Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 3 (2016), pp. 2053–2065

http://www.ripublication.com/gjpam.htm

Spectral gap of a mean field type tri-color exclusion process

Toumi Manel
LaPS laboratory, Badji Mokhtar-university,
BP 12, Annaba 23000, Algeria.

Abstract

We consider a mean field type tri-color exclusion process in a volume Λ of $\mathbb{Z}^d$. In this model the jump rate is general and the particles of each color are subjected to a speed different from 1. The aim of the present paper is to compute the spectral gap of a mean field tri–color exclusion process.

AMS subject classification: 60K35, 60F10, 60J25.
Keywords: Markov generator, configuration, reversible measure, spectral gap, exclusion processes.

1. Introduction

One of the difficulties in studying the hydrodynamic limit is to give the speed of convergence to equilibrium of conservative stochastic dynamics related to finite cubes. This requires an estimate of the spectral gap of the corresponding Markov generator. More precisely, the spectral gap shrinks at a rate slower than N^{-2}, where N is the side-length of a cube as has been proven by Kipnis and Ladim [7].

Caputo [1] gives a general strategy which can be generalized to many situations including the case of a multi-colored disorder lattice gas exclusion process (or even multi-species exclusion process). This opens the way to Dermoune and Heinrich [3], where in a first step to get the hydrodynamic limit of this colored simple exclusion process similar to the one described by Faggionnato and Martinelli [6], they estimate the spectral gap by using the matrix representation of the dynamic. Another step was reached by Zeghdoudi and Boutabia [9], where they compute the spectral gap of exclusion processes related to a colored lattice gas in the case where the jump rate equal to 1.

Dermoune and Heinrich [5] have in another paper, consider the case of the non homgeneous nearest neighbor process and showed that if the particles are subjected to a speed equal to 1 and the collection of probabilities of occupation is periodic, then the spectral gap is bounded by $C\rho_0 N^{-2}$, where ρ_0 is the density of empty sites. Nagahata and Sasada [8] extends the previous results to a multi-species exclusion process in the homogeneous and non-homogeneous hypercubes in $\mathbb{Z}^d$, with the assumption that the dynamic satisfies a condition of irreducibility. It appears through these works that the new destination in the study of exclusion processes is that the spectral gap depends on ρ_0.

In this work we give the exact value of the spectral gap of a mean field type homogeneous bi-color exclusion process similar to the one used by Nagahata and Sasada [8], without any assumption. The rate of probability that a particle jumps from one site to another is different from 1 and the speed of the particles is general. An intermediate result was obtained, which is to explicit the canonical measures of configurations. The matricial representation of the dynamics and a transformation of the covariance matrix of the marginal were the key of these results.

The paper is organized as follows. In Section 2 we introduce the concepts needed for the model studied, such as the jump rate and the generator. In section 3, we give the canonical measures of configurations and the matricial representation of the Dirichlet form. In section 4, we state our main results.

2. Preliminaries

We consider a finite subset Λ of the d-dimensionl lattice $\mathbb{Z}^d$, that cardinality N. As described by Nagahata and Sasada [8] the mean field type multi-color exlusion process is defined as follows. Let us consider the number of colors $r \geq 1$, configuration is denoted by $\eta \in \{0, \ldots, r\}^{\Lambda}$ with interpertation that $\eta(x) = 0$ means that the site x is empty, where as $\eta(x) = i$ for $i \in \{1, \ldots, r\}$ means that x is occupied by a particle of the $i-th$ color. It is obvious that the numbers of particles are conserved by the dynamic. Let $k = (k_0, \ldots, k_r) \in \mathbb{N}^{r+1}$ such that $\sum_{i=0}^{r} k_i = N$ and $1 \leq k_i$, $N_i(\eta)$ for $i \in \{0, 1, \ldots, r\}$ is the number of particles of the $i-th$ color on the configuration η (0 is considered as a color), and let

$$S_k = \left\{\eta \in \{0, \ldots, r\}^{\Lambda} : N_i(\eta) = k_i,\ i = 0, 1, \ldots, r\right\}.$$

Let $g : \{0, \ldots, r\} \to \mathbb{R}$ be a function satisfying $g(0) = 0$, $g(i) > 0$ for all $i \in \{1, \ldots, r\}$. The generator acting on the function $f : \{0, \ldots, r\}^{\Lambda} \to \mathbb{R}$ as

$$(Lf)(\eta) = \sum_{x,y\in\Lambda} g(\eta_x) c_{x,y}(\eta)\, \pi^{x,y} f(\eta),$$

here $\pi^{x,y}$ is the operator defined by

$$\pi^{x,y} = f\left(\eta^{x,y}\right) - f(\eta),$$

$\eta^{x,y}$ is the configuration obtained from η letting η_x and η_y be exchanged, that is,

$$\left(\eta^{x,y}\right)_z = \begin{cases} \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y \\ \eta_z & \text{otherwise,} \end{cases}$$

and $c_{x,y}(\eta)$ the jump rate for a collection of occupation probability $\left\{q_x,\ x \in \mathbb{Z}^d\right\}$ is defined by

$$c_{x,y}(\eta) = \begin{cases} q_x\left(1-q_y\right) & \text{if } \left(\xi_x(\eta), \xi_y(\eta)\right) = (0,1) \\ q_y\left(1-q_x\right) & \text{if } \left(\xi_x(\eta), \xi_y(\eta)\right) = (1,0) \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi_x(\eta) = \mathbf{1}_{\{\eta_x \geq 1\}} \in \{0,1\}^\Lambda$ for all $x \in \Lambda$ is the configuration of occupied sites associated to η.

Each particle of the i-th color at x jump to y at rate $g(i)c_{x,y}(\eta)$ if y is empty. Here the constant $g(i)$ represents the speed of the particle of the i-th color.

The generator L defines a Markov process $\left\{\eta(t), t \in \mathbb{R}^+\right\}$ on $\{0,\ldots,r\}^\Lambda$ called mean field type non homogeneous multi-color exclusion process with parameter $\left\{g, \{q_x\}_{x\in\mathbb{Z}^d}\right\}$. When $q_x = q$ does not depend on x we say that we are in the homogeneous case. Actually, in that case

$$g\left(\eta_x\right) c_{x,y}(\eta) = g\left(\eta_x\right) q\,(1-q)\,\mathbf{1}_{\{\eta_y=0\}}.$$

3. The Dirichlet form

We first describe reversible measures of the process. Let $(s_i)_{1\leq i\leq r}$ be a probability distribution on the set $\{1,\ldots,r\}$ and μ be the product probability measure on $\{0,\ldots,r\}^\Lambda$ defined by

$$\mu(\eta) = \prod_{i=1}^{r} s_i^{N_i(\eta)} \prod_{z\in\Lambda} q^{\xi_z(\eta)}\,(1-q)^{(1-\xi_z(\eta))}.$$

We denote the canonical measure associated to μ by $\nu_k(.) := \mu(.\mid S_k)$ which is indeed independent of choise of $(s_i)_{1\leq i\leq r}$. More precisely ν_k is the uniforme measure on S_k. It easy to see that the generator L is reversible w.r.t μ and the same is true for the measure ν_k since $\nu_k(\eta)\,c_{x,y}(\eta) = \nu_k\left(\eta^{x,y}\right) c_{x,y}\left(\eta^{x,y}\right)$ for all x, y and η. We set for simplicity

$$\eta_x^i = \mathbf{1}_{\{\eta^x=i\}} \text{ and } \overline{\eta_x^i} = \eta_x^i - \nu_k(\eta_x^i) \text{ for } x \in \Lambda \text{ and } i \in \{0,1,\ldots,r\}.$$

The following theorem explicite the canonical measures of configurations.

Theorem 3.1. For all i, $j \in \{0,1,\ldots,r\}$ and for all $x, y \in \Lambda$, we have for $N \geq 3$

$$\nu_k\left(\eta_x^i \eta_y^j\right) = \frac{k_i k_j}{N\,(N-1)} \tag{1}$$

$$\nu_k\left(\eta_x^i\eta_y^i\right)=\frac{k_i\left(k_i-1\right)}{N\left(N-1\right)}$$

and

$$\nu_k\left(\eta_x^i\right)=\frac{k_i}{N}.$$

Remark 3.2. For all $i, j \in \{0, 1, \ldots, r\}$ and for all $x, y \in \Lambda$, we have for $N \geq 3$

$$\nu_k\left(\eta_x^i;\eta_y^j\right)=\frac{k_ik_j}{N^2\left(N-1\right)}$$

$$\nu_k\left(\eta_x^i;\eta_y^i\right)=\frac{k_i\left(k_i-N\right)}{N^2\left(N-1\right)}$$

$$\nu_k\left(\eta_x^i;\eta_x^i\right)=\frac{k_i\left(N-k_i\right)}{N^2}$$

and

$$\nu_k\left(\eta_x^i;\eta_x^j\right)=-\frac{k_ik_j}{N^2},$$

where $\nu_k(f;g)$ stands for the covariance w.r.t. ν, that is $\nu_k(f;g)=\nu_k(fg)-\nu_k(f)\,\nu_k(g)$.

The Dirichlet form is defined by

$$\mathcal{D}(f)=\frac{1}{N}\sum_{x,y\in\Lambda}\nu_k\left(g(\eta_x)c_{x,y}(\eta)\left(\pi^{x,y}f(\eta)\right)^2\right).$$

In the homogeneous case we write $\mathcal{D}_q(f)$. The following lemma gives the matricial representation of $\mathcal{D}_q(f)$ in the case $r=3$ (three colors). For this purpose, we consider the $3N$ by $3N$ matrix

$$X=\begin{pmatrix} k_1g(1)M_N & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & k_2g(2)M_N & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & k_3g(3)M_N\end{pmatrix}, \tag{2}$$

such that

$$M_N=\left(m_{x,y}\right)\text{ with }m_{x,x}=\frac{2k_0q\left(1-q\right)}{N^2}\text{ and }m_{x,y}=-\frac{k_0q\left(1-q\right)}{N^2\left(N-1\right)}\text{ if }x\neq y$$

and $\mathbf{O}$ is the zero matrix.

Lemma 3.3. Assume that $r=3$ and $N\geq 3$. Let $f=\sum_{x\in\Lambda}f_x^1\overline{\eta_x^1}+f_x^2\overline{\eta_x^2}+f_x^3\overline{\eta_x^3}$, where $f_x^i(x\in\Lambda, i=\overline{1,3})$ are real numbers. Then we have

$$\mathcal{D}_q(f)=fXf^T, \tag{3}$$

where f is identified with the row

$$\left(\left(f_x^1\right)_{x\in\Lambda}, \left(f_x^2\right)_{x\in\Lambda}, \left(f_x^3\right)_{x\in\Lambda}\right)$$

in the second member of (3).

Proof. We have

$$\begin{aligned}\nu_k\left(g(\eta_x)c_{x,y}(\eta)\left(\pi^{x,y}f(\eta)\right)^2\right) &= q(1-q)\,\nu_k\left(g(\eta_x)1_{\{\eta_y=0\}}\left(\pi^{x,y}f(\eta)\right)^2\right)\\ &= q(1-q)\sum_{i=1}^{3} g(i)\nu_k\left(1_{\{\eta_x=i\}}1_{\{\eta_y=0\}}\left(\pi^{x,y}f(\eta)\right)^2\right).\end{aligned}$$

The definition of $\eta^{x,y}$ implies that

$$\pi^{x,y}f(\eta) = \left(f_x^1 - f_y^1\right)\left(\overline{\eta_y^1} - \overline{\eta_x^1}\right) + \left(f_x^2 - f_y^2\right)\left(\overline{\eta_y^2} - \overline{\eta_x^2}\right) + \left(f_x^3 - f_y^3\right)\left(\overline{\eta_y^3} - \overline{\eta_x^3}\right)$$

Note that, since $\nu_k\left(\eta_y^i\right) = \nu_k\left(\eta_x^i\right)$, then

$$\overline{\eta_y^1} - \overline{\eta_x^1} = \eta_y^1 - \nu_k\left(\eta_y^1\right) - \eta_x^1 + \nu_k\left(\eta_x^1\right) = -1$$

and

$$\begin{cases}\overline{\eta_y^2} - \overline{\eta_x^2} = \eta_y^2 - \nu_k\left(\eta_y^2\right) - \eta_x^2 + \nu_k\left(\eta_x^2\right) = 0\\ \overline{\eta_y^3} - \overline{\eta_x^3} = \eta_y^3 - \nu_k\left(\eta_y^3\right) - \eta_x^3 + \nu_k\left(\eta_x^3\right) = 0\end{cases}$$

on $\left\{\eta_x = 1, \eta_y = 0\right\}$, which impies that

$$\nu_k\left(\mathbf{1}_{\{\eta_x=1\}}\mathbf{1}_{\{\eta_y=0\}}\left(\pi^{x,y}f(\eta)\right)^2\right) = \nu_k\left(\mathbf{1}_{\{\eta_x=1\}}\mathbf{1}_{\{\eta_y=0\}}\left(f_x^1 - f_y^1\right)^2\right),$$

similarly we have

$$\nu_k\left(\mathbf{1}_{\{\eta_x=2\}}\mathbf{1}_{\{\eta_y=0\}}\left(\pi^{x,y}f(\eta)\right)^2\right) = \nu_k\left(\mathbf{1}_{\{\eta_x=2\}}\mathbf{1}_{\{\eta_y=0\}}\left(f_x^2 - f_y^2\right)^2\right).$$

and

$$\nu_k\left(\mathbf{1}_{\{\eta_x=3\}}\mathbf{1}_{\{\eta_y=0\}}\left(\pi^{x,y}f(\eta)\right)^2\right) = \nu_k\left(\mathbf{1}_{\{\eta_x=3\}}\mathbf{1}_{\{\eta_y=0\}}\left(f_x^3 - f_y^3\right)^2\right).$$

Then we have

$$\begin{aligned}&\nu_k\left(g(\eta_x)c_{x,y}(\eta)\left(\pi^{x,y}f(\eta)\right)^2\right)\\ &\quad= \left[\alpha_1\left(f_x^1 - f_y^1\right)^2 + \alpha_2\left(f_x^2 - f_y^2\right)^2 + \alpha_3\left(f_x^3 - f_y^3\right)^2\right],\end{aligned}$$

where, taking into acount the formula (1)

$$\alpha_i = g(i)q\,(1-q)\,\nu_k\left(\mathbf{1}_{\{\eta_x=i\}}\mathbf{1}_{\{\eta_y=0\}}\right) = g(i)q\,(1-q)\,\frac{k_0k_i}{N\,(N-1)} \text{ for } i=\overline{1,3}.$$

Therefore

$$\begin{aligned}
\mathrm{D}_q\,(f) &= \frac{1}{N}\sum_{x,y\in\Lambda:x\neq y}\left[\alpha_1\left(f_x^1-f_y^1\right)^2+\alpha_2\left(f_x^2-f_y^2\right)^2+\alpha_3\left(f_x^3-f_y^3\right)^2\right]\\
&= \frac{1}{N}\Bigg[(N-1)\sum_{x\in\Lambda}\alpha_1\left(f_x^1\right)^2+(N-1)\sum_{y\in\Lambda}\alpha_1\left(f_y^1\right)^2-2\sum_{x,y\in\Lambda:x\neq y}\alpha_1 f_x^1 f_y^1\\
&\quad+(N-1)\sum_{x\in\Lambda}\alpha_2\left(f_x^2\right)^2+(N-1)\sum_{y\in\Lambda}\alpha_2\left(f_y^2\right)^2-2\sum_{x,y\in\Lambda:x\neq y}\alpha_2 f_x^2 f_y^2\\
&\quad+(N-1)\sum_{x\in\Lambda}\alpha_3\left(f_x^3\right)^2+(N-1)\sum_{y\in\Lambda}\alpha_3\left(f_y^3\right)^2-2\sum_{x,y\in\Lambda:x\neq y}\alpha_3 f_x^3 f_y^3\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{D}_q\,(f) &= \sum_{x\in\Lambda}\frac{2\alpha_1\,(N-1)}{N}\left(f_x^1\right)^2-2\sum_{x,y\in\Lambda:x\neq y}\frac{\alpha_1}{N}f_x^1 f_y^1+\sum_{y\in\Lambda}\frac{2\alpha_2\,(N-1)}{N}\left(f_y^2\right)^2\\
&\quad-2\sum_{x,y\in\Lambda:x\neq y}\frac{\alpha_2}{N}f_x^2 f_y^2+\sum_{x\in\Lambda}\frac{2\alpha_3\,(N-1)}{N}\left(f_x^3\right)^2-2\sum_{x,y\in\Lambda:x\neq y}\frac{\alpha_3}{N}f_x^3 f_y^3
\end{aligned}$$

$$\begin{aligned}
&= \sum_{x\in\Lambda}\frac{2k_0k_1g(1)q\,(1-q)}{N^2}\left(f_x^1\right)^2-2\sum_{x,y\in\Lambda:x\neq y}\frac{k_0k_1g(1)q\,(1-q)}{N^2\,(N-1)}f_x^1 f_y^1\\
&\quad+\sum_{y\in\Lambda}\frac{2k_0k_2g(2)q\,(1-q)}{N^2}\left(f_y^2\right)^2-2\sum_{x,y\in\Lambda:x\neq y}\frac{k_0k_2g(2)q\,(1-q)}{N^2\,(N-1)}f_x^2 f_y^2,\\
&\quad+\sum_{y\in\Lambda}\frac{2k_0k_3g(3)q\,(1-q)}{N^2}\left(f_y^3\right)^2-2\sum_{x,y\in\Lambda:x\neq y}\frac{k_0k_3g(3)q\,(1-q)}{N^2\,(N-1)}f_x^3 f_y^3,
\end{aligned}$$

and so

$$\mathrm{D}_q\,(f) = \sum_{x,y\in\Lambda}\sum_{i,j=1}^{3} f_x^i f_y^j m_{x,y}\delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker symbol, which implies the formula (3) via the identification of f and the row $\left(\left(f_x^1\right)_{x\in\Lambda},\left(f_x^2\right)_{x\in\Lambda},\left(f_x^3\right)_{x\in\Lambda}\right)$. ■

4. The spectral gap

Let $\lambda(N,k)$ be the spectral gap of $-L$ defined by

$$\lambda(N,k) = \inf_{f\in L^2(\nu_k):\nu_k(f)=0} \frac{\mathcal{D}(f)}{\nu_k\left(f^2\right)},$$

where $\nu_k\left(f^2\right)$ is the variance of f w.r.t. ν_k. In the homogeneous case the spectral gap $\lambda_q(N,k)$ is defined as $\lambda(N,k)$ with $\mathcal{D}_q(f)$ instead of $\mathcal{D}(f)$. We are now able to state the main result of this paper.

Theorem 4.1. Assume that $N \geq 3$ and $r = 3$. Then we have

$$\lambda_q(N,k) = q(1-q)(2N-3)\rho_0 \min\left\{\frac{g(1)}{N-k_1}, \frac{g(2)}{N-k_2}, \frac{g(3)}{N-k_3}\right\}, \tag{4}$$

where $\rho_0 = \dfrac{k_0}{N}$ is the density of vacant sites.

Proof. In order to compute $\lambda_q(N,k)$ let us consider the set

$$F = \left\{ f = \sum_{x\in\Lambda} f_x^1\overline{\eta_x^1} + f_x^2\overline{\eta_x^2} + f_x^3\overline{\eta_x^3}\ ,\ f_x^i \in \mathbb{R}\ (x \in \Lambda, i = \overline{1,3}) \right\}$$

and the infimum on F :

$$\gamma_q(N,k) = \inf\left\{\frac{\mathcal{D}_q(f)}{\nu_k\left(f^2\right)} : f \in F\right\}.$$

It is clear that $\lambda_q(N,k) \leq \gamma_q(N,k)$. We will compute $\gamma_q(N,k)$ and prove that $\lambda_q(N,k) = \gamma_q(N,k)$ in three steps.

Step 1: Change of variables in $\nu_k\left(f^2\right)$ and $\mathcal{D}_q(f)$, $f \in F$.
We take the matrix representation of $\nu_k(f^2)$ used by Dermoune and Einrich [4] page 6 subsection 3.2. We can write

$$\nu_k\left(f^2\right) = \sum_{x,y\in\Lambda}\sum_{i,j=1}^{3} f_x^i f_y^j \nu_k\left(\overline{\eta_x^i}\,\overline{\eta_y^j}\right) = fCf^T,$$

where $C = \left(C_{xy}^{ij}\right)$ is the $3N$ by $3N$ covariance matrix:

$$C_{xy}^{ij} = \nu_k\left(\eta_x^i;\eta_y^j\right)$$

and so

$$C=\begin{pmatrix} C^{11} & C^{12} & C^{13} \\ C^{21} & C^{22} & C^{23} \\ C^{31} & C^{32} & C^{33} \end{pmatrix}.$$

Since C is a symmetric matrix then $C^{ij}=C^{ji}$. By the Remark 2, we have

$$C^{ij}=\left(c^{ij}_{x,y}\right),\text{ where } c^{ij}_{x,x}=-\frac{k_ik_j}{N^2}\text{ and } c^{ij}_{x,y}=\frac{k_ik_j}{N^2(N-1)}\text{ if } x\neq y,$$

and

$$C^{ii}=\left(c^{i,i}_{x,y}\right),\text{ where } c^{i,i}_{x,x}=\frac{k_i(N-k_i)}{N^2}\text{ and } c^{i,i}_{x,y}=\frac{k_i(N-k_i)}{N^2(N-1)}\text{ if } x\neq y.$$

Now, due to the fact that the matrix C is not invertible, we shall find a matrix Γ and a row h such that

$$\nu_k\left(f^2\right)=h\Gamma h^T \text{ and } \mathcal{D}_q(f)=hh^T. \tag{5}$$

Matrices M_N and X defined by the formula (2) are symmetric and positive definite, then they admits square roots $M_N^{\frac{1}{2}}$ and $X^{\frac{1}{2}}$, that is, $M_N=M_N^{\frac{1}{2}}\left(M_N^{\frac{1}{2}}\right)^T$ and $X=X^{\frac{1}{2}}\left(X^{\frac{1}{2}}\right)^T$. Note that Since M_N is invertible then it is the same for $M_N^{\frac{1}{2}}$ and $X^{\frac{1}{2}}$. It is immediate that

$$X^{\frac{1}{2}}=\begin{pmatrix} (k_1g(1))^{\frac{1}{2}}M_N^{\frac{1}{2}} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & (k_2g(2))^{\frac{1}{2}}M_N^{\frac{1}{2}} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & (k_3g(3))^{\frac{1}{2}}M_N^{\frac{1}{2}} \end{pmatrix}$$

and

$$\left(X^{\frac{1}{2}}\right)^{-1}=\begin{pmatrix} (k_1g(1))^{-\frac{1}{2}}\left(M_N^{\frac{1}{2}}\right)^{-1} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & (k_2g(2))^{-\frac{1}{2}}\left(M_N^{\frac{1}{2}}\right)^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & (k_3g(3))^{-\frac{1}{2}}\left(M_N^{\frac{1}{2}}\right)^{-1} \end{pmatrix}.$$

Then simply put

$$h=fX^{\frac{1}{2}},\text{ and } \Gamma=\left(X^{\frac{1}{2}}\right)^{-1}C\left(\left(X^{\frac{1}{2}}\right)^{-1}\right)^T.$$

Step 2: The relation between $\lambda_q(N,k)$ and the largest eigenvalue of Γ.
By the formula (4) and the the definition of $\gamma_q(N,k)$, we see that

$$\gamma_q(N,k)=\inf_{h\in\mathbb{R}^{2N}:h\neq 0}\frac{hh^T}{h\Gamma h^T}.$$

There exists an orthogonal matrix $\mathbf{P}$ such that $\Gamma = \mathbf{P}^{-1}\mathbf{SP}$, where

$$\mathbf{S} = Diag(\lambda_1, \lambda_2, \ldots, \lambda_{3N})$$

is a diagonal matrix whose diagonal elements are the eigenvalues of Γ. If $\mathbf{P}$ is partitioned as $\mathbf{P} = (e_1, e_2, \ldots, e_{3N})$ where e_i $(i = 1, 2, \ldots, 3N)$ are the eigenvectors of Γ, then for each $h = \sum_{i=1}^{3N} h_i e_i \neq 0$ we have

$$\frac{hh^T}{h\Gamma h^T} = \frac{\sum_{i=1}^{3N} h_i^2}{\sum_{i=1}^{3N} \lambda_i h_i^2} \geq \frac{\sum_{i=1}^{3N} h_i^2}{\lambda_{\max} \sum_{i=1}^{3N} h_i^2} = \frac{1}{\lambda_{\max}},$$

where $\lambda_{\max}$ is the largest eigenvalue of Γ. Now, if we choose $h = e_{\max}$ the corresponding eigenvector to $\lambda_{\max}$, we see that

$$\frac{hh^T}{h\Gamma h^T} = \frac{1}{\lambda_{\max}}$$

and so

$$\gamma_q (N, k) = \frac{1}{\lambda_{\max}}.$$

On the other hand the randon variable

$$f = \sum_{x \in \Lambda} f_x^1 \overline{\eta_x^1} + f_x^2 \overline{\eta_x^2} + f_x^3 \overline{\eta_x^3}$$

which corresponds to the row

$$\left(\left(f_x^1\right)_{x\in\Lambda}, \left(f_x^2\right)_{x\in\Lambda}, \left(f_x^3\right)_{x\in\Lambda} \right) = e_{\max}^T \left(X^{\frac{1}{2}} \right)^{-1}$$

satisfy $f \in L^2(\nu_k)$, $\nu_k(f) = 0$ and

$$\frac{\mathcal{D}_q(f)}{\nu_k\left(f^2\right)} = \frac{1}{\lambda_{\max}},$$

this maens that the infimum is reached on F and

$$\lambda_q (N, k) = \frac{1}{\lambda_{\max}}. \tag{6}$$

Step 3:The spectrum of Γ. It is obvious that

$$\Gamma = \begin{pmatrix} \theta_1 A^{-1} C^{11} \left(A^{-1}\right)^T & \theta_4 A^{-1} C^{12} \left(A^{-1}\right)^T & \theta_5 A^{-1} C^{13} \left(A^{-1}\right)^T \\ \theta_4 A^{-1} C^{21} \left(A^{-1}\right)^T & \theta_2 A^{-1} C^{22} \left(A^{-1}\right)^T & \theta_6 A^{-1} C^{23} \left(A^{-1}\right)^T \\ \theta_5 A^{-1} C^{31} \left(A^{-1}\right)^T & \theta_6 A^{-1} C^{32} \left(A^{-1}\right)^T & \theta_3 A^{-1} C^{33} \left(A^{-1}\right)^T \end{pmatrix},$$

where

$$A = M_N^{\frac{1}{2}}, \theta_1 = (k_1 g\,(1))^{-1},$$

$$\theta_2 = (k_2 g\,(2))^{-1},$$

$$\theta_3 = (k_3 g\,(3))^{-1},$$

$$\theta_4 = (g\,(1)\,g\,(2)\,k_1 k_2)^{-\frac{1}{2}},$$

$$\theta_5 = (g\,(1)\,g\,(3)\,k_1 k_3)^{-\frac{1}{2}}$$

and

$$\theta_6 = (g\,(2)\,g\,(3)\,k_2 k_3)^{-\frac{1}{2}}.$$

Let $P(\lambda)$ (resp. $Q_i(\lambda)$ for $i = \overline{1,3}$) be the characteristic polynomial of Γ (resp. $A^{-1}C^{ii}\left(A^{-1}\right)^T$ for $i = \overline{1,3}$). Then we have:

$$\begin{aligned}
P(\lambda) &= \det\begin{pmatrix} \theta_1 A^{-1}C^{11}\left(A^{-1}\right)^T - \lambda Id_N & \theta_4 A^{-1}C^{12}\left(A^{-1}\right)^T & \theta_5 A^{-1}C^{13}\left(A^{-1}\right)^T \\ \theta_4 A^{-1}C^{21}\left(A^{-1}\right)^T & \theta_2 A^{-1}C^{22}\left(A^{-1}\right)^T - \lambda Id_N & \theta_6 A^{-1}C^{23}\left(A^{-1}\right)^T \\ \theta_5 A^{-1}C^{31}\left(A^{-1}\right)^T & \theta_6 A^{-1}C^{32}\left(A^{-1}\right)^T & \theta_3 A^{-1}C^{33}\left(A^{-1}\right)^T - \lambda Id_N \end{pmatrix} \\
&= \det\left(\theta_1 A^{-1}C^{11}\left(A^{-1}\right)^T - \lambda Id_N\right) \\
&\quad \times \det\begin{pmatrix} \theta_2 A^{-1}C^{22}\left(A^{-1}\right)^T - \lambda Id_N & \theta_6 A^{-1}C^{23}\left(A^{-1}\right)^T \\ \theta_6 A^{-1}C^{32}\left(A^{-1}\right)^T & \theta_3 A^{-1}C^{33}\left(A^{-1}\right)^T - \lambda Id_N \end{pmatrix} \\
&\quad - \det \theta_4 A^{-1}C^{12}\left(A^{-1}\right)^T \det\begin{pmatrix} \theta_4 A^{-1}C^{21}\left(A^{-1}\right)^T & \theta_6 A^{-1}C^{23}\left(A^{-1}\right)^T \\ \theta_5 A^{-1}C^{31}\left(A^{-1}\right)^T & \theta_3 A^{-1}C^{33}\left(A^{-1}\right)^T - \lambda Id_N \end{pmatrix} \\
&\quad + \theta_5 A^{-1}C^{21}\left(A^{-1}\right)^T \det\begin{pmatrix} \theta_4 A^{-1}C^{21}\left(A^{-1}\right)^T & \theta_2 A^{-1}C^{22}\left(A^{-1}\right)^T - \lambda Id_N \\ \theta_5 A^{-1}C^{31}\left(A^{-1}\right)^T & \theta_6 A^{-1}C^{32}\left(A^{-1}\right)^T \end{pmatrix}. \\
&= \det\left(\theta_1 A^{-1}C^{11}\left(A^{-1}\right)^T - \lambda Id_N\right)\det\left(\theta_2 A^{-1}C^{22}\left(A^{-1}\right)^T - \lambda Id_N\right) \\
&\quad \times \det\left(\theta_3 A^{-1}C^{33}\left(A^{-1}\right)^T - \lambda Id_N\right) \\
&\quad + 2\det\left(\theta_4 A^{-1}C^{21}\left(A^{-1}\right)^T\right)\det\left(\theta_5 A^{-1}C^{31}\left(A^{-1}\right)^T\right)\det\left(\theta_6 A^{-1}C^{23}\left(A^{-1}\right)^T\right) \\
&\quad - \det\left(\theta_1 A^{-1}C^{11}\left(A^{-1}\right)^T - \lambda Id_N\right)\det\left(\theta_6 A^{-1}C^{32}\left(A^{-1}\right)^T\right)^2 \\
&\quad - \det\left(\theta_2 A^{-1}C^{22}\left(A^{-1}\right)^T - \lambda Id_N\right)\det\left(\theta_5 A^{-1}C^{31}\left(A^{-1}\right)^T\right)^2 \\
&\quad - \det\left(\theta_3 A^{-1}C^{33}\left(A^{-1}\right)^T - \lambda Id_N\right)\det\left(\theta_4 A^{-1}C^{21}\left(A^{-1}\right)^T\right)^2,
\end{aligned}$$

determinant: Thus

$$\begin{aligned} P(\lambda) &= (\theta_1\theta_2\theta_3)^N Q_1\left(\frac{\lambda}{\theta_1}\right) Q_2\left(\frac{\lambda}{\theta_2}\right) Q_3\left(\frac{\lambda}{\theta_3}\right) \\ &\quad +2(\theta_4\theta_5\theta_6)^N \left(\det\left(A^{-1}C^{21}\left(A^{-1}\right)^T\right)\right. \\ &\quad\quad \left.\times \det\left(A^{-1}C^{31}\left(A^{-1}\right)^T\right)\det\left(A^{-1}C^{23}\left(A^{-1}\right)^T\right)\right) \\ &\quad - (\theta_1\theta_6)^N Q_1\left(\frac{\lambda}{\theta_1}\right)\det\left(A^{-1}C^{32}\left(A^{-1}\right)^T\right)^2 \\ &\quad - (\theta_2\theta_5)^N Q_2\left(\frac{\lambda}{\theta_2}\right)\det\left(A^{-1}C^{31}\left(A^{-1}\right)^T\right)^2 \\ &\quad - (\theta_3\theta_4)^N Q_3\left(\frac{\lambda}{\theta_3}\right)\det\left(A^{-1}C^{21}\left(A^{-1}\right)^T\right)^2 \end{aligned}$$

In fact; there exist a diagonal matrix D and an orthogonal matrix O ($i.e$ $O^T = O^{-1}$) such that $M_N = ODO^T$. In this case $A = OD^{\frac{1}{2}}$, where

$$D^{\frac{1}{2}} = \left(\sqrt{\lambda_x}\delta_{x,y}\right) \text{ if } D = \left(\lambda_x\delta_{x,y}\right).$$

We have for $i = \overline{1,3}$

$$\begin{aligned} A^{-1}C^{ii}\left(A^{-1}\right)^T - \lambda Id_N &= A^{-1}C^{ii}\left(A^{-1}\right)^T - \lambda A^{-1}AA^T\left(A^T\right)^{-1} \\ &= A^{-1}\left(C^{ii} - \lambda AA^T\right)\left(A^{-1}\right)^T \\ &= \left(D^{\frac{1}{2}}\right)^{-1} O^T\left(C^{ii} - \lambda M_N\right) O\left(D^{\frac{1}{2}}\right)^{-1}, \end{aligned}$$

which implies that

$$\begin{aligned} Q_i(\lambda) &= \det\left(D^{\frac{1}{2}}\right)^{-2}\det\left(C^{ii} - \lambda M_N\right) \\ &= (\det M)^{-1}\det\left(C^{ii} - \lambda M_N\right). \end{aligned}$$

To compute $\det\left(C^{ii} - \lambda M\right)$ and $\det\left(C^{12}\right)$ we will need the exact value of the following determinant of order N

$$B_N = \det\begin{pmatrix} \alpha & \beta & . & \beta \\ \beta & . & . & . \\ . & . & . & \beta \\ \beta & . & \beta & \alpha \end{pmatrix} \quad (\alpha, \beta \in \mathbb{R}).$$

By subtracting the last line by the others we obtain

$$\begin{aligned} B_N &= \det \begin{pmatrix} \alpha-\beta & 0 & . & 0 & \beta-\alpha \\ 0 & . & . & . & . \\ . & . & . & 0 & . \\ 0 & . & 0 & \alpha-\beta & \beta-\alpha \\ \beta & . & . & \beta & \alpha \end{pmatrix} \\ &= (\alpha-\beta)\det(B_{N-1}) \\ &\quad + (-1)^{N+1}\beta \det \begin{pmatrix} 0 & 0 & . & 0 & \beta-\alpha \\ \alpha-\beta & . & . & . & . \\ 0 & . & . & 0 & . \\ . & 0 & \alpha-\beta & 0 & \beta-\alpha \\ 0 & . & 0 & \alpha-\beta & \beta-\alpha \end{pmatrix} \\ &= (\alpha-\beta)(B_{N-1}) + \beta(\beta-\alpha)^{N-1}. \end{aligned}$$

Finally we get by induction

$$B_N = (\alpha-\beta)^{N-1}(\alpha+(N-1)\beta). \tag{7}$$

By replacing α by $\frac{k_i(N-k_i)}{N^2} - \lambda m_{x,x}$ $\left(\text{resp. } -\frac{k_i k_j}{N^2}\right)$ and β by $\frac{k_i^2 - k_i N}{N^2(N-1)} - \lambda m_{x,y}$ $\left(\text{resp. } \frac{k_i k_j}{N^2(N-1)}\right)$ in the formula (8), we get

$$Q_i(\lambda) = \left(\frac{k_i(N-k_i)}{N(N-1)} - \lambda\frac{q(1-q)k_0(2N-3)}{N^2(N-1)}\right)^{N-1}\left(-\lambda\frac{k_0 q(1-q)}{N^2\det M}\right)$$

$$(\text{resp.} \det A^{-1}C^{ij}\left(A^{-1}\right)^T = 0). \text{ for } i,j=\overline{1,3} \text{ and } i\neq j$$

Then we obtain by using the formula (7),

$$P(\lambda) = \left(\lambda\frac{k_0 q(1-q)}{N^2\det M}\right)^3 \prod_{i=1}^{3}\theta_i^{N-1}\left(\frac{k_i(N-k_i)}{N(N-1)} - \frac{\lambda}{\theta_i}\frac{q(1-q)k_0(2N-3)}{N^2(N-1)}\right)^{N-1},$$

which implies, taking into account of the values of θ_1 and θ_2, that the spectrum of the matrix Γ is the set:

$$\left\{0, \frac{N(N-k_1)}{q(1-q)k_0 g(1)(2N-3)}, \frac{N(N-k_2)}{q(1-q)k_0 g(2)(2N-3)}, \frac{N(N-k_3)}{q(1-q)k_0 g(3)(2N-3)}\right\}.$$

Thus

$$\lambda_{\max} = \frac{1}{q(1-q)(2N-3)\rho_0}\max\left\{\frac{N-k_1}{g(1)}, \frac{N-k_2}{g(2)}, \frac{N-k_3}{g(3)}\right\}. \tag{8}$$

The formula (5) follows from (6) and (9), which completes the proof. ■

References

[1] Caputo P., Spectral gap inequalities in product spaces with conservation laws. Stochastic analysis on large scale interacting systems, *Adv. Stud. Pure Math.*, 39, 53–88 (2004).

[2] Caputo P., On the spectral gap of the Kac waik and other binary collision processes. *ALEA Lat. Am. J. Prob. Math. Stat.*, 4, 205–222 (2008).

[3] Dermoune A., Heinrich P., A small step towards the hydrodynamic limit of a colored disordered lattice gas. *C.R. Math. Acad. Sci. Paris*, 339, 507–511 (2004).

[4] Dermoune A., Heinrich P., Spectral gap inequality for a colored disordered lattice gas. Sem. Prob. XLI Lecture Notes in Maths Springer Verlag, 1934, 1–18 (2008).

[5] Dermoune A., Heinrich P., Spectral gap for multicolor nearest neighbor exclusion process with site desorder. *J. Stat. Phys.*, 131, 117–125 (2008).

[6] Fagginatto A., Martinelli F., Hydrodynamic limit of a disordered lattice gas. *Probab. Theory Related Fields*, 127, 535–608 (2003).

[7] Kipnis C., Landim C., Scaling Limits of Interacting Particles Systems, Springer Berlin, 1999.

[8] Nagahata Y., Sasada M., Spectral Gap of Multi-species Exclusion Processus. *J. Stat. Phys.*, 143, 381–398 (2011).

[9] Zeghdoudi H. Boutabia H., Computation for the canonical measures of a colored disordered lattice gas and spectral gap. *J. Math. Phys.*,50, 623–679 (2009).