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The Role of Fuzzy ℓ-ideals in a Commutative ℓ-group

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Abstract

This paper investigates the concept of fuzzy ℓ-ideals of commutative ℓ-group. We obtain a characterization theorem for fuzzy ℓ-ideals. Moreover, some results and properties on fuzzy ℓ-ideals are discussed.

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Keywords: Fuzzy sub ℓ-group, fuzzy ℓ-ideal, Level ℓ-ideal.

1. Introduction

One of the most important facts of fuzzy logic is constitute by fuzzy set theory. L.A. Zadeh [18] introduced the notion of a fuzzy set of a set X as a function from X into [0,1] and Rosenfeld [14] applied it to group theory and developed the theory of fuzzy groups. Das [3] introduced fuzzy subgroups by their level subgroups. Since then researchers in various disciplines of Mathematics have been trying to extend their ideas to the broader frame work of the fuzzy setting. R. Natarajan and J. Vimala [11] introduced the concept of ideals and distributive ℓ-ideals in commutative lattice ordered groups. The concepts of fuzzy sublattice, fuzzy ideal, fuzzy prime ideal in lattice were introduced by many authors. U.M. Swamy and D. Viswananda Raju [16] developed the theory of fuzzy ideals and gave some interesting results. The partially ordered algebraic systems play vital role in algebra. Lattice ordered groups and lattice ordered rings are Some important concepts in partially ordered systems. These concepts play a prominent role in with wide ranging applications in many disciplines.

In algebra, a fundamental domain is the lattice theory. The formation of lattices is an important feature of many structures such as subgroups of a group, ideals of a ring, submodules of a module or a ideals of a lattice. This provides sufficient motivation to researchers to analyse the results from the real of abstract algebra in the broader framework of fuzzy setting. G.S.V. Sathya Saibaba [15] considered fuzzy lattice ordered group as a mapping from lattice ordered group into a complete lattice and he introduced the notion of ℓ-fuzzy ideal of a lattice ordered group. In this paper some properties and important results on fuzzy ℓ-ideal of commutative lattice ordered group are given.

2. Preliminaries

Definition 2.1. [1] A non-empty set G is called a commutative ℓ-group iff

(i) $(G, +)$ is a commutative group

(ii) $(G, \leq)$ is a lattice

(iii) $x \leq y$ implies $a + x + b \leq a + y + b$ for all a, b in G.

Definition 2.2. [1] A non-empty set G is called a commutative ℓ-group iff

(i) $(G, +)$ is a commutative group.

(ii) $(G, \vee, \wedge)$ is a lattice.

(iii) $a + (x \vee y) = (a + x) \vee (a + y)$ and $a + (x \wedge y) = (a + x) \wedge (a + y)$ for all a, b in G.

Result 2.3. [1] The above two definitions of commutative ℓ-group are equivalent.

Definition 2.4. [4] Let G be a commutative ℓ-group. A non-empty subset I of G is called an ℓ-ideal of G if

(i) I is a subgroup of G.

(ii) I is a sublattice of G.

(iii) $0 < x < a$ and $a \in I$ implies $x \in I$.

Definition 2.5. [18] A fuzzy set is a pair (X, μ) where X is any non empty set and $\mu : X \to [0, 1]$.

Definition 2.6. [18] Let (X, μ) be a fuzzy set and $t \in [0, 1]$. Then the set $\mu_t = \{x \in X/\mu(x) \geq t\}$ is called the level set of μ.

Definition 2.7. [18] Let (X, μ) be a fuzzy set. Then the set $\{\mu(x)/x \in X\}$ is called the image of μ and is denoted by $Im(\mu)$.

Definition 2.8. [18] Let (X, μ) be a fuzzy set. The set $\{x/x \in X, \mu(x) > 0\}$ is called the support of μ and it is denoted by supp(μ).

Definition 2.9. [15] Let $G = (G, +, \wedge, \vee)$ be a commutative ℓ-group. A fuzzy set (G, μ) said to be fuzzy sub ℓ-group of G if

(i) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$

(ii) $\mu(-x) = \mu(x)$

(iii) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$

(iv) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$ for all x, y in G.

Result 2.10. [15] Let G be a commutative ℓ-group and (G, μ) be a fuzzy sub ℓ-group. Then $\mu(0) \geq \mu(x)$ for all $x \in G$.

Result 2.11. [15] Let G be a commutative ℓ-group and (G, μ) be a fuzzy sub ℓ-group. If $\mu(x - y) = \mu(0)$ then $\mu(x) = \mu(y)$ where $x, y \in G$.

Result 2.12. [15] Let G be a commutative ℓ-group and μ be a fuzzy sub ℓ-group of G. Then $\mu(x + y) = \mu(y)$ for all $x, y \in G$ iff $\mu(x) = \mu(0)$.

Result 2.13. [15] Let G be a commutative ℓ-group. Then the intersection of any family of fuzzy sub ℓ-groups of G is a fuzzy sub ℓ-group.

3. Fuzzy ℓ-ideals in Commutative ℓ-groups

Definition 3.1. Let G be a commutative ℓ-group. A fuzzy set (G, μ) is called a fuzzy l-ideal of G if

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$

(ii) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$

(iii) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$

(iv) $0 < x < a \Rightarrow \mu(x) \geq \mu(a)$ for $x, y, a, b \in G$.

Proposition 3.2. Let G be a commutative ℓ-group. If μ is a fuzzy ℓ-ideal of G then supp(μ) is an ℓ-ideal of G if Supp$\neq \phi$.

Proof. We have Supp(μ) $= \{x/x \in G, \mu(x) > 0\}$. Let $x, y \in Supp(\mu)$.
$\Rightarrow \mu(x) > 0$ and $\mu(y) > 0$
$\Rightarrow \mu(x \wedge y) \geq \mu(x) \wedge \mu(y) > 0$ and $\mu(x \vee y) \geq \mu(x) \vee \mu(y) > 0$
$\Rightarrow x \wedge y, x \vee y \in Supp(\mu)$.
Also $\mu(x - y) \geq \mu(x) \wedge \mu(y) > 0$

$\Rightarrow x - y \in Supp(\mu)$
$\Rightarrow Supp(\mu)$ is a ℓ-subgroup of G.
Let $0 < x < a$ and $a \in Supp(\mu)$. Since μ is a fuzzy ℓ-ideal, $\mu(x) \geq \mu(a)$.
$a \in Supp(\mu) \Rightarrow \mu(a) > 0$
$\Rightarrow \mu(x) \geq \mu(a) > 0$
$\Rightarrow \mu(x) > 0$
$\Rightarrow x \in Supp(\mu)$
$\Rightarrow$ Supp(μ) is an ℓ-ideal of G. ■

Proposition 3.3. Let μ be a fuzzy subset of a commutative ℓ-group G. If two level sets μ_t, μ_s for $t, s \in [0, 1]$ are equal with $t < s$ then there is no x in G such that $t < \mu(x) < s$.

Proof. Let

$$\mu_t = \mu_s \tag{1}$$

Suppose there exist $x \in G$ such that $t < \mu(x) < s$.
(1) $\Rightarrow \mu_s \subseteq \mu_t$,
If $x \in \mu_t$ then x may not belongs to μ_s which is a contradiction to (1).
There is no $x \in G$ such that $t < \mu(x) < s$. ■

Proposition 3.4. (CHARACTERIZATION THEOREM) Let G be a commutative ℓ-group. A fuzzy set (G, μ) is a fuzzy ℓ-ideal of G if and only if the set $\mu_t = \{x \in G/\mu(x) \geq t\}$ is an ℓ-ideal of G for all $t \in [0, 1]$ with $\mu_t \neq \phi$.

Proof. Let μ be a fuzzy ℓ-ideal of G. Let $\mu_t = \{xinG/\mu(x) \geq t, tin[0, 1]\}$. For all $x \in G$, $\mu(x) \leq \mu(0)$
$\Rightarrow \mu(0) \geq t$
$\Rightarrow 0 \in \mu_t$
$\Rightarrow \mu_t \neq \phi$
To prove μ_t is an ℓ-ideal of G.

(i) μ_t is a subgroup of G

(ii) μ_t is a sublattice of G.

(iii) $0 < x < a$ and $a \in \mu_t \Rightarrow x \in \mu_t$.

Let $x, y \in \mu_t$ be arbitrary.
$\Rightarrow \mu(x) \geq t$ and $\mu(y) \geq t$ (1)
Now $\mu(x - y) \geq min\{\mu(x), \mu(y)\} \geq t$ by (1).
$\Rightarrow \mu(x - y) \geq t$
$x - y \in \mu_t$.
Also $0 \in \mu_t \Rightarrow$ Identity element exist in μ_t.
$\Rightarrow \mu_t$ is a subgroup of G

Let $x, y \in \mu_t$.
$\Rightarrow \mu(x) \geq t$ and $\mu(y) \geq t$
$\Rightarrow \mu(x) \wedge \mu(y) \geq t$ and $\mu(x) \vee \mu(y) \geq t$.
$\Rightarrow \mu(x \wedge y) \geq min\{\mu(x), \mu(y)\} \geq t$
$\Rightarrow x \vee y \in \mu_t$ and $\mu(x \wedge y) \geq min\{\mu(x), \mu(y)\} \geq t$.
$\Rightarrow x \wedge y \in \mu_t$
$\Rightarrow$ For $x, y \in \mu_t, x \wedge y, x \vee y \in \mu_t$
$\Rightarrow \mu_t$ is a sublattice.
Let a $\in \mu_t$
Assume that $0 < x < a$.
$a \in \mu_t \Rightarrow \mu(a) \geq t$
Since $x < a, \mu(x) \geq \mu(a)$
$\Rightarrow \mu(x) \geq \mu(a) \geq t$
$\Rightarrow \mu(x) \geq t \Rightarrow x \in \mu_t$.
Hence μ_t is an ℓ-ideal of G. Conversely assume that μ_t is an ℓ-ideal of G. To Prove μ is a fuzzy ℓ-ideal of G.

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$

(ii) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$

(iii) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$

(iv) $0 < x < a \Rightarrow \mu(x) \geq \mu(a)$ for $x, a, b \in G$.

Let $min\{\mu(a), \mu(b)\} = r$.
Either $\mu(a) = r, \mu(b) \geq \mu(a)$ or $\mu(b) = r, \mu(a) \geq \mu(b)$
$\Rightarrow \mu(a) \geq r$ and $\mu(b) \geq r$
$\Rightarrow a, b \in \mu_r$
$\Rightarrow a - b, a \wedge b, a \vee b \in \mu_r$
$\Rightarrow \mu(a - b) \geq r, \mu(a \wedge b) \geq r$ and $\mu(a \vee b) \geq r$.
Let $a, x \in G$ and $0 < x < a$.
Let $\mu(a) = t$.
$\Rightarrow a \in \mu_t$ and $x \in \mu_t$.
$\Rightarrow \mu(x) \geq t = \mu(a)$
$\Rightarrow \mu(x) \geq \mu(a)$.
Hence μ is a fuzzy ℓ-ideal of G. ■

Proposition 3.5. Let G be a commutative ℓ-group and μ be a fuzzy ℓ-ideal of G. If image of $\mu = \{t_0, t_1, \ldots t_n\}$ with $t_0 > t_1 > t_n$ then we have $\mu_{t_0} \subseteq \mu_{t_1} \ldots \subseteq \mu_{t_n}$.

Proof. Let μ be the fuzzy ℓ-ideal of a commutative ℓ-group G. Assume that $t_0 > t_1 \cdots > t_n$.
Let $x_1 \in \mu_{t_0}$
$\Rightarrow x_1 \in G$ such that $\mu(x_1) \geq t_0$
$\Rightarrow x_1 \in G$ such that $\mu(x_1) \geq t_0 > t_1$

$\Rightarrow x_1 \in \mu_{t_0}$
$\Rightarrow \mu_{t_0} \subseteq \mu_{t_1}$
Again let $x_2 \in \mu_{t_1}$.
$\Rightarrow x_2 \in G$ such that $\mu(x_2) \geq t_1$
$\Rightarrow x_2 \in G$ such that $\mu(x_2) \geq t_1 > t_2$
$\Rightarrow x_2 \in \mu_{t_2}$
$\Rightarrow \mu_{t_1} \subseteq \mu_{t_2}$.
Hence we get $\mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2}$.
Proceeding like this we get $\mu_{t_0} \subseteq \mu_{t_1} \cdots \subseteq \mu_{t_n}$. ∎

Definition 3.6. The union of two fuzzy ℓ-ideals μ_1 and μ_2 of a commutative ℓ-group G denoted by $(\mu_1 \cup \mu_2)$ is a fuzzy subset of G defined by $(\mu_1 \cup \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}$ for all $x \in G$.
The intersection of two fuzzy ℓ-ideals μ_1 and μ_2 of a commutative ℓ-group G denoted by $(\mu_1 \cap \mu_2)$ is a fuzzy subset of G defined by
$(\mu_1 \cap \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}$ for all $x \in G$.

Definition 3.7. Let μ_1 and μ_2 be any two fuzzy ℓ-ideals of a commutative ℓ-group G. Then μ_1 is said to be contained in μ_2 denoted by $\mu_1 \subseteq \mu_2$ if $\mu_1(x) \leq \mu_2(x)$ for all $x \in G$. If $\mu_1(x) = \mu_2(x)$ for all $x \in G$ then μ_1 and μ_2 are said to be equal and we can write $\mu_1 = \mu_2$.

Proposition 3.8. Let μ_1 and μ_2 be any two fuzzy ℓ-ideals of a commutative ℓ-group G. If $\mu_1 \subseteq \mu_2$ then $\mu_1 \cup \mu_2 = \mu_2$ and $\mu_1 \cap \mu_2 = \mu_1$.

Proof. Given μ_1 and μ_2 are any two fuzzy ℓ-ideals of a ℓ-group G. Assume that $\mu_1 \subseteq \mu_2$. Let $x \in G$ be arbitrary. Then $\mu_1(x) \leq \mu_2(x)$ for all $x \in G$. We have $\mu_1 \cup \mu_2 = \max\{\mu_1(x), \mu_2(x)\}$ for all $x \in G = \mu_2(x) \Rightarrow \mu_1 \cup \mu_2 = \mu_2$. Also $\mu_1 \cap \mu_2(x) = \min\{\mu_1(x), \mu_2(x)\} = \mu_1(x) \Rightarrow \mu_1 \cap \mu_2 = \mu_1$. ∎

Proposition 3.9. Let μ_1 and μ_2 be any two fuzzy ℓ-ideals of an ℓ-group G. Then $\mu_1 \cup \mu_2 \supseteq \mu_1 \cap \mu_2$.

Proof. Let $x \in G$ be arbitrary. Then

$$\begin{aligned}(\mu_1 \cup \mu_2)(x) &= \max\{\mu_1(x), \mu_2(x)\}\\ &\geq \min\{\mu_1(x), \mu_2(x)\}\\ &= (\mu_1 \cap \mu_2)(x) \Rightarrow \mu_1 \cup \mu_2 \subseteq \mu_1 \cap \mu_2.\end{aligned}$$

∎

Proposition 3.10. Intersection of any two fuzzy ℓ-ideals of an commutative ℓ-group is a fuzzy ℓ-ideal.

Proposition 3.11. Union of two fuzzy ℓ-ideals need not be a fuzzy ℓ-ideal.

Proposition 3.12. If μ_1 and μ_2 are any two fuzzy ℓ-ideals of the commutative ℓ-group G then $\mu_1 \wedge \mu_2 = \mu_1 \cap \mu_2$ where $\mu_1 \wedge \mu_2$ is defined by $(\mu_1 \wedge \mu_2) = \sup_{x=y\wedge z} \{\min\{\mu_1(y), \mu_2(z)\}\}$ where $x, yz \in G$.

Proof.

$$\begin{aligned}\mu_1 \wedge \mu_2(x) = &\sup_{x=y\wedge z} \{min\{\mu_1(y), \mu_2(z)\}\}\\ &\geq \min\{\mu_1(y), \mu_2(z)\} = (\mu_1 \cap \mu_2)(x)\}\\ &\Rightarrow (\mu_1 \wedge \mu_2)(x) \geq (\mu_1 \cap \mu_2)(x) \qquad (1)\end{aligned}$$

Again $(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\}$. Let $x = a \wedge b$. We have $a \wedge b \leq a$ and $a \wedge b \leq b$.
$\Rightarrow x \leq a$ and $x \leq b$.
Since μ_1 is a fuzzy ideal, $\mu_1(x) \geq \mu_1(a)$ and μ_2 is a fuzzy ideal, $\mu_2(x) \geq \mu_2(b)$.

$$\begin{aligned}&\Rightarrow \min\{\mu_1(a), \mu_2(b)\} \leq \min \mu_1(x), \mu_2(x)\\ &\Rightarrow \min\{\mu_1(a), \mu_2(b)\} \leq (\mu_1 \cap \mu_2)(x).\\ &\Rightarrow (\mu_1 \cap \mu_2)(x) \geq \min\{\mu_1(a), \mu_2(b)\}\\ &\Rightarrow (\mu_1 \cap \mu_2)(x) \geq \sup\{\min\{\mu_1(a), \mu_2(b)\}\}.\\ &\Rightarrow (\mu_1 \cap \mu_2)(x) \geq (\mu_1 \wedge \mu_2)(x). \qquad (2)\end{aligned}$$

From (1) and (2) $\mu_1 \wedge \mu_2 = \mu_1 \cap \mu_2$. ■

Proposition 3.13. If μ_1 and μ_2 are any two fuzzy ℓ-ideals of the commutative ℓ-group G then $\mu_1 \wedge \mu_2$ is also a fuzzy ℓ-ideal of G.

Conflict of interest

The authors declare that there are no conflicts of interest.

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