Inverse Restrained Domination in Graphs

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Abstract
Let $G$ be a connected simple graph. A set $S \subseteq V(G)$ is a restrained dominating set if every vertex not in $S$ is adjacent to a vertex in $S$ and to a vertex in $V(G) \setminus S$. Let $D$ be a minimum restrained dominating set in $G$. A restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an inverse restrained dominating set of $G$ with respect to $D$. The inverse restrained domination number of $G$ denoted by $\gamma_r^{-1}(G)$ is the minimum cardinality of an inverse restrained dominating set of $G$. An inverse restrained dominating set of cardinality $\gamma_r^{-1}(G)$ is called $\gamma_r^{-1}$-set. In this paper, we show that every integers $k$ and $n$ with $1 \leq k < n$ is realizable as inverse restrained domination number and order of $G$ respectively. Further, we give the characterization of the inverse restrained dominating set with inverse restrained domination numbers of one and two and give some important results.

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1. Introduction

The concept of domination in graphs introduced by Claude Berge in 1958 and Oystein Ore in 1962 [5] is currently receiving much attention in literature. Following the article of Ernie Cockayne and Stephen Hedetniemi [1], the domination in graphs became an area of study by many researchers. One type of domination parameter is the restrained domination number in a graph. This was introduced by Telle and Proskurowski [4].

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indirectly as a vertex partitioning problem. One practical application of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. To effect security, each prisoner’s position is observed by a guard’s position. To protect the rights of prisoners, each prisoner’s position is seen by at least one other prisoner’s position. To be cost effective, it is desirable to place a few guards as possible. In [2], Enriquez and Canoy, introduced a variant of domination in graphs, the concept of restrained convex domination in a graph. The inverse domination in a graph was first found in the paper of Kulli [6]. Moreover, for the general concepts not mentioned, readers may refer to [3].

Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $S \subseteq V(G)$, then the open neighborhood of $S$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$. The closed neighborhood of $S$ is $N_G[S] = N[S] = S \cup N(S)$. A subset $S$ of $V(G)$ is a dominating set of $G$ if for every $v \in (V(G) \setminus S)$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The domination number $\gamma(G)$ of $G$ is the smallest cardinality of a dominating set of $G$. A set $S \subseteq V(G)$ is a restrained dominating set if every vertex not in $S$ is adjacent to a vertex in $S$ and to a vertex in $V(G) \setminus S$. The restrained domination number of $G$, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of $G$. A subset $S$ of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. Let $D$ be a minimum dominating set in $G$. The dominating set $S \subseteq V(G) \setminus D$ is called an inverse dominating set with respect to $D$. The minimum cardinality of inverse dominating set is called an inverse domination number of $G$ and is denoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is called $\gamma^{-1}$-set of $G$.

Motivated by definition of inverse dominating set, we define the following variant of inverse domination in graphs. Let $D$ be a minimum restrained dominating set in $G$. A restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an inverse restrained dominating set of $G$ with respect to $D$. The inverse restrained domination number of $G$ denoted by $\gamma^{-1}_r(G)$ is the minimum cardinality of an inverse restrained dominating set of $G$. An inverse restrained dominating set of cardinality $\gamma^{-1}_r(G)$ is called $\gamma^{-1}_r$-set.

2. Results

One of the classical results in the domination theory which was introduced by Ore in 1962 state the following theorem:

**Theorem 2.1.** [5] Let $G$ be a graph with no isolated vertex. If $S \subseteq V(G)$ is a $\gamma$-set, then $V(G) \setminus S$ is also a dominating set in $G$.

This motivate us to introduce a variant of inverse domination in graphs, the inverse restrained domination in graphs. Theorem 2.1 guarantees the existence of $\gamma^{-1}_r$-set in some graph $G$. Since the inverse restrained dominating set of any graph $G$ of order $n$
cannot be $V(G)$, it follows that $\gamma_{r}^{-1}(G) \neq n$ and hence $\gamma_{r}^{-1}(G) < n$.

Since $\gamma_{r}^{-1}(G)$ does not always exists in a connected nontrivial graph $G$, we denote by $\mathcal{R}(G)$ be a family of all graphs with inverse restrained dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{R}(G)$. From the definitions, the following result is immediate.

**Remark 2.2.** Let $G$ be a connected graph of order $n \geq 4$. If $D$ is a $\gamma_r$-set and $S$ is an inverse restrained dominating set of $G$, then $D \cap S = \emptyset$.

**Remark 2.3.** Let $G$ be a connected graph of order $n \geq 4$. Then

(i) $\gamma_{r}^{-1}(G) \in \{1, 2, \ldots, n-3, n-2\}$, and

(ii) $\gamma(G) \leq \gamma_r(G) \leq \gamma_{r}^{-1}(G)$.

The next result says that the value of the parameter $\gamma_{r}^{-1}$ ranges over all positive integers.

**Theorem 2.4. (Realization Problem)** Given positive integers $k$ and $n$ such that $n \geq 4$ and $k \in \{1, 2, \ldots, n-3, n-2\}$, there exists a connected nontrivial graph $G$ with $|V(G)| = n$ and $\gamma_{r}^{-1}(G) = k$.

**Proof.** Consider the following cases:

**Case 1.** Suppose $k = 1$.
Let $G = K_n$. Then, clearly, $|V(G)| = n$ and $\gamma_{r}^{-1}(G) = 1$.

**Case 2.** Suppose $2 \leq k < n-2$.
Let $H = K_r$ ($r \geq 3$) and $P_m = [a_1, a_2, \ldots, a_m]$ ($m \geq 2$). Consider the graph $G$ obtained from $H$ by adding the edges $va_1, va_2, \ldots, va_m$ (see Figure 1).

![Figure 1: A graph $G$ with $\gamma_{r}^{-1}(G) = k$](image)

Let $n = m + r$. If $m = 3s - 1$ for some $s \in \mathbb{N}$, then let $k = (m+4)/3$. The set $D = \{v\}$ is a $\gamma_r$-set of $G$ and $S = \{a_{3j-1} : j = 1, 2, \ldots, \frac{m+1}{3}\} \cup \{u\}$ is a $\gamma_{r}^{-1}$-set of $G$. Thus, $\gamma_{r}^{-1}(G) = \frac{m+1}{3} + 1 = k$. If $m = 3s+1$ for some $s \in \mathbb{N}$, then let $k = (m+5)/3$. The set $D = \{v\}$ is a $\gamma_r$-set of $G$ and $S = \{a_{3j-2} : j = 1, 2, \ldots, \frac{m+2}{3}\} \cup \{u\}$ is a $\gamma_{r}^{-1}$-set of
Theorem 2.6. Let $D = \gamma(H)$ has no isolated vertices. This implies that $G$ only if $\gamma(H)$ sets in $\gamma$. Thus, $G$.

Corollary 2.7. The difference $G$ if and only if $\gamma(H)$ is a $\gamma_r$-set of $G$ and $S = \{a_{3j-1} : j = 1, 2, \ldots, \frac{m}{3}\} \cup \{u\}$ is a $\gamma_r^{-1}$-set of $G$. Thus, $\gamma_r^{-1}(G) = \frac{m}{3} + 1 = k$. Moreover, $|V(G)| = r + m = n$.

Let $G = C_n$ where $n = 4$ (see Figure 2).

![Figure 2: A graph $G$ with $\gamma_r^{-1}(G) = n - 2$](image)

The set $D = \{a_1, a_2\}$ is a $\gamma_r$-set and $S = \{a_3, a_4\}$ is a $\gamma_r^{-1}$-set of $G$. Thus, $|V(G)| = 4 = n$ and $\gamma_r^{-1}(G) = 2 = n - 2$.

This proves the assertion. ■

Corollary 2.5. The difference $\gamma_r^{-1} - \gamma_r$ can be made arbitrarily large.

Proof. Let $k$ be a positive integer. By Theorem 2.4, there exists a connected graph $G$ such that $\gamma_r^{-1}(G) = k + 1$ and $\gamma_r(G) = 1$. Thus, $\gamma_r^{-1}(G) - \gamma_r(G) = k$. Therefore, $\gamma_r^{-1} - \gamma_r$ can be made arbitrarily large. ■

Theorem 2.6. Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_r^{-1}(G) = 1$ if and only if $G = K_1 + H$ where $\gamma(H) = 1$.

Proof. Suppose that $\gamma_r^{-1}(G) = 1$. Let $S = V(K_1)$ be a $\gamma_r^{-1}$-set of $G$. Set $V(H) = V(G) \setminus S$. Since $\gamma_r(G) \leq \gamma_r^{-1}(G) = 1$ by Remark 2.3, it follows that $\gamma_r(G) = 1$. Let $D = \{x\}$ be a $\gamma_r$-set of $G$. Since $D \cap S = \emptyset$ by Remark 2.2, $D \subset V(H)$, that is, $\gamma(H) = 1$. Therefore, $G = K_1 + H$ where $\gamma(H) = 1$.

For the converse, suppose that $G = K_1 + H$ where $\gamma(H) = 1$. Let $D = V(K_1) = \{x\}$ be a $\gamma_r$-set of $G$ and let $S = \{y\}$ be a dominating set of $H$. Since $D$ is a dominating set of $G$ and $n \geq 3$, $xz \in E(G)$ for every $z \in V(G) \setminus S$ ($x \neq z$). Thus, $\langle V(G) \setminus S \rangle$ has no isolated vertices. This implies that $S$ is a restrained dominating set of $G$. Since $D \cap S = \emptyset, S \subseteq (V(G) \setminus D)$, that is, $S$ is a $\gamma_r^{-1}$-set of $G$. Hence, $\gamma_r^{-1}(G) = 1$. ■

The following result is a direct consequence of Theorem 2.6.

Corollary 2.7. Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_r^{-1}(G) = 1$ if and only if $G = K_2 + H$ for some subgraph $H$.

Suppose that $\gamma(H_1) = 1 = \gamma(H_2)$. Let $S_1 = \{a\}$ and $S_2 = \{b\}$ be dominating sets in $H_1$ and $H_2$ respectively. Then the graph $G = H_1 + H_2$ may be expressed as
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\[ G = \langle S_1 \rangle + (\langle S_2 \rangle + J) \text{ where } V(J) = V(H_1) \setminus S_1 \text{ and } V(I) = V(H_2) \setminus S_2. \]  
Thus, \( G = \langle S_1 \rangle + (\langle S_2 \rangle + J + I) = K_1 + H \) where \( \gamma(H) = 1 \). Thus the following result is a direct consequence of Theorem 2.6.

**Corollary 2.8.** Let \( G \) and \( H \) be connected graphs of orders \( n \geq 2 \) and \( m \geq 1 \) (or \( n \geq 1 \) and \( m \geq 2 \)) respectively. Then \( \gamma_r^{-1}(G + H) = 1 \) if and only if \( \gamma(G) = \gamma(H) = 1 \).

**Remark 2.9.** If \( G \) is a complete graph of order \( n \geq 3 \), then \( \gamma_r^{-1}(G) = 1 \).

**Theorem 2.10.** Let \( G \) be a connected non-complete graph of order \( n \geq 4 \). Then \( \gamma_r^{-1}(G) = 2 \) if and only if \( G \neq K_2 + H \) for any subgraph \( H \) and there exist distinct vertices \( x \) and \( y \) that dominate \( G \) such that \( \langle V(G) \setminus \{x, y\} \rangle \) has no isolated vertices and satisfies one of the following:

(i) There exists \( a \in V(G) \setminus \{x, y\} \) that dominate \( G \) and \( \langle V(G) \setminus \{a\} \rangle \) has no isolated vertices.

(ii) \( \gamma(\langle N(x) \setminus \{y\} \rangle) = 1 \) and

\( \gamma(\langle N(y) \setminus \{x\} \rangle) = 1 \) or

\( \gamma(\langle (N(y) \setminus \{x\}) \setminus \{c : c \notin N(b) \text{ for some } b, c \in N(y) \setminus \{x\} \rangle) = 1 \) where \( c \in N(a) \).

(iii) \( \gamma(\langle (N(x) \setminus \{y\}) \setminus \{d : d \notin N(a) \text{ for some } a, d \in N(x) \setminus \{y\}\rangle) = 1 \) where \( d \in N(b) \) and

\( \gamma(\langle N(y) \setminus \{x\} \rangle) = 1 \) or

\( \gamma(\langle (N(y) \setminus \{x\}) \setminus \{c : c \notin N(b) \text{ for some } b, c \in N(y) \setminus \{x\} \rangle) = 1 \) where \( c \in N(a) \).

(iv) \( \gamma(\langle N(x) \rangle) = 1 \) and

\( \gamma(\langle N(y) \rangle) = 1 \); or

\( \gamma(\langle (N(y) \setminus \{N(a) \setminus \{b\}) \rangle = 1 \) for some vertex \( a \in N(x) \) with \( ab \in E(G) \).

(v) \( \gamma(\langle (N(x) \setminus \{N(b) \setminus \{a\}) \rangle = 1 \) for some vertex \( b \in N(y) \) with \( ab \in E(G) \) and

\( \gamma(\langle N(y) \rangle) = 1 \); or

\( \gamma(\langle (N(y) \setminus \{N(a) \setminus \{b\}) \rangle = 1 \) for some vertex \( a \in N(x) \) with \( ab \in E(G) \).

(vi) \( \gamma(\langle (N(x) \setminus \{N(b)\}) \rangle = 1 \) for some \( b \in N(y) \) and \( \gamma(\langle (N(y) \setminus \{N(a)\}) \rangle = 1 \) for some \( a \in N(x) \) with \( ab \notin E(G) \); or

(vii) there exists \( a \in N(x) \) and \( b \in N(y) \) such that \( ab \notin E(G) \) and

\( \gamma(\langle (N(x) \setminus \{N(b)\}) \rangle = 1 \) and \( \gamma(\langle N(x) \rangle) = 1 \); or
Thus, \( \gamma(\langle N(x) \rangle) = 1 \) and \( \gamma(\langle N(y) \setminus N(a) \rangle) = 1 \).

Proof. Suppose that \( \gamma_r^{-1}(G) = 2 \). Let \( S = \{x, y\} \) be a \( \gamma_r^{-1}-set \) of \( G \). Then \( x \) and \( y \) dominate \( G \) such that \( \langle V(G) \setminus \{x, y\} \rangle \) has no isolated vertices. Suppose that \( G = K_2 + H \) for some subgraph \( H \). Then \( \gamma_r^{-1}(G) = 1 \) by Corollary 2.7 contrary to our assumption. Thus, \( G \neq K_2 + H \) for any subgraph \( H \). Now, by Remark 2.3, \( \gamma_r(G) \leq \gamma_r^{-1}(G) = 2 \). Consider the following cases:

Case 1. Suppose that \( \gamma_r(G) = 1 \).

Let \( D = \{a\} \) be a \( \gamma_r\)-set of \( G \). In view of Remark 2.2, \( D \cap S = \emptyset \). This implies that \( a \in V(G) \setminus S \) dominate \( G \). Since \( S \) is dominating set of \( G \), for every \( u \in V(G) \setminus \{a\} \), there exists \( v \in S \) such that \( uv \in E(G) \). Thus, \( \langle V(G) \setminus \{a\} \rangle \) has no isolated vertices. This proves (i).

Case 2. Suppose that \( \gamma_r(G) = 2 \).

Let \( D = \{a, b\} \) be a \( \gamma_r\)-set. Then \( \langle V(G) \setminus D \rangle \) has no isolated vertices. Consider the following subcases.

Subcase 1. Suppose that \( xy \in E(G) \).

Since \( D \) is a dominating set in \( G \), let \( \{a\} \) be a dominating set of \( N(x) \setminus \{y\} \). Then \( \gamma(\langle N(x) \setminus \{y\} \rangle) = 1 \). If \( \{b\} \) is a dominating set of \( N(y) \setminus \{x\} \), then \( \gamma(\langle N(y) \setminus \{x\} \rangle) = 1 \). This proves (iiia). If \( \{b\} \) is not a dominating set of \( N(y) \setminus \{x\} \), then there exists \( c \in N(y) \setminus \{x\} \) such that \( c \notin N(b) \) for some \( b \in N(y) \setminus \{x\} \). Thus, \( \gamma(\langle N(y) \setminus \{x\} \rangle \setminus \{c : c \notin N(b) \text{ for some } b, c \in N(y) \setminus \{x\}\}) = 1 \). Since \( D \) is dominating, \( c \in N(a) \). This proves (iiib).

Now, if \( \{a\} \) is not a dominating set of \( N(x) \setminus \{y\} \), then there exists \( d \in N(x) \setminus \{y\} \) such that \( d \notin N(a) \) for some \( a \in N(x) \setminus \{y\} \). Thus, \( \gamma(\langle N(x) \setminus \{y\} \rangle \setminus \{d : d \notin N(a) \text{ for some } a, d \in N(x) \setminus \{y\}\}) = 1 \). Since \( D \) is dominating, \( d \in N(b) \). If \( \{b\} \) is a dominating set of \( N(y) \setminus \{x\} \), then \( \gamma(\langle N(y) \setminus \{x\} \rangle) = 1 \). This proves (iiic). If \( \{b\} \) is not a dominating set of \( N(y) \setminus \{x\} \), then \( \gamma(\langle N(y) \setminus \{x\} \rangle \setminus \{c : c \notin N(b) \text{ for some } b, c \in N(y) \setminus \{x\}\}) = 1 \) where \( c \in N(a) \) by similar arguments used in (iiic). This proves (iiid).

Subcase 2. Suppose that \( xy \notin E(G) \).

Consider \( ab \in E(G) \). Let \( D_a = \{a\} \) be a dominating set of \( \langle N(x) \rangle \). Then \( \gamma(\langle N(x) \rangle) = 1 \). If \( D_b = \{b\} \) is a dominating set of \( \langle N(y) \rangle \), then \( \gamma(\langle N(y) \rangle) = 1 \). This proves (iua). Suppose that \( D_b \) is not a dominating set of \( \langle N(y) \rangle \). Then there exists \( c \in N(y) \) such that \( c \notin N(b) \). Since \( D = \{a, b\} \) is a dominating set of \( G \), it follows that \( c \in N(a) \). Thus, \( D_b \) is a dominating set of \( \langle N(y) \setminus (N(a) \setminus \{b\}) \rangle \), that is, \( \gamma(\langle N(y) \setminus (N(a) \setminus \{b\}) \rangle) = 1 \) for some \( a \in N(x) \) with \( ab \in E(G) \). This proves (ivb).

Similarly, if \( D_a \) is not a dominating set in \( \langle N(x) \rangle \), then \( \gamma(\langle N(x) \setminus (N(b) \setminus \{a\}) \rangle) = 1 \) for some vertex \( b \in N(y) \) with \( ab \in E(G) \). If \( D_b \) is a dominating set of \( \langle N(y) \rangle \), then \( \gamma(\langle N(y) \rangle) = 1 \), proving (va). If \( D_b \) is not a dominating set of \( \langle N(y) \rangle \), then \( \gamma(\langle N(y) \setminus (N(a) \setminus \{b\}) \rangle) = 1 \) for some \( a \in N(x) \) with \( ab \in E(G) \). This proves (vib).

Consider \( ab \notin E(G) \). Suppose that \( D_a = \{a\} \) is not a dominating set of \( \langle N(x) \rangle \). Then there exists \( c \in N(x) \) such that \( c \notin N(a) \). Since \( D = \{a, b\} \) is a dominating
set of $G$, $c \in N(b)$. Thus $\gamma((N(x) \setminus N(b))) = 1$. If $D_b = \{b\}$ is not a dominating set of $\langle N(y) \setminus N(a) \rangle$, then there exists $d \in N(y)$ such that $d \notin N(b)$. Since $D = \{a, b\}$ is a dominating set of $G$, $d \in N(a)$. Thus $\gamma((N(y) \setminus N(a))) = 1$. This shows (vi). If $D_b = \{b\}$ is a dominating set of $\langle N(y) \setminus N(a) \rangle$, then $\gamma(N(y)) = 1$. This proves (viia). Now, suppose that $D_a = \{a\}$ is a dominating set of $\langle N(x) \setminus N(a) \rangle$. Then $\gamma(\langle N(x) \setminus N(a) \rangle) = 1$. If $D_b = \{b\}$ is not a dominating set of $\langle N(y) \setminus N(a) \rangle$, $\gamma(\langle N(y) \setminus N(a) \rangle) = 1$ by similar arguments used above. This proves (viib).

For the converse, suppose that $G \neq K_2 + H$ for any subgraph $H$ and there exist distinct vertices $x$ and $y$ that dominate $G$ such that $\langle V(G) \setminus \{x, y\} \rangle$ has no isolated vertices and satisfies (i), (ii), (iii), (iv), (v), (vi) or (vii).

Suppose first that (i) holds. Let $D = \{a\}$ be a $\gamma_r$-set and $S = \{x, y\}$ be a restrained dominating set of $G$. Since $a \in V(G) \setminus S$, it follows that $S \subseteq (V(G) \setminus D)$. Thus, $S$ is an inverse restrained dominating set of $G$, that is, $\gamma_r^{-1}(G) \leq |S| = 2$. Suppose that $\gamma_r^{-1}(G) = 1$. Then there exist a vertex in $S$, say $x$, such that $x$ dominate $G$. Since $x \neq a$, it follows that $\{x\}$ and $\{a\}$ are dominating sets of $G$. This implies that $G = K_2 + H$ for some subgraph $H$ contrary to our assumption. Thus, $\gamma_r^{-1}(G) = 2$.

Next, suppose that (iiia) holds. Then $xy \in E(G)$. Let $S = \{x, y\}$ and let $D_a = \{a\}$ be a dominating set of $\langle N(x) \setminus \{y\} \rangle$ and $D_b = \{b\}$ be a dominating set of $\langle N(y) \setminus \{x\} \rangle$. Then, $N[a] = N[x] \setminus \{y\}$ and $N[b] = N[y] \setminus \{x\}$. Thus,

$$N[a] \cup N[b] = (N[x] \setminus \{y\}) \cup (N[y] \setminus \{x\}) = N[x] \cup N[y] = V(G).$$

This implies that $D = \{a, b\}$ is a dominating set of $G$. Now, let $u, v \in V(G) \setminus D$. If $u = x$ and $v = y$, then $uv \in E(G)$. Suppose that $u = x$ and $v \neq y$. If $v \in N(x) \setminus \{y\}$, then $xv = uv \in E(G)$. If $v \in N(y) \setminus \{x\}$, then $vy, uy \in E(G)$. This implies that $u-v$ is a path in $G$. Similarly, if $u \neq x$ and $v = y$, then $u-v$ is path in $G$. Moreover, suppose that $u \neq x$ and $v \neq y$. If $u \in N(x) \setminus \{y\}$ and $v \in N(y) \setminus \{x\}$, then $ux, xy, yv \in E(G)$. Thus, $u-v$ is a path in $G$. If $u, v \in N(x) \setminus \{y\}$ or $u, v \in N(y) \setminus \{x\}$, then it can be shown $u-v$ is a path in $G$. In any case, $\langle V(G) \setminus D \rangle$ has no isolated vertices. This implies that $D$ is a restrained dominating set in $G$. Thus, $\gamma_r(G) \leq |D| = 2$. Suppose that $\gamma_r(G) = 1$. Let $D_a = \{a\}$ be a $\gamma_r$-set of $G$. Then, $\gamma_r^{-1}(G) = 2$ by following similar arguments used in (i). Suppose that $\gamma_r(G) = 2$ and let $D = \{a, b\}$ be a $\gamma_r$-set of $G$. By hypothesis, $S = \{x, y\}$ is a restrained dominating set of $G$, and by (iiia), $S \cap D = \emptyset$. This implies that $S \subseteq (V(G) \setminus D)$, that is $S$ is an inverse restrained dominating set of $G$ with respect to $D$. Since $\gamma_r(G) = 2$, it follows that $S = \{x, y\}$ is the minimum inverse restrained dominating set of $G$ with respect to $D$ by Remark 2.3. Hence, $\gamma_r^{-1}(G) = 2$.

Suppose that (iiib) holds. Then $xy \in E(G)$. Let $D_a = \{a\}$ be a dominating set of $\langle N(x) \setminus \{y\} \rangle$ and $D_b = \{b\}$ be a dominating set of $\langle N(y) \setminus \{x\} \rangle \setminus C$ where $C = \{c : c \notin N(b) \text{ for some } b, c \in N(y) \setminus \{x\}\}$ and $c \in N(a)$. Then, $N[a] = (N[x] \setminus \{y\}) \cup C$
Theorem 2.12. Let \( \gamma \) be the number of vertices. Hence, \( \gamma \) is an inverse dominating set of \( G \). This implies that \( \gamma \) is a minimum dominating set of \( V(G) \).

Suppose that \( (iii) \) holds. By using similar arguments in \( (ii) \), it can be shown that \( \gamma = 2 \). Finally, if any of the conditions \( (iv) \) or \( (v) \) or \( (vi) \) or \( (vii) \) holds, then it is clear that \( \gamma = 2 \).

The following result is a direct consequence of Theorem 2.10.

Corollary 2.11. Let \( G = K_2 \) and \( H \) be connected graphs of order \( m \geq 2 \). Then \( \gamma(G \circ H) = 2 \) if and only if \( \gamma(H) = 1 \).

Corollary 2.11, can be generally stated by the following result.

Theorem 2.12. Let \( G \) and \( H \) be connected graphs of orders \( n \) and \( m \geq 2 \) respectively. Then \( \gamma(G \circ H) = n \) if and only if \( \gamma(H) = 1 \).

Proof. Suppose that \( \gamma(G \circ H) = n \). Let \( S = V(G) \) be a \( \gamma(G \circ H) \)-set of \( G \circ H \). Then \( S \subseteq V(G) \) where \( D \) is a \( \gamma(G \circ H) \)-set of \( G \circ H \). In view of Remark 2.3, \( |D| \leq |S| \). If \( |D| < |S| \), then \( D \) is not a dominating set of \( G \circ H \) since there exists \( v \in V(G) \) such that \( H^v \) is not dominated by element of \( D \). Thus, \( |D| = |S| \), that is, \( D = \bigcup_{i=1}^{n} x_i : x_i \in V(H^v), v_i \in V(G) \). This implies that \( x_i \) dominate \( V(H^v) \) for each \( v_i \in V(G) \) where \( i = 1, 2, \ldots, n \). Hence \( \gamma(H^v) = 1 \) for each \( v_i \in V(G) \) \( (i = 1, 2, \ldots, n) \), that is, \( \gamma(H) = 1 \).

For the converse, suppose that \( \gamma(H) = 1 \). Let \( x \in V(H^v) \) dominate \( H^v \) for each \( v \in V(G) \). Then \( \{x \} \subseteq V(v + H^v) \) is a minimum dominating set of \( v + H^v \) for each \( v \in V(G) \). This implies that \( D = \bigcup_{i=1}^{n} x_i : x_i \in V(H^v), v_i \in V(G) \) is a minimum dominating set of \( V(G \circ H) \). Let \( v \in V(G) \). Since the order of \( H \) is \( m \geq 2 \), for each \( u \in V(H^v) \setminus \{x\} \) where \( x \in D \), \( uu \in E(v + H^v) \). Thus, \( \langle V(v + H^v) \setminus \{x\} \rangle \) has no isolated vertices for each \( v \in V(G) \). This implies that \( \langle V(G \circ H) \setminus D \rangle \) has no isolated vertices. Hence \( D \) is a \( \gamma(G \circ H) \)-set of \( G \circ H \). Since \( V(G) \subseteq \langle V(G \circ H) \setminus D \rangle \), it follows that \( V(G) \) is an inverse dominating set of \( V(G \circ H) \). Since \( \langle V(G \circ H) \setminus V(G) \rangle = H \) has no isolated vertices, \( V(G) \) is an inverse restrained dominating set of \( G \circ H \). Since \( |V(G)| \) is a minimum dominating set of \( G \circ H \), it follows that \( V(G) \) is a \( \gamma(G \circ H) \)-set of \( G \circ H \).

Hence, \( \gamma(G \circ H) = n \).
References


