)$ holds. The proof is completed. ■

The following result is a quick consequence of Theorem 2.11.

**Corollary 2.12.** Let $G$ and $H$ be connected non-complete graphs and let $S_G \subset V(G)$ and $S_H \subset V(H)$. Then

$$\gamma_{cls}(G+H) = \begin{cases} 2, \text{ if } \gamma(G) = 1 = \gamma(H) \text{ or } \gamma_{cl}(G) = 2 \text{ or } \gamma_{cl}(H) = 2 \\ 3, \text{ if } \gamma(G) = 2(\text{or } 3)\text{or } \gamma(H) = 2(\text{ or } 3) \\ 4, \text{ if otherwise.} \end{cases}$$

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