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Inverse Restrained Domination in Graphs

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Abstract

Let G be a connected simple graph. A set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Let D be a minimum restrained dominating set in G. A restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an *inverse restrained dominating set* of G with respect to D. The *inverse restrained domination number* of G denoted by $\gamma_r^{-1}(G)$ is the minimum cardinality of an inverse restrained dominating set of G. An inverse restrained dominating set of cardinality $\gamma_r^{-1}(G)$ is called γ_r^{-1}-set. In this paper, we show that every integers k and n with $1 \leq k < n$ is realizable as inverse restrained domination number and order of G respectively. Further, we give the characterization of the inverse restrained dominating set with inverse restrained domination numbers of one and two and give some important results.

AMS subject classification: 05C69.
Keywords: Dominating set, restrained dominating set, inverse restrained dominating set, inverse restrained domination number.

1. Introduction

The concept of domination in graphs introduced by Claude Berge in 1958 and Oystein Ore in 1962 [5] is currently receiving much attention in literature. Following the article of Ernie Cockayne and Stephen Hedetniemi [1], the domination in graphs became an area of study by many researchers. One type of domination parameter is the restrained domination number in a graph. This was introduced by Telle and Proskurowski [4]

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indirectly as a vertex partitioning problem. One practical application of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. To effect security, each prisoner's position is observed by a guard's position. To protect the rights of prisoners, each prisoner's position is seen by at least one other prisoner's position. To be cost effective, it is desirable to place a few guards as possible. In [2], Enriquez and Canoy, introduced a variant of domination in graphs, the concept of restrained convex domination in a graph. The inverse domination in a graph was first found in the paper of Kulli [6]. Moreover, for the general concepts not mentioned, readers may refer to [3].

Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $S \subseteq V(G)$, then the *open neighborhood* of S is the set $N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$. The *closed neighborhood* of S is $N_G[S] = N[S] = S \cup N(S)$. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in (V(G) \backslash S)$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. A set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. A subset S of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. Let D be a minimum dominating set in G. The dominating set $S \subseteq V(G) \setminus D$ is called an *inverse dominating set* with respect to D. The minimum cardinality of inverse dominating set is called an *inverse domination number* of G and is denoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is called γ^{-1}-*set* of G.

Motivated by definition of inverse dominating set, we define the following variant of inverse domination in graphs. Let D be a minimum restrained dominating set in G. A restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an *inverse restrained dominating set* of G with respect to D. The *inverse restrained domination number* of G denoted by $\gamma_r^{-1}(G)$ is the minimum cardinality of an inverse restrained dominating set of G. An inverse restrained dominating set of cardinality $\gamma_r^{-1}(G)$ is called γ_r^{-1}-*set*.

2. Results

One of the classical results in the domination theory which was introduced by Ore in 1962 state the following theorem:

Theorem 2.1. [5] Let G be a graph with no isolated vertex. If $S \subseteq V(G)$ is a γ-*set*, then $V(G) \setminus S$ is also a dominating set in G.

This motivate us to introduce a variant of inverse domination in graphs, the inverse restrained domination in graphs. Theorem 2.1 guarantees the existence of γ_r^{-1}-*set* in some graph G. Since the inverse restrained dominating set of any graph G of order n

cannot be $V(G)$, it follows that $\gamma_r^{-1}(G) \neq n$ and hence $\gamma_r^{-1}(G) < n$.

Since $\gamma_r^{-1}(G)$ does not always exists in a connected nontrivial graph G, we denote by $\mathcal{R}(G)$ be a family of all graphs with inverse restrained dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{R}(G)$. From the definitions, the following result is immediate.

Remark 2.2. Let G be a connected graph of order $n \geq 4$. If D is a γ_r-*set* and S is an inverse restrained dominating set of G, then $D \cap S = \varnothing$.

Remark 2.3. Let G be a connected graph of order $n \geq 4$. Then

(i) $\gamma_r^{-1}(G) \in \{1, 2, .., n-3, n-2\}$, and

(ii) $\gamma(G) \leq \gamma_r(G) \leq \gamma_r^{-1}(G)$.

The next result says that the value of the parameter γ_r^{-1} ranges over all positive integers.

Theorem 2.4. (Realization Problem) Given positive integers k and n such that $n \geq 4$ and $k \in \{1, 2, \ldots, n-3, n-2\}$, there exists a connected nontrivial graph G with $|V(G)| = n$ and $\gamma_r^{-1}(G) = k$.

Proof. Consider the following cases:

Case 1. Suppose $k = 1$.
Let $G = K_n$. Then, clearly, $|V(G)| = n$ and $\gamma_r^{-1}(G) = 1$.

Case 2. Suppose $2 \leq k < n-2$.
Let $H = K_r$ $(r \geq 3)$ and $P_m = [a_1, a_2, \ldots, a_m]$ $(m \geq 2)$. Consider the graph G obtained from H by adding the edges va_1, va_2, ..., and va_m (see Figure 1).

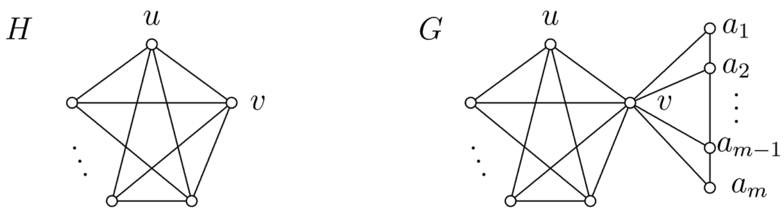

Figure 1: A graph G with $\gamma_r^{-1}(G) = k$

Let $n = m + r$. If $m = 3s - 1$ for some $s \in \mathbb{N}$, then let $k = (m+4)/3$. The set $D = \{v\}$ is a γ_r-*set* of G and $S = \left\{a_{3j-1} : j = 1, 2, \ldots, \frac{m+1}{3}\right\} \cup \{u\}$ is a γ_r^{-1}-*set* of G. Thus, $\gamma_r^{-1}(G) = \frac{m+1}{3} + 1 = k$. If $m = 3s + 1$ for some $s \in \mathbb{N}$, then let $k = (m+5)/3$. The set $D = \{v\}$ is a γ_r-*set* of G and $S = \left\{a_{3j-2} : j = 1, 2, \ldots, \frac{m+2}{3}\right\} \cup \{u\}$ is a γ_r^{-1}-*set* of

G. Thus, $\gamma_r^{-1}(G) = \dfrac{m+2}{3} + 1 = k$. If $m = 3s$ for some $s \in \mathbb{N}$, then let $k = (m+3)/3$. The set $D = \{v\}$ is a γ_r-*set* of G and $S = \left\{a_{3j-1} : j = 1, 2, \ldots, \dfrac{m}{3}\right\} \cup \{u\}$ is a γ_r^{-1}-*set* of G. Thus, $\gamma_r^{-1}(G) = \dfrac{m}{3} + 1 = k$. Morover, $|V(G)| = r + m = n$.

Case 3. Suppose $k = n - 2$.
Let $G = C_n$ where $n = 4$ (see Figure 2).

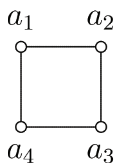

Figure 2: A graph G with $\gamma_r^{-1}(G) = n - 2$

The set $D = \{a_1, a_2\}$ is a γ_r-*set* and $S = \{a_3, a_4\}$ is a γ_r^{-1}-*set* of G. Thus, $|V(G)| = 4 = n$ and $\gamma_r^{-1}(G) = 2 = n - 2$.

This proves the assertion. ■

Corollary 2.5. The difference $\gamma_r^{-1} - \gamma_r$ can be made arbitrarily large.

Proof. Let k be a positive integer. By Theorem 2.4, there exists a connected graph G such that $\gamma_r^{-1}(G) = k + 1$ and $\gamma_r(G) = 1$. Thus, $\gamma_r^{-1}(G) - \gamma_r(G) = k$. Therefore, $\gamma_r^{-1} - \gamma_r$ can be made arbitrarily large. ■

Theorem 2.6. Let G be a connected graph of order $n \geq 3$. Then $\gamma_r^{-1}(G) = 1$ if and only if $G = K_1 + H$ where $\gamma(H) = 1$.

Proof. Suppose that $\gamma_r^{-1}(G) = 1$. Let $S = V(K_1)$ be a γ_r^{-1}-*set* of G. Set $V(H) = V(G) \setminus S$. Since $\gamma_r(G) \leq \gamma_r^{-1}(G) = 1$ by Remark 2.3, it follows that $\gamma_r(G) = 1$. Let $D = \{x\}$ be a γ_r-*set* of G. Since $D \cap S = \varnothing$ by Remark 2.2, $D \subset V(H)$, that is, $\gamma(H) = 1$. Therefore, $G = K_1 + H$ where $\gamma(H) = 1$.

For the converse, suppose that $G = K_1 + H$ where $\gamma(H) = 1$. Let $D = V(K_1) = \{x\}$ be a γ_r-*set* of G and let $S = \{y\}$ be a dominating set of H. Since D is a dominating set of G and $n \geq 3$, $xz \in E(G)$ for every $z \in V(G) \setminus S$ $(x \neq z)$. Thus, $\langle V(G) \setminus S \rangle$ has no isolated vertices. This implies that S is a restrained dominating set of G. Since $D \cap S = \varnothing$, $S \subseteq (V(G) \setminus D)$, that is, S is a γ_r^{-1}-*set* of G. Hence, $\gamma_r^{-1}(G) = 1$. ■

The following result is a direct consequence of Theorem 2.6.

Corollary 2.7. Let G be a connected graph of order $n \geq 3$. Then $\gamma_r^{-1}(G) = 1$ if and only if $G = K_2 + H$ for some subgraph H.

Suppose that $\gamma(H_1) = 1 = \gamma(H_2)$. Let $S_1 = \{a\}$ and $S_2 = \{b\}$ be dominating sets in H_1 and H_2 respectively. Then the graph $G = H_1 + H_2$ may be expressed as

$G = (\langle S_1 \rangle + J) + (\langle S_2 \rangle + I)$ where $V(J) = V(H_1) \setminus S_1$ and $V(I) = V(H_2) \setminus S_2$. Thus, $G = \langle S_1 \rangle + (\langle S_2 \rangle + J + I) = K_1 + H$ where $\gamma(H) = 1$. Thus the following result is a direct consequence of Theorem 2.6.

Corollary 2.8. Let G and H be connected graphs of orders $n \geq 2$ and $m \geq 1$ (or $n \geq 1$ and $m \geq 2$) respectively. Then $\gamma_r^{-1}(G + H) = 1$ if and only if $\gamma(G) = 1 = \gamma(H)$.

Remark 2.9. If G is a complete graph of order $n \geq 3$, then $\gamma_r^{-1}(G) = 1$.

Theorem 2.10. Let G be a connected non-complete graph of order $n \geq 4$. Then $\gamma_r^{-1}(G) = 2$ if and only if $G \neq K_2 + H$ for any subgraph H and there exist distinct vertices x and y that dominate G such that $\langle V(G) \setminus \{x, y\} \rangle$ has no isolated vertices and satisfies one of the following:

(i) There exists $a \in V(G) \setminus \{x, y\}$ that dominate G and $\langle V(G) \setminus \{a\} \rangle$ has no isolated vertices.

(ii) $\gamma(\langle N(x) \setminus \{y\} \rangle) = 1$ and

- (a) $\gamma(\langle N(y) \setminus \{x\} \rangle) = 1$ or
- (b) $\gamma(\langle (N(y) \setminus \{x\}) \setminus \{c : c \notin N(b)$ for some $b, c \in N(y) \setminus \{x\}\} \rangle) = 1$ where $c \in N(a)$.

(iii) $\gamma(\langle (N(x) \setminus \{y\}) \setminus \{d : d \notin N(a)$ for some $a, d \in N(x) \setminus \{y\}\} \rangle) = 1$ where $d \in N(b)$ and

- (a) $\gamma(\langle N(y) \setminus \{x\} \rangle) = 1$ or
- (b) $\gamma(\langle (N(y) \setminus \{x\}) \setminus \{c : c \notin N(b)$ for some $b, c \in N(y) \setminus \{x\}\} \rangle) = 1$ where $c \in N(a)$.

(iv) $\gamma(\langle N(x) \rangle) = 1$ and

- (a) $\gamma(\langle N(y) \rangle) = 1$; or
- (b) $\gamma(\langle N(y) \setminus (N(a) \setminus \{b\}) \rangle) = 1$ for some vertex $a \in N(x)$ with $ab \in E(G)$.

(v) $\gamma(\langle N(x) \setminus (N(b) \setminus \{a\}) \rangle) = 1$ for some vertex $b \in N(y)$ with $ab \in E(G)$ and

- (a) $\gamma(\langle N(y) \rangle) = 1$; or
- (b) $\gamma(\langle N(y) \setminus (N(a) \setminus \{b\}) \rangle) = 1$ for some vertex $a \in N(x)$ with $ab \in E(G)$.

(vi) $\gamma(\langle N(x) \setminus (N(b) \rangle) = 1$ for some $b \in N(y)$ and $\gamma(\langle N(y) \setminus N(a) \rangle) = 1$ for some $a \in N(x)$ with $ab \notin E(G)$; or

(vii) there exists $a \in N(x)$ and $b \in N(y)$ such that $ab \notin E(G)$ and

- (a) $\gamma(\langle N(x) \setminus N(b) \rangle) = 1$ and $\gamma(\langle N(x) \rangle) = 1$; or

(b) $\gamma(\langle N(x)\rangle) = 1$ and $\gamma(\langle N(y) \setminus N(a)\rangle) = 1$.

Proof. Suppose that $\gamma_r^{-1}(G) = 2$. Let $S = \{x, y\}$ be a γ_r^{-1}-*set* of G. Then x and y dominate G such that $\langle V(G)\setminus\{x, y\}\rangle$ has no isolated vertices. Suppose that $G = K_2+H$ for some subgraph H. Then $\gamma_r^{-1}(G) = 1$ by Corollary 2.7 contrary to our assumption. Thus, $G \neq K_2 + H$ for any subgraph H. Now, by Remark 2.3, $\gamma_r(G) \leq \gamma_r^{-1}(G) = 2$. Consider the following cases:

Case 1. Suppose that $\gamma_r(G) = 1$.

Let $D = \{a\}$ be a γ_r-*set* of G. In view of Remark 2.2, $D \cap S = \varnothing$. This implies that $a \in V(G) \setminus S$ dominate G. Since S is dominating set of G, for every $u \in V(G) \setminus \{a\}$, there exists $v \in S$ such that $uv \in E(G)$. Thus, $\langle V(G) \setminus \{a\}\rangle$ has no isolated vertices. This proves (i).

Case 2. Suppose that $\gamma_r(G) = 2$.

Let $D = \{a, b\}$ be a γ_r-*set*. Then $\langle V(G) \setminus D\rangle$ has no isolated vertices. Consider the following subcases.

Subcase 1. Suppose that $xy \in E(G)$.

Since D is a dominating set in G, let $\{a\}$ be a dominating set of $N(x) \setminus \{y\}$. Then $\gamma(\langle N(x)\setminus\{y\}\rangle) = 1$. If $\{b\}$ is a dominating set of $N(y)\setminus\{x\}$, then $\gamma(\langle N(y)\setminus\{x\}\rangle) = 1$. This proves (iia). If $\{b\}$ is not a dominating set of $N(y) \setminus \{x\}$, then there exists $c \in N(y) \setminus \{x\}$ such that $c \notin N(b)$ for some $b \in N(y) \setminus \{x\}$. Thus, $\gamma(\langle(N(y) \setminus \{x\}) \setminus \{c : c \notin N(b)$ for some $b, c \in N(y) \setminus \{x\}\}\rangle) = 1$. Since D is dominating, $c \in N(a)$. This proves (iib).

Now, if $\{a\}$ is not a dominating set of $N(x) \setminus \{y\}$, then there exists $d \in N(x) \setminus \{y\}$ such that $d \notin N(a)$ for some $a \in N(x) \setminus \{y\}$. Thus, $\gamma(\langle(N(x) \setminus \{y\}) \setminus \{d : d \notin N(a)$ for some $a, d \in N(x)\} \setminus \{y\}\rangle) = 1$. Since D is dominating, $d \in N(b)$. If $\{b\}$ is a dominating set of $N(y)\setminus\{x\}$, then $\gamma(\langle N(y)\setminus\{x\}\rangle) = 1$. This proves $(iiia)$. If $\{b\}$ is not a dominating set of $N(y) \setminus \{x\}$, then $\gamma(\langle(N(y) \setminus \{x\}) \setminus \{c : c \notin N(b)$ for some $b, c \in N(y) \setminus \{x\}\}\rangle) = 1$ where $c \in N(a)$ by similar arguments used in $(iiia)$. This proves $(iiib)$.

Subcase 2. Suppose that $xy \notin E(G)$.

Consider $ab \in E(G)$. Let $D_a = \{a\}$ be a dominating set of $\langle N(x)\rangle$. Then $\gamma(\langle N(x)\rangle) = 1$. If $D_b = \{b\}$ is a dominating set of $\langle N(y)\rangle$, then $\gamma(\langle N(y)\rangle) = 1$. This proves (iva). Suppose that D_b is not a dominating set of $\langle N(y)\rangle$. Then there exists $c \in N(y)$ such that $c \notin N(b)$. Since $D = \{a, b\}$ is a dominating set of G, it follows that $c \in N(a)$. Thus, D_b is a dominating set of $\langle N(y) \setminus (N(a) \setminus \{b\})\rangle$, that is, $\gamma(\langle N(y) \setminus (N(a) \setminus \{b\})\rangle) = 1$ for some $a \in N(x)$ with $ab \in E(G)$. This proves (ivb).

Similarly, if D_a is not a dominating set in $\langle N(x)\rangle$, then $\gamma(\langle N(x) \setminus (N(b) \setminus \{a\})\rangle) = 1$ for some vertex $b \in N(y)$ with $ab \in E(G)$. If D_b is a dominating set of $\langle N(y)\rangle$, then $\gamma(\langle N(y)\rangle) = 1$, proving (va). If D_b is not a dominating set of $\langle N(y)\rangle$, then $\gamma(\langle N(y) \setminus (N(a) \setminus \{b\})\rangle) = 1$ for some $a \in N(x)$ with $ab \in E(G)$. This proves (vb).

Consider $ab \notin E(G)$. Suppose that $D_a = \{a\}$ is not a dominating set of $\langle N(x)\rangle$. Then there exists $c \in N(x)$ such that $c \notin N(a)$. Since $D = \{a, b\}$ is a dominating set of G, $c \in N(b)$. Thus $\gamma(\langle N(x) \setminus N(b)\rangle) = 1$. If $D_b = \{b\}$ is not a dominating set of $\langle N(y)\rangle$, then there exists $d \in N(y)$ such that $d \notin N(b)$. Since $D = \{a, b\}$ is a dominating set of G, $d \in N(a)$. Thus $\gamma(\langle N(y) \setminus N(a)\rangle) = 1$. This shows (vi). If $D_b = \{b\}$ is a dominating set of $\langle N(y)\rangle$, then $\gamma(N(y)) = 1$. This proves $(viia)$. Now, suppose that $D_a = \{a\}$ is a dominating set of $\langle N(x)\rangle$. Then $\gamma(\langle N(x)\rangle) = 1$. If $D_b = \{b\}$ is not a dominating set of $\langle N(y)\rangle$, $\gamma(\langle N(y) \setminus N(a)\rangle) = 1$ by similar arguments used above. This proves $(viib)$.

For the converse, suppose that $G \neq K_2 + H$ for any subgraph H and there exist distinct vertices x and y that dominate G such that $\langle V(G) \setminus \{x, y\}\rangle$ has no isolated vertices and satisfies (i), (ii), (iii), (iv), (v), (vi) or (vii).

Suppose first that (i) holds. Let $D = \{a\}$ be a γ_r-*set* and $S = \{x, y\}$ be a restrained dominating set of G. Since $a \in V(G) \setminus S$, it follows that $S \subseteq (V(G) \setminus D)$. Thus, S is an inverse restrained dominating set of G, that is, $\gamma_r^{-1}(G) \leq |S| = 2$. Suppose that $\gamma_r^{-1}(G) = 1$. Then there exist a vertex in S, say x, such that x dominate G. Since $x \neq a$, it follows that $\{x\}$ and $\{a\}$ are dominating sets of G. This implies that $G = K_2 + H$ for some subgraph H contrary to our assumption. Thus, $\gamma_r^{-1}(G) = 2$.

Next, suppose that (iia) holds. Then $xy \in E(G)$. Let $S = \{x, y\}$ and let $D_a = \{a\}$ be a dominating set of $\langle N(x) \setminus \{y\}\rangle$ and $D_b = \{b\}$ be a dominating set of $\langle N(y) \setminus \{x\}\rangle$. Then, $N[a] = N[x] \setminus \{y\}$ and $N[b] = N[y] \setminus \{x\}$. Thus,

$$\begin{aligned} N[a] \cup N[b] &= (N[x] \setminus \{y\}) \cup (N[y] \setminus \{x\}) \\ &= N[x] \cup N[y] \\ &= V(G). \end{aligned}$$

This implies that $D = \{a, b\}$ is a dominating set of G. Now, let $u, v \in V(G) \setminus D$. If $u = x$ and $v = y$, then $uv \in E(G)$. Suppose that $u = x$ and $v \neq y$. If $v \in N(x) \setminus \{y\}$, then $xv = uv \in E(G)$. If $v \in N(y) \setminus \{x\}$, then $vy, uy \in E(G)$. This implies that u-v is a path in G. Similarly, if $u \neq x$ and $v = y$, then u-v is path in G. Moreover, suppose that $u \neq x$ and $v \neq y$. If $u \in N(x) \setminus \{y\}$ and $v \in N(y) \setminus \{x\}$, then $ux, xy, yv \in E(G)$. Thus, u-v is a path in G. If $u, v \in N(x) \setminus \{y\}$ or $u, v \in N(y) \setminus \{x\}$, then it can be shown u-v is a path in G. In any case, $\langle V(G) \setminus D\rangle$ has no isolated vertices. This implies that D is a restrained dominating set in G. Thus, $\gamma_r(G) \leq |D| = 2$. Suppose that $\gamma_r(G) = 1$. Let $D_a = \{a\}$ be a γ_r-*set* of G. Then, $\gamma_r^{-1}(G) = 2$ by following similar arguments used in (i). Suppose that $\gamma_r(G) = 2$ and let $D = \{a, b\}$ be a γ_r-*set* of G. By hypothesis, $S = \{x, y\}$ is a restrained dominating set of G, and by (iia), $S \cap D = \varnothing$. This implies that $S \subseteq (V(G) \setminus D)$, that is S is an inverse restrained dominating set of G with respect to D. Since $\gamma_r(G) = 2$, it follows that $S = \{x, y\}$ is the minimum inverse restrained dominating set of G with respect to D by Remark 2.3. Hence, $\gamma_r^{-1}(G) = 2$.

Suppose that (iib) holds. Then $xy \in E(G)$. Let $D_a = \{a\}$ be a dominating set of $\langle N(x) \setminus \{y\}\rangle$ and $D_b = \{b\}$ be a dominating set of $\langle (N(y) \setminus \{x\}) \setminus C\rangle$ where $C = \{c : c \notin N(b)$ for some $b, c \in N(y) \setminus \{x\}\}$ and $c \in N(a)$. Then, $N[a] = (N[x] \setminus \{y\}) \cup C$

and $N[b] = (N[y] \setminus \{x\}) \setminus C$. Thus,

$$\begin{aligned} N[a] \cup N[b] &= (N[x] \setminus \{y\}) \cup C \cup (N[y] \setminus \{x\}) \setminus C \\ &= N[x] \cup N[y] \\ &= V(G). \end{aligned}$$

This implies that $D = \{a, b\}$ is a dominating set of G. Following similar arguments in (iia), $\gamma_r^{-1}(G) = 2$.

Suppose that (iii) holds. By using similar arguments in (ii), it can be shown that $\gamma_r^{-1}(G) = 2$. Finally, if any of the conditions (iv) or (v) or (vi) or (vii) holds, then it is clear that $\gamma_r^{-1}(G) = 2$. ■

The following result is a direct consequence of Theorem 2.10.

Corollary 2.11. Let $G = K_2$ and H be connected graphs of order $m \geq 2$. Then $\gamma_r^{-1}(G \circ H) = 2$ if and only if $\gamma(H) = 1$.

Corollary 2.11, can be generally stated by the following result.

Theorem 2.12. Let G and H be connected graphs of orders n and $m \geq 2$ respectively. Then $\gamma_r^{-1}(G \circ H) = n$ if and only if $\gamma(H) = 1$.

Proof. Suppose that $\gamma_r^{-1}(G \circ H) = n$. Let $S = V(G)$ be a γ_r^{-1}-*set* of $G \circ H$. Then $S \subseteq V(G \circ H) \setminus D$ where D is a γ_r-*set* of $G \circ H$. In view of Remark 2.3, $|D| \leq |S|$. If $|D| < |S|$, then D is not a dominating set of $G \circ H$ since there exists $v \in V(G)$ such that H^v is not dominated by element of D. Thus, $|D| = |S|$, that is, $D = \bigcup_{i=1}^{n}\{x_i : x_i \in V(H^{v_i}), v_i \in V(G)\}$. This implies that x_i dominate $V(H^{v_i})$ for each $v_i \in V(G)$ where $i = 1, 2, \ldots, n$. Hence $\gamma(H^{v_i}) = 1$ for each $v_i \in V(G)$ $(i = 1, 2, \ldots, n)$, that is, $\gamma(H) = 1$.

For the converse, suppose that $\gamma(H) = 1$. Let $x \in V(H^v)$ dominate H^v for each $v \in V(G)$. Then $\{x\} \subset V(v + H^v)$ is a minimum dominating set of $v + H^v$ for each $v \in V(G)$. This implies that $D = \bigcup_{i=1}^{n}\{x_i : x_i \in V(H^{v_i}), v_i \in V(G)\}$ is a minimum dominating set of $V(G \circ H)$. Let $v \in V(G)$. Since the order of H is $m \geq 2$, for each $u \in V(H^v) \setminus \{x\}$ where $x \in D$, $uv \in E(v + H^v)$. Thus, $\langle V(v + H^v) \setminus \{x\}\rangle$ has no isolated vertices for each $v \in V(G)$. This implies that $\langle V(G \circ H) \setminus D\rangle$ has no isolated vertices. Hence D is a γ_r-*set* of $G \circ H$. Since $V(G) \subseteq (V(G \circ H) \setminus D)$, it follows that $V(G)$ is an inverse dominating set of $V(G \circ H)$. Since $\langle V(G \circ H) \setminus V(G)\rangle = H$ has no isolated vertices, $V(G)$ is an inverse restrained dominating set of $G \circ H$. Since $|V(G)|$ is a minimum dominating set of $G \circ H$, it follows that $V(G)$ is a γ_r^{-1}-*set* of $G \circ H$. Hence, $\gamma_r^{-1}(G \circ H) = n$. ■

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