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Application of the Basic Optimal Homotopy Analysis Method to Fingering Phenomenon

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Abstract

The problem of fingering phenomenon in immiscible fluid flow through homogeneous porous medium has been discussed by developing its mathematical model. The resulting nonlinear partial differential equation with appropriate boundary conditions has been solved using basic optimal homotopy analysis method.The optimal value of converenge-control parameter is obtained using discrete squared residul error.The numerical values and graphical presentation are given using Mathematica.

AMS subject classification:
Keywords: Fingering phenomenon, fluid flow through porous media, optimal homotopy analysis method, discrete squared residual error.

1. Introduction

In primary oil recovery process, oil is pushed to the surface of earth by natural pressure of the reservoir. It allows about 5% to 10% of the oil in the reservoir to be extracted. In secondary oil recovery process water or gas is injected to drive the residual oil remaining after the primary recovery process to the surface wells.This allows about 25% to 30% of the oil in the reservoir to be extracted.

In this paper, we consider the fingering phenomenon occurred during water injection in secondary oil recovery process as shown in figure 1.When a fluid contained in a porous medium is displaced by another of lesser viscosity, instead of regular displacement of the whole front, protuberances may occur which shoot through the porous medium at relatively great speed. This phenomenon is called fingering phenomenon and the protuberances which represent instabilities in the displacement problem are called fingers. The displacement problems of this type (particularly, the stabilization of fingers) have much current importance in the secondary recovery processes of petroleum technology.

This instability phenomenon is discussed by many researchers from different points of view. Saffman and Taylor [21] derived a classical result for the shape of fingers in the absence of capillary. Recently many researchers have discussed the shape, size and velocity prediction of fingers under different situation with different view (Lange et al. [6]; Brailovsky et al. [3]; Zhan and Yortsos [28]; Wang and Feyen [26]). Joshi and Mehta [17] have discussed the solution by the group invariant method of the instability phenomenon arising in fluid flow through porous media. Kinjal [16] has obtained the power series solution of fingering phenomenon. Prajapati and Desai [20] has obtained approximate analytical solution of the instability phenomenon using similarity transformation.

In the present paper, our interest is to solve nonlinear partial differential equation for one dimensional fingering phenomenon arising in homogeneous porous medium during secondary oil recovery process. The solution of this equation has been obtained by using basic optimal homotopy analysis method and the optimal value of convergence-control parameter is obtained by minimizing discrete squared residual error.

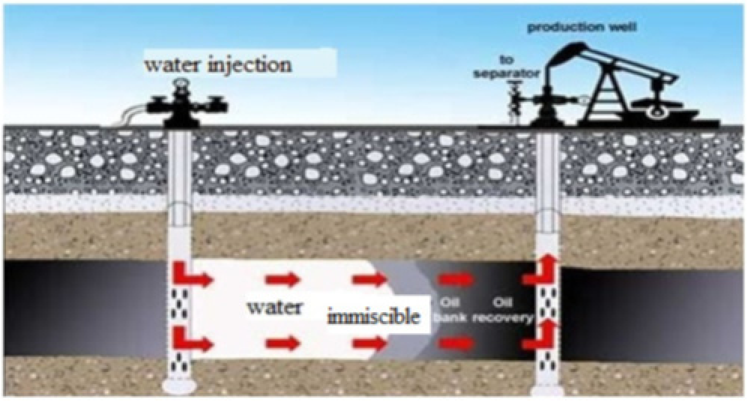

Figure 1: Secondary oil recovery process in oil reservoir.

2. Problem Formulation

To study the phenomenon of fingering in homogeneous porous medium, we choose a cylindrical piece of porous matrix of length L whose three sides are impermeable except one end from where water is injected (Figure 2). For mathematical study, we take vertical

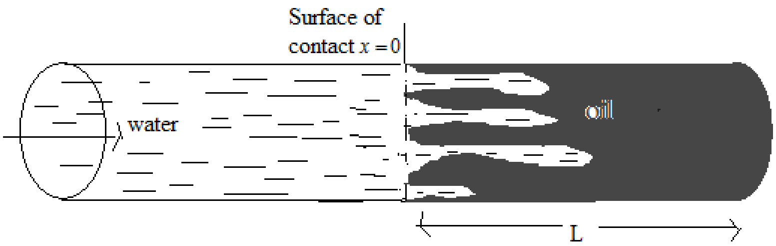

Figure 2: The formation of fingers in the cylindrical piece of porous medium

cross-sectional area of this cylindrical piece of porous matrix which is rectangle and open end will be the common interface $x = 0$.

Let the water be injected at $x = 0$, then due to the injecting force and viscosity difference the protuberances or instability may arise which is due to the displacement of oil by water injection through inter connected capillaries. The length x of the fingers is being measured in the direction of displacement. Scheidegger and Johnson suggested replacing these irregular fingers by schematic fingers of rectangular size (Figure 3).

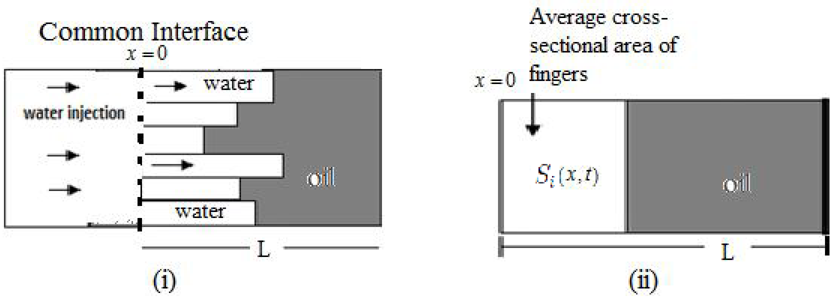

Figure 3: (i) Schematic representation of the fingers and (ii) Average cross-sectional area of the fingers in the rectangular porous matrix.

For the sake of mathematical study, consider the average cross-sectional area occupied by schematic fingers as a saturation of injected water at level x and time $t > 0$.

By Darcy's law, the velocities V_i and V_n of injected and native fluids are:

$$V_i = -\frac{k_i}{\mu_i} K \frac{\partial p_i}{\partial x} \tag{1}$$

$$V_n = -\frac{k_n}{\mu_n} K \frac{\partial p_n}{\partial x} \tag{2}$$

where K is the permeability of the homogenous porous medium, k_n and k_i are relative permeabilities of native fluid (oil) and injected fluid (water) which are functions of saturations S_n and S_i of oil and water respectively. p_i and p_n denote the pressures of water and oil respectively while μ_i and μ_n are the viscosities of the water and oil respectively. Regarding phase densities as constant,the equations of continuity(mass balance equations) are

$$P\frac{\partial S_i}{\partial t}+\frac{\partial V_i}{\partial x}=0 \tag{3}$$

$$P\frac{\partial S_n}{\partial t}+\frac{\partial V_n}{\partial x}=0 \tag{4}$$

where P is the porosity of the medium regarded as constant. The porous medium is considered to be fully saturated. From the definition of phase saturation (Scheidegger (1960)),

$$S_i+S_n=1 \tag{5}$$

If the substances are immiscible, then there is a surface tension occurring at their contact line which creates a capillary pressure

$$p_c(S_i)=p_n-p_i \tag{6}$$

It is well known that relative permeability is the function of displacing fluid saturation. Then at this stage for definiteness of the mathematical analysis, we assume standard forms of Scheidegger and Johnson (1961) [23] for the analytical relationship between the relative permeability, phase saturation and capillary pressure as

$$k_i=S_i \quad and \quad k_n=S_n=1-S_i \tag{7}$$

$$p_c=-\beta S_i \tag{8}$$

where β is a constant.
Using (1) and (2) in (3) and (4), we have

$$P\frac{\partial S_i}{\partial t}=\frac{\partial}{\partial x}\left\{K\frac{k_i}{\mu_i}\frac{\partial p_i}{\partial x}\right\} \tag{9}$$

$$P\frac{\partial S_n}{\partial t}=\frac{\partial}{\partial x}\left\{K\frac{k_n}{\mu_n}\frac{\partial p_n}{\partial x}\right\} \tag{10}$$

From (6) and (9),we get,

$$P\frac{\partial S_i}{\partial t}=\frac{\partial}{\partial x}\left[K\frac{k_i}{\mu_i}\left\{\frac{\partial p_n}{\partial x}-\frac{\partial p_c}{\partial x}\right\}\right] \tag{11}$$

Using (10) and (11), we get

$$\frac{\partial}{\partial x}\left[K\left\{\frac{k_i}{\mu_i}+\frac{k_n}{\mu_n}\right\}\frac{\partial p_n}{\partial x}-K\frac{k_i}{\mu_i}\frac{\partial p_c}{\partial x}\right]=0 \tag{12}$$

Integrating both sides with respect to x, we have

$$K\left\{\frac{k_i}{\mu_i}+\frac{k_n}{\mu_n}\right\}\frac{\partial p_n}{\partial x}-K\frac{k_i}{\mu_i}\frac{\partial p_c}{\partial x}=-M \tag{13}$$

where M is integrating constant. Now equation (13) can be rewritten as

$$\frac{\partial p_n}{\partial x}=\frac{-M}{K\left\{\frac{k_i}{\mu_i}+\frac{k_n}{\mu_n}\right\}}+\frac{\frac{\partial p_c}{\partial x}}{1+\frac{k_n}{k_i}\frac{\mu_i}{\mu_n}} \tag{14}$$

Using (11) and (14) we obtain

$$P\frac{\partial S_i}{\partial t}+\frac{\partial}{\partial x}\left\{\frac{K\frac{k_n}{\mu_n}\frac{\partial p_c}{\partial x}}{1+\frac{k_n}{k_i}\frac{\mu_i}{\mu_n}}+\frac{M}{1+\frac{k_n}{k_i}\frac{\mu_i}{\mu_n}}\right\}=0 \tag{15}$$

The value of the pressure of native fluid p_n can be written as

$$p_n=\frac{p_n+p_i}{2}+\frac{p_n-p_i}{2}=\overline{p}+\frac{1}{2}p_c \tag{16}$$

where $\overline{p}$ is the constant mean pressure.

From (16)

$$\frac{\partial p_n}{\partial x}=\frac{1}{2}\frac{\partial p_c}{\partial x} \tag{17}$$

Using (17) in (13),we get,

$$M=\frac{K}{2}\left\{\frac{k_i}{\mu_i}-\frac{k_n}{\mu_n}\right\}\frac{\partial p_c}{\partial x} \tag{18}$$

Substituting value of M in equation (15), we obtain

$$P\frac{\partial S_i}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}\left\{K\frac{k_i}{\mu_i}\frac{dp_c}{dS_i}\frac{\partial S_i}{\partial x}\right\}=0 \tag{19}$$

Since $k_i=S_i$ and $p_c=-\beta S_i$, we have

$$P\frac{\partial S_i}{\partial t}-\frac{\beta}{2}\frac{K}{\mu_i}\frac{\partial}{\partial x}\left\{S_i\frac{\partial S_i}{\partial x}\right\}=0 \tag{20}$$

The equation (20) is a nonlinear partial differential equation which describes the immiscible oil-water fluid flow through porous media.

Using dimensionless variables

$$X=\frac{x}{L},\, T=\frac{K\beta t}{2\mu_i L^2 P}, \tag{21}$$

equation (20) reduces to

$$\frac{\partial S_i}{\partial T} = \frac{\partial}{\partial X}\left\{S_i \frac{\partial S_i}{\partial X}\right\} \tag{22}$$

To solve this nonlinear partial differential equation for fingering phenomenon, we choose the set of appropriate boundary conditions.

Let at the common interface saturation of injected water be linear function of time T, that is, at $X = 0$,

$$S_i(0, T) = 0.0001T \quad for\ T > 0 \tag{23}$$

Also let the saturation of injected water at the end $X = L$ be

$$S_i(1, T) = 0.001T \quad for\ T > 0 \tag{24}$$

We solve equation (22) together with (23) and (24) using basic optimal homotopy analysis method.

3. Solution by Basic Optimal Homotopy Analysis Method

We choose

$$S_{i_0}(X, T) = 0.0001T + 0.0009TX \tag{25}$$

as the initial approximation of $S_i(X, T)$. Besides we choose the linear operator as

$$\mathcal{L}[\phi(X, T; q)] = \frac{\partial^2 \phi(X, T; q)}{\partial X^2} \tag{26}$$

with the property

$$\mathcal{L}[f] = 0 \quad when\ f = 0. \tag{27}$$

Furthermore, based on governing equation (22), we define such a nonlinear operator

$$\mathcal{N}[\phi(X, T; q)] = \phi(X, T; q)\frac{\partial^2 \phi(X, T; q)}{\partial X^2} + \left\{\frac{\partial \phi(X, T; q)}{\partial X}\right\}^2 - \frac{\partial \phi(X, T; q)}{\partial T} \tag{28}$$

Let c_0 denote a nonzero auxiliary parameter.Then we construct the zeroth order deformation equation

$$(1 - q)\mathcal{L}[\phi(X, T; q) - S_{i_0}(X, T)] = c_0 q H(X, T)\mathcal{N}[\phi(X, T; q)] \tag{29}$$

where $q \in [0, 1]$ is the embedding parameter, $H(X, T)$ is nonzero auxiliary function and $\phi(X, T; q)$ is an unknown function. Obviously, when $q = 0$ and $q = 1$, we have from (27) and (29),

$$\phi(X, T; 0) = S_{i_0}(X, T) \tag{30}$$

and

$$\phi(X, T; 1) = S_i(X, T) \tag{31}$$

Therefore, according to equations (30) and (31), the solution $\phi(X, T; q)$ varies from the initial guess $S_{i_0}(X, T)$ to the solution $S_i(X, T)$ of the equation (22) as the embedding parameter q increases from 0 to 1.

Obviously, $\phi(X, T; q)$ is determined by the auxiliary linear operator $\mathcal{L}$, the initial guess $S_{i_0}(X, T)$ and the convergence-control parameter c_0. We have great freedom to select all of them. Assuming that all of them are so properly chosen that the Taylor series

$$\phi(X, T; q) = S_{i_0}(X, T) + \sum_{m=1}^{\infty} S_{i_m}(X, T)q^m \tag{32}$$

exists and besides converges at $q = 1$, we have the homotopy-series solution

$$S_i(X, T) = S_{i_0}(X, T) + \sum_{m=1}^{\infty} S_{i_m}(X, T) \tag{33}$$

where

$$S_{i_m}(X, T) = \frac{1}{m!} \frac{\partial^m \phi(X, T; q)}{\partial q^m}\bigg|_{q=0} \tag{34}$$

Differentiating the zeroth order deformation equation (29) m times with respect to the embedding parameter q and then dividing them by $m!$ and finally setting $q = 0$,we have the so called high order deformation equation

$$\mathcal{L}[S_{i_m}(X, T) - \chi_m S_{i_{m-1}}(X, T)] = c_0 H(X, T) R_m(X, T) \tag{35}$$

subject to the conditions

$$S_{i_m}(0, T) = 0, \quad S_{i_m}(1, T) = 0, \quad m \geq 1 \tag{36}$$

where

$$R_m(X, T) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(X, T; q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{37}$$

and

$$\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases} \tag{38}$$

For simplicity, assume $H(X, T) = 1$; hence the solution of (22) can be expressed in the form

$$S_m(X, T) = \chi_m S_{m-1}(X, T) + c_0\mathcal{L}^{-1}[R_m(X, T)] + C_1X + C_2 \tag{39}$$

where the constants C_1 and C_2 are determined using (36). Taking $m = 1$ in (39), we have,

$$\begin{aligned} S_1(X, T) &= c_0\Big[0.000000405T^2X^2 - 0.00005X^2 - 0.00015X^3\Big] \\ &\quad + c_0\Big[0.0002X - 0.000000405T^2X\Big] \end{aligned} \tag{40}$$

Therefore, the saturation of injected fluid is

$$\begin{aligned} S_i(X, T) &= 0.0001T + 0.0009TX \\ &\quad + c_0\Big[0.000000405T^2X^2 - 0.00005X^2 - 0.00015X^3\Big] \\ &\quad + c_0\Big[0.0002X - 0.000000405T^2X\Big] + \cdots \end{aligned} \tag{41}$$

As given by Liao [14], the averaged (or discrete) squared residual error at the mth order of approximation is

$$E_m = \frac{1}{(M+1)(N+1)} \sum_{i=0}^{M}\sum_{j=0}^{N} \left\{\mathcal{N}\left[\sum_{n=0}^{m} S_{i_n}\left(\frac{i}{M}, \frac{j}{N}\right)\right]\right\}^2 \tag{42}$$

Since the squared residual E_m is dependent upon c_0, the optimal homotopy approximation is gained by

$$\frac{dE_m(c_0)}{dc_0} = 0 \tag{43}$$

Here the optimal value of c_0 is determined by the minimum of E_{25} corresponding to the nonlinear algebraic equation $E'_{25} = 0$ using equation (42) for $M = 50$ and $N = 50$ and it is $c_0 = -0.0801986$. For this value of c_0, we get the convergent homotopy-series solution. The curves of the averaged squared residual E_m versus c_0 are as shown in Figure 4. It is found that by means of $c_0 = -0.0801986$, the corresponding 25th-order homotopy-approximation is accurate enough with the averaged squared residual 5.99246×10^{-6} as shown in Table 1.

4. Results and Discussion

The BVPh, a Mathematica package, is used to obtain numerical presentation. Table 2 indicates the numerical values of saturation of injected water for different distance X and time T.

Figure 5 represents the graph of $S_i(X, T)$ versus distance X for fixed values of time $T = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$. Numerical values of Table 2 are used for Figure 5. From this graph, we can see that the saturation $S_i(X, T)$ of water increases as the distance X increases for fixed value of time T.

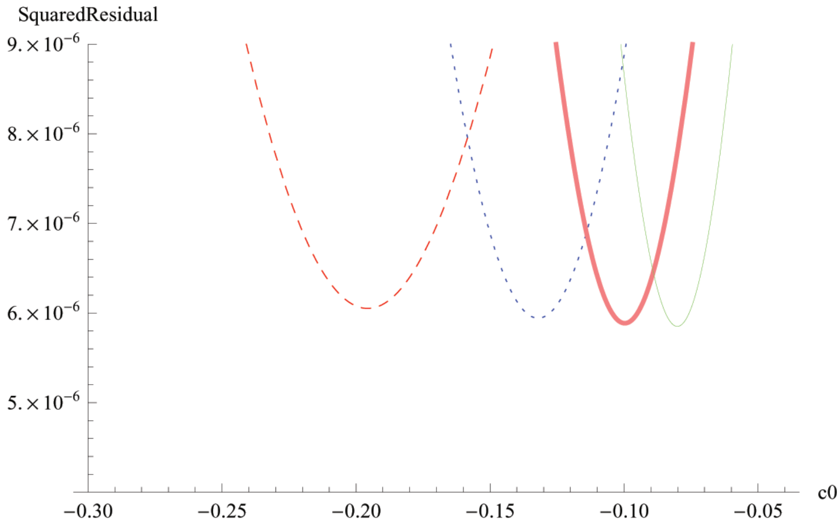

Figure 4: Averaged squared residual E_m versus $c0$. Dashed Red: E_{10}, Dotted Blue: E_{15}, Thick Pink: E_{20}, Thin Green: E_{25}.

Table 1: Averaged squared residual E_m of governing equation (22) by means of $c_0 = -0.0801986$.

Order of approximation m	Averaged squared residual E_m
3	2.70641×10^{-5}
5	2.56849×10^{-5}
7	2.37520×10^{-5}
9	2.13907×10^{-5}
11	1.87538×10^{-5}
13	1.60113×10^{-5}
15	1.33375×10^{-5}
17	1.08997×10^{-5}
19	8.84599×10^{-6}
21	7.29540×10^{-6}
23	6.33049×10^{-6}
25	5.99246×10^{-6}

5. Conclusion

The basic optimal HAM is applied to find the solution of the nonlinear differential equation (22). The value of the convergence-control parameter is obtained by the minimum of the discrete squared residual. It is found that the saturation of the injected fluid $S_i(X, T)$ increases smoothly with increase in distance X for fixed time T.

Table 2: Numerical values of the Saturation $S_i(X, T)$ of injected water.

X	$T = 0.4$	$T = 0.5$	$T = 0.6$	$T = 0.7$	$T = 0.8$	$T = 0.9$	$T = 1$
0	0.000040	0.000050	0.000060	0.000070	0.000080	0.000090	0.000100
0.1	0.000042	0.000056	0.000075	0.000094	0.000113	0.000132	0.000151
0.2	0.000047	0.000066	0.000094	0.000122	0.000150	0.000178	0.000206
0.3	0.000057	0.000082	0.000119	0.000156	0.000193	0.000230	0.000267
0.4	0.000074	0.000105	0.000151	0.000197	0.000243	0.000289	0.000335
0.5	0.000099	0.000137	0.000192	0.000247	0.000302	0.000357	0.000412
0.6	0.000133	0.000180	0.000244	0.000308	0.000373	0.000437	0.000501
0.7	0.000179	0.000237	0.000310	0.000383	0.000456	0.000529	0.000602
0.8	0.000238	0.000307	0.000389	0.000471	0.000553	0.000635	0.000718
0.9	0.000311	0.000394	0.000486	0.000577	0.000668	0.000759	0.000850
1	0.000400	0.000500	0.000600	0.000700	0.000800	0.000900	0.001000

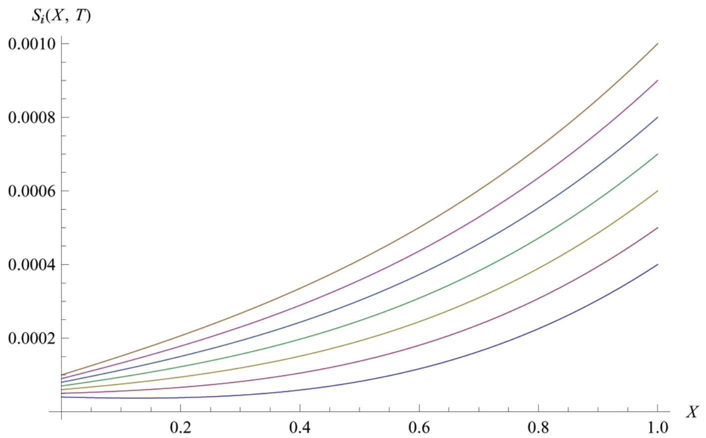

Figure 5: The saturation of water versus distance X for fixed values of times $T = 0.4(lowermost\ graph), 0.5, 0.6, 0.7, 0.8, 0.9, 1(uppermost\ graph)$.

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