Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 2 (2016), pp. 1921-

http://www.ripublication.com/gjpam.htm

The K-Exponential Matrix to solve systems of differential equations deformed

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Abstract

With this work we introduce exponential matrix notions and deformed logarithms, wherof we realize a complete study of their properties with a deformation generator introduced by G. Kaniadakis 1. Furthermore, we establish deformed differential equations and some of their solution techniques, which permit to solve common physical problems which are modeled with the mentioned equations. We show an application of K-differential equations to a physical application of basic engineering demonstrating its effectiveness for engineering calculations.

AMS subject classification:
Keywords:

1. Introduction

In this article we define the k-exponential matrix and the k-logarithmic matrix of a square matrix A, which we will denote *expk (A)*, and *lnk (A)* respectively. Additionally, we will present some convergence properties and criteria for the same matrices [1], [2]. In the first part we will discuss the deformation generator according to G. Kaniadakis, some aspects of deformed k-algebra and properties of the k-exponential and k-logarithmic functions [3], [4], [5]. In the second part we will define the k-exponential and k-logarithmic matrices of an A-square matrix by using their performances in a series of convergent matrices. Finally, in the last part, we define the k-differential equations, understood as differential equations with kaniadakis deviations and k-deformation parameters. Also, we present some techniques for solving k-differential equations and k-differential equation systems, where the k-exponential matrix forms part of the solutions for some of these systems.

2. Deformed mathematics after Kaniadakis

Deformation Generator

In [1] a real g function is defined, which depends on $k \in \mathbb{R}$ parameter. A deformation generator has to show the following properties:

i) $g(x) \in C^{\infty}(\mathbb{R})$

ii) $g(-x) = -g(x)$

iii) $\frac{d}{dx}\, g(x) > 0$

iv) $g(\pm\infty) = \pm\infty$

v) $g(x) \approx x$ when $x \to 0$.

G. Kaniadiakis [1] defines the real function as $x_k = \frac{1}{k} arcsenhg(kx)$ and its inverse $x^k = \frac{1}{k} g^{-1}\left(\sinh(kx)\right)$ which complies with the following:

i) $x_k \in C^{\infty}(\mathbb{R})$

ii) $(-x)_k = -x_k$

iii) $\frac{d}{dx} x_k > 0$

iv) $(\pm\infty)_k = \pm\infty$

v) $x_{-k} = x_k$

vi) $x_k \approx x$ when $k \to 0 (x_0 = x)$

vii) $x_k \approx x$ when $k \to 0 (0_k = 0)$.

Deformed K-Algebra

In [15] the construction of a deformed algebra and calculus to solve problems outlined thermostatic not extended. Similarly, in [1] the k-sum is defined which generalizes the sum of real numbers by means of

$$x \overset{k}{\oplus} y = (x_k + y_k)^k = \frac{1}{k} g^{-1}[senh[arcsenh(g(kx)) + arcsenh(g(ky))] \tag{2.1}$$

The following formulas are $k-$ sum properties as demonstrated by [1] and [3].

i) $(x \overset{k}{\oplus} y) = x_k + y_k$

ii) $x \overset{0}{\oplus} y = x + y$

iii) $x \overset{k}{\oplus} y = y \overset{k}{\oplus} x$

iv) $(x \overset{k}{\oplus} y) \overset{k}{\oplus} z = x \overset{k}{\oplus} (y \overset{k}{\oplus} z)$

v) $x \overset{k}{\oplus} 0 = x$

vi) $x \overset{k}{\oplus} (-k) = (-x) \overset{k}{\oplus} x = 0$

Remark 2.1. The $x \overset{k}{\oplus} (-y)$ is written as $x \overset{k}{\ominus} y$ and it is called k-difference.

The $k-$product is also introduced in the following way:

$$x \overset{k}{\otimes} y = (x_k \cdot y_k)^k = \frac{1}{k} g^{-1}(senh[arcsenh(g(kx)) \cdot arcsenh(g(ky))]) \tag{2.2}$$

The following properties relate k-sum and k-product, [1]:

i) $x \overset{0}{\otimes} y = xy$

ii) $(x \overset{k}{\otimes} y) \overset{k}{\otimes} z = x \overset{k}{\otimes} (y \overset{k}{\otimes} z)$

iii) $z \overset{k}{\otimes} (x \overset{k}{\oplus} y) = (z \overset{k}{\otimes} x) \overset{k}{\oplus} (z \overset{k}{\otimes} y)$

iv) $(\mathbb{R} - \{0\}, \overset{k}{\otimes})$ is an abelian group [4].

v) $I = 1^k$ is such as $x \overset{k}{\otimes} I = x$.

vi) $\overline{x} = \left(\frac{1}{x_k}\right)^k$ appears like in $x \overset{k}{\otimes} \overline{x} = \overline{x} \overset{k}{\otimes} x = I$

vii) $x \overset{k}{\oslash} y = x \overset{k}{\otimes} \overline{y} = x \overset{k}{\otimes} \left(\frac{1}{y_k}\right)^k$.

In [6] the exponential deformed k-function or k-exponential is defined as

$$exp_k(x) = exp\left(\frac{1}{k}arcsenh(kx)\right) = (\sqrt{1+k^2x^2}+kx)^{1/k} \tag{2.3}$$

with $0 < k < 1$ [13].

The $exp_k(x)$ function consists of real numbers, is part of the $C^\infty(\mathbb{R})$ classification, is strictly growing and obeys the exponential rules when $k \to 0$. Furthermore, the k-derivation and the usual derivation show the following properties:

i) $\lim_{x\to-\infty}(exp_k(x)) = 0$

ii) $\lim_{x\to\infty}(exp_k(x)) = \infty$

iii) $\frac{d}{dx}(exp_x(x)) = \frac{1}{\sqrt{1+k^2x^2}}\,\mathrm{exp}_k(x)$ [14],

iv) $\int exp_k(x)dx = \left(\frac{k^2x-\sqrt{1+x^2k^2}}{k^2-1}\right)exp_k(x) + C$

v) $exp_{-k}(x) = exp_k(x)$

vi) $exp_k(x)exp_k(-x) = 1$

vii) $(exp_k(x))^r = exp_{k/r}(rx),\ (r \neq 0)$

viii) $exp_k(0) = 1$

ix) $exp_k(x)exp_k(y) = exp_k(x \overset{k}{\oplus} y)$

x) $exp_k(x)/exp_k(y) = exp_k(x \overset{k}{\ominus} y)$.

The inverse k-exponential function equals the k-logarithmic function, denoted as $ln_k(x)$, defined in real positive numbers and for $k = 0$ it behaves as $ln_0(x) = lnx$. When $k \neq 0$ it is defined as,

$$ln_k(x) = \frac{1}{k}senh(klnx) = \frac{x^k - x^{-k}}{2k} \tag{2.4}$$

The $ln_k(x)$ function [2] belongs to the $\mathbb{C}^\infty(\mathbb{R}^+)$ class is strictly growing, concave and adheres to:

i) $\lim_{x\to0^+}(ln_k(x)) = -\infty$

ii) $\lim_{x\to\infty}(ln_k(x)) = \infty$

iii) $ln_k(1) = 0$

iv) $ln_{-k}(x) = ln_k(x)$

v) $ln_k(\frac{1}{x}) = -ln_k(x)$ [14],

vi) $ln_k(x^r) = r ln_{rk}(x)$; with $r \in \mathbb{R}$

vii) $ln_k(xy) = ln_k(x) \overset{k}{\oplus} ln_k(y)$

viii) $ln_k(x/y) = ln_k(x) \overset{k}{\ominus} ln_k(y)$

ix) $\frac{d}{dt} ln_k(tb) = \frac{1}{t}\left(\frac{ln_{2k}(tb)}{ln_k(tb)}\right)$

x) $\int ln_k(tb)dt = \frac{t}{1-k^2}\left(ln_k(tb) - \frac{ln_{2k}(tb)}{ln_k(tb)}\right).$

Returning to the k-exponential [7], one can use the Taylor expansion series for $exp(\cdot)$ and proceed with $x_0 = 0$, which is given by:

$$exp_k(x) = \sum_{n=0}^{\infty} a_n(k)\frac{x^n}{n!}, \quad k^2x^2 < 1 \tag{2.5}$$

where the coefficients a_n are defined by $a_0(k) = a_1(k) = 1$ and

$$a_{2m}(k) = \prod_{j=0}^{m-1}[1-(2j)^2k^2]; \quad a_{2m+1}(k) = \prod_{j=1}^{m}[1-(2j-1)^2k^2] \tag{2.6}$$

It is important to note that $a_n(0) = 1$ and $a_n(-k) = a_n(k)$. According to [5], the Taylor expansion of $ln_k(1+x)$ [7] converges if $-1 < x \leq 1$ and if it has the following form:

$$ln_k(1+x) = \sum_{n=1}^{\infty} b_n(k)(-1)^{n-1}\frac{x^n}{n} \tag{2.7}$$

with $b_n(k) = 1$, $b_n(0) = 1$ $b_n(-k) = b_n(k)$. for $n > 1$

$$b_n(k) = \frac{1}{2}(1-k)\left(1-\frac{k}{2}\right)\cdots\left(1-\frac{k}{n-1}\right) + \frac{1}{2}(1+k)\left(1+\frac{k}{2}\right)\cdots\left(1+\frac{k}{n-1}\right) \tag{2.8}$$

3. K-Exponential matrix

In this part we explain how the definition of the k-exponential matrix behaves when joined with an A-square matrix, denoted as $exp_k(A)$, as an extension of the execution of a series of k-exponential matrix, in such a way that when $k \longrightarrow 0$ then the usual exponential function of a matrix is obtained [10]. Moreover, some important properties of an k-exponential matrix are verified.

Theorem 3.1. The matrix series $\sum_{p=0}^{\infty} \frac{a_p(k)}{p!} A^p$ converges if $\|A\| < \frac{1}{|k|}$ with $a_p(K)$ as in (2.6).

Proof. Note that $\sum_{p=0}^{\infty} \left\| \frac{a_p(k)}{p!} A^p \right\| = \sum_{p=0}^{\infty} \left| \frac{a_p(k)}{p!} \right| \|A^p\| \le \sum_{p=0}^{\infty} \left| \frac{a_p(k)}{p!} \right| \|A\|^p.$

Effectivley,

$$\sum_{p=0}^{\infty} \left| \frac{a_p(k)}{p!} \right| \|A\|^p = \sum_{m=0}^{\infty} \left| \frac{a_{2m}(k)}{2m!} \right| \|A\|^{2m} + \sum_{n=1}^{\infty} \left| \frac{a_{2n+1}(k)}{(2n+1)!} \right| \|A\|^{2n+1}$$

$$\left| \frac{b_{m+1}}{b_m} \right| = \left| \frac{\frac{a_{2m+2}(k)\|A\|^{2m}\|A\|^2}{(2m+2)(2m+1)(2m)!}}{\frac{a_{2m}(k)\|A\|^{2m}}{(2m)!}} \right| = \left| \frac{a_{2m+2}(k)\|A\|^2}{(2m+1)(2m+2)a_{2m}(k)} \right|$$

$$= \left| \frac{[1-(2m)^2k^2]\|A\|^2}{(2m+2)(2m+1)} \right|$$

Then,

$$\begin{aligned} \lim_{m\to\infty} \left| \frac{b_{m+1}}{b_m} \right| &= \lim_{m\to\infty} \left(\frac{[1-(2m)^2k^2]}{(2m+2)(2m+1)} \|A\|^2 \right) \\ &= \lim_{m\to\infty} \left(\frac{\left[\frac{1}{(2m)^2} - k^2\right]}{\left(1+\frac{2}{2m}\right)\left(1+\frac{1}{2m}\right)} \|A\|^2 \right) \\ &= |-k^2|\|A\|^2 \end{aligned}$$

Therefore, the series is convergent, when $k^2\|A\|^2 < 1$, which is the same as $\|A\| < \frac{1}{|k|}$.

Similarly, it shows that the series $\sum_{n=1}^{\infty} \left| \frac{a_{2n+1}(k)}{(2n+1)!} \right| \|A\|^{2n+1}$ is convergent [11]. ∎

Definition 3.2. [k-Exponential of a matrix] Given a matrix $A \in \mathcal{C}^{n\times n}$, with $\|A\| < \frac{1}{|k|}$, we will define the k-exponential $exp_k(A)$ as the matrix $n \times n$ given by the convergent series:

$$exp_k(A) := \sum_{p=0}^{\infty} \frac{a_p(k)}{p!} A^p \text{ with } a_p(k) \text{ as in (2.6)} \tag{3.9}$$

Note that A is a nilpotent matrix, therefore, $exp_k(A)$ is a finite series and, consequently, convergent.

Proposition 3.3. If $O = [0] \in \mathbb{C}^{n\times n}$ is the zero matrix, than $exp_k(O) = I_n$.

Proof.

$$exp_k(O) = I_n + \frac{O}{1!} + \frac{a_2(k)}{2!}O^2 + \frac{a_3(k)}{3!}O^3 + \cdots = I_n \text{ where } k^2\|O\| = 0 < 1$$

■

Proposition 3.4. When $D \in \mathbb{C}^{n\times n}$ is a diagonal matrix, $D = diag\{d_1, \ldots, d_n\}$ with $|d_i| < \frac{1}{|k|}$, for $i = 1, 2, 3, \ldots, n$ so that, $exp_k(D)$ converges to $diag\{exp_k(d_1), \ldots, exp_k(d_n)\}$.

Proof.

$$\begin{aligned} exp_k(D) &= I_n + D + \frac{a_2(k)}{2!}D^2 + \frac{a_3(k)}{3!}D^3 + \cdots \\ &= \begin{pmatrix} 1 + d_1 + a_2(k)d_1^2/2! + \cdots & & 0 \\ & \ddots & \\ 0 & & 1 + d_n + a_2(k)d_n^2/2! + \cdots \end{pmatrix} \\ &= \begin{pmatrix} exp_k(d_1) & & 0 \\ & \ddots & \\ 0 & & exp_k(d_n) \end{pmatrix} = diag\,\{exp_k(d_1), \ldots, exp_k(d_n)\} \text{ with } |d_i| < \frac{1}{|k|}, \\ &for\ i = 1, 2, 3, \ldots, n \end{aligned}$$

■

Proposition 3.5.

$$\frac{d}{dt}\left[exp_k(tB)\right] = B\left(\sum_{p=0}^{\infty} \frac{a_{p+1}(k)}{p!}(tB)^p\right).$$

Proof. This time $b_{ij}^{(m)}$ shall be the element i, j of $(tB)^m$, therefore the element i, j of $\frac{(tB)^m}{m!}$ is $\frac{t^m}{m!}b_{ij}^{(m)}$, consequently,

$$\frac{d}{dt}\left[1 + tb_{ij}^{(1)} + \frac{a_2(k)}{2!}t^2 b_{ij}^{(2)} + \frac{a_3(k)}{3!}t^3 b_{ij}^{(3)} + \cdots\right] = \left[\sum_{p=0}^{\infty}\frac{a_{p+1}(k)}{p!}t^p b_{ij}^{p+1}\right] \text{ then,}$$

$$\frac{d}{dt}\left[exp_k(tB)\right] = \sum_{p=0}^{\infty}\frac{a_{p+1}(k)}{p!}t^p B^{p+1} = B\sum_{p=0}^{\infty}\frac{a_{p+1}(k)}{p!}(tB)^p$$

■

Proposition 3.6. For $D = diag\{d_1, d_2, \ldots, d_n\}$ the derivative of the k-exponential matrix $exp_k(tD)$ is given by:

$$\frac{d}{dt}\left[exp_k(tD)\right] = D\begin{pmatrix} \frac{1}{\sqrt{1+(ktd_1)^2}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{1+(ktd_n)^2}} \end{pmatrix} exp_k(tD) \qquad (3.10)$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\left(exp_k(tD)\right) &= D\left(\sum_{p=0}^{\infty}\frac{a_{p+1}(k)}{p!}(tD)^p\right) \\
&= D\begin{pmatrix} \sum_{p=0}^{\infty}\frac{a_{p+1}(k)}{p!}(td_1)^p & & 0 \\ & \ddots & \\ 0 & & \sum_{p=0}^{\infty}\frac{a_{p+1}(k)}{p!}(td_n)^p \end{pmatrix} \\
&= D\begin{pmatrix} \frac{1}{\sqrt{1+(ktd_1)^2}}exp_k(td_1) & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{1+(ktd_n)^2}}exp_k(td_n) \end{pmatrix} \\
&= D \cdot diag\left(\frac{1}{\sqrt{1+(ktd_i)^2}}\right) \cdot exp_k(tD)
\end{aligned}$$

■

When $k \to 0$, one obtains,

$$\frac{d}{dt}\left(exp(tD)\right) = D\, exp(tD).$$

Proposition 3.7. For a diagonalizable matrix $B = SDS^{-1}$ we obtain $exp_k = S\,[exp_k(D)]\,S^{-1}$.

Proof.

$$\begin{aligned} exp_k(B) &= \sum_{p=0}^{\infty} \frac{a_p(k)}{p!} B^p = \sum_{p=0}^{\infty} \frac{a_p(k)}{p!} \left(SDS^{-1}\right) \\ &= S\left(\sum_{p=0}^{\infty} \frac{a_p(k)D}{p!}\right) S^{-1} \\ &= S\, exp_k(D)\, S^{-1} \end{aligned}$$

■

And also

$$\begin{aligned} \frac{d}{dt}[exp_k(tB)] &= \frac{d}{dt}[S\,(exp_k(tD))\,S^{-1}] \\ &= SD \cdot diag\left(\frac{1}{\sqrt{1+(ktd_i)^2}}\right) \cdot exp_k(tD) \end{aligned}$$

Proposition 3.8.

$$\int exp_k(tB)dt = B^{-1} \sum_{p=1}^{\infty} \frac{a_{p-1}}{p!}(tB)^p + C$$

where B and C are matrices with the size of $n \times n$.

Proof.

$$\begin{aligned} \int exp_k(tB)dt &= \int \left(\sum_{p=0}^{\infty} \frac{a_p(k)}{p!}(tB)^p\right) dt \\ &= \int \sum_{p=0}^{\infty} \frac{a_p(k)}{p!}(t)^p B^p dt \end{aligned} \tag{3.11}$$

one element (i,j) out of (3.11) has the following form,

$$\int \left[1 + tb_{ij}^{(1)} + \frac{a_2(k)}{2!} t^2 b_{ij}^{(2)} + \frac{a_3(k)}{3!} t^3 b_{ij}^{(3)} + \cdots \right] dt$$

$$= \left[t + \frac{t^2}{2} b_{ij}^{(1)} + \frac{a_2(k)}{2!} \frac{t^3}{3} b_{ij}^{(2)} + \frac{a_3(k)}{3!} \frac{t^4}{4} b_{ij}^{(3)} + ... + C_{ij} \right] = \sum_{p=1}^{\infty} \frac{a_{p-1}\, t^p}{p!} b_{ij}^{(p-1)}$$

Consequently,

$$\int exp_k(tB)dt = \sum_{p=1}^{\infty} \frac{a_{p-1}(k) t^p}{p!} B^{p-1} + C$$

$$= B^{-1} \sum_{p=1}^{\infty} \frac{a_{p-1}}{p!} (tB)^p + C$$

■

Then, for a matrix $D = diag\{d_1, \ldots, d_n\}$ one obtains,

$$\int exp_k(tD)dt = D^{-1} \begin{pmatrix} \frac{k^2 t^2 d_1^2 - \sqrt{1 + k^2 t^2 d_1^2}}{k^2 - 1} & & 0 \\ & \ddots & \\ 0 & & \frac{k^2 t^2 d_n^2 - \sqrt{1 + k^2 t^2 d_n^2}}{k^2 - 1} \end{pmatrix} exp_k(tD) + C$$

When $k \to 0$ one obtains $\int exp(tD)dt = D^{-1} exp(tD) + C$.

4. K-Logarithmic matrix

Through the series performance of the deformed k-logarithms we define a square matrix A as the logarithmic matrix and we call it $\ln_k(A)$. Also, we define some properties of the logarithmic k-matrix which reduce the usual properties of logarithmic performance of a square matrix.

Proposition 4.1. The matrix series

$$\sum_{p=1}^{\infty} \frac{b_p(k)(-1)^{p-1}}{p} (A - I_n)^p$$

is convergent when $\|A - I_n\| < 1$. where $b_p(K)$ as in (2.8).

Proof.

$$\sum_{p=1}^{\infty}\left\|\frac{b_p(k)(-1)^{p-1}}{p}(A-I_n)^n\right\| \leq \sum_{p=1}^{\infty}\frac{b_p(k)}{p}\|A-I_n\|^p$$

Given

$$a_p = \frac{b_p(k)}{p}\|A-I_n\|^p,$$

then:

$$\begin{aligned}\left|\frac{a_{p+1}}{a_p}\right| &= \left|\frac{pb_{p+1}(k)\|A-I_n\|^{p+1}}{(p+1)b_p(k)\|A-I_n\|^p}\right| \\ &= \left|\frac{pb_{p+1}(k)}{(p+1)b_p(k)}\right|\|A-I_n\|\end{aligned}$$

When $p \longrightarrow \infty$, $\left(1 \pm \dfrac{k}{p}\right) \longrightarrow 1$, if $b_{p+1}(k) = b^-_{p+1}(k) + b^+_{p+1}(k)$, with,

$$b^-_{p+1}(k) = \frac{1}{2}(1-k)\cdots\left(1-\frac{k}{p}\right)$$

and

$$b^+_{p+1}(k) = \frac{1}{2}(1+k)\cdots\left(1+\frac{k}{p}\right).$$

Furthermore,

$$\begin{aligned}&\lim_{p\to\infty}\left(\frac{b^-_{p+1}}{b^-_p}\right) = \lim_{p\to\infty}(1-k/p) = 1 \\ &y \lim_{p\to\infty}\left(\frac{b^+_{p+1}}{b^+_p}\right) \\ &= \lim_{p\to\infty}(1+k/p) = 1,\end{aligned}$$

therefore

$$\begin{aligned}&\lim_{p\to\infty}\left(\frac{b^-_{p+1}+b^+_{p+1}}{b^-_p+b^+_p}\right) \\ &= \frac{b^-_p+b^+_p}{b^-_p+b^+_p} = 1,\end{aligned}$$

then

$$\lim_{p\to\infty}\left|\frac{a_{p+1}}{a_p}\right| \\ = \|A - I_n\|$$

Consequently, the series

$$\sum_{p=1}^{\infty}\frac{b_p(k)}{p}(A - I_n)^p$$

converges when $\|A - I_n\| < 1$. ■

Definition 4.2. [k-logarithms of a matrix] Given a matrix $A \in M_{n\times n}(\mathbb{C})$, with $\|A - I_n\| < 1$. We define the k-logarithms of a $ln_k(A)$ matrix as the $n \times n$ matrix given by the convergent series:

$$ln_k(A) := \sum_{p=1}^{\infty}\frac{b_p(k)(-1)^{p-1}}{p}(A - I_n)^p, \quad \text{for } n - 1 < \|A\| < n + 1$$

Proposition 4.3. Proves that $ln_k(I_n) = 0$, where 0 is the zero matrix with the size of $n \times n$.

Proof.

$$\begin{aligned} ln_k(I_n) &= \sum_{p=1}^{\infty}\frac{b_p(k)(-1)^{p-1}}{p}(I_n - I_n)^p \\ &= \sum_{p=1}^{\infty}\frac{b_p(k)(-1)^{p-1}}{p}(0)^p = 0 \end{aligned}$$

■

Proposition 4.4. When D is a diagonal matrix $D = diag\{d_1, d2, \ldots, d_n\}$ then $ln_k(D) = diag\{ln_k(d_1), \ldots, ln_k(d_n)\}$.

Proof.

$$ln_k(D) = \sum_{p=1}^{\infty}\frac{b_n(k)(-1)^{p-1}}{p}A^p$$

with

$$A^p = \begin{pmatrix} (d_1 - 1)^p & & 0 \\ & \ddots & \\ 0 & & (d_n - 1)^p \end{pmatrix}$$

one element (i, j) of

$$\sum_{p=1}^{\infty} \frac{b_n(k)(-1)^{p-1}}{p} A^p$$

has the form

$$\sum_{p=1}^{\infty} \frac{b_p(k)(-1)^{p-1}}{p} (d_i - 1)^p \text{ for } i = j$$

and "0" for $i \neq j$.

Then

$$\sum_{p=1}^{\infty} \frac{b_p(k)(-1)^{p-1}}{p} (d_i - 1)^p = ln_k(1 + (d_i - 1)) = ln_k(d_i).$$

■

Proposition 4.5. For a diagonal matrix $D = diag(d_1, \ldots, d_n)$ one obtains $ln(exp_k(tD)) = tD$.

Proof.

$$\begin{aligned} ln_k(exp_k(tD)) &= ln_k(diag(exp_k(td_1), \ldots, exp_k(td_n))) \\ &= diag(ln_k(exp_k(td_1)), \ldots, ln_k(exp_k(td_n))) = tD. \end{aligned}$$

■

Proposition 4.6. For a diagonizable matrix $A = RDR^{-1}$ with $D = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we conclude that $ln_k(A)$ is convergent when $0 < \lambda_i \leq 2$, where λ_i are values belonging to A.

Proof.

$$\begin{aligned} A - I_n = RDR^{-1} - I_n &= RDR^{-1} - RR^{-1} = R(D - I_n)R^{-1} \\ &= R\, diag\{\lambda_1 - 1, \ldots, \lambda_n - 1\}R^{-1} \end{aligned}$$

Then

$$\begin{aligned} ln_k(A) &= \sum_{p=1}^{\infty} \frac{b_p(k)(-1)^{p-1}}{p} [R(D - I_n)R^{-1}]^p \\ &= R \left(\sum_{p=1}^{\infty} \frac{b_p(k)(-1)^{p-1}}{p} (D - I_n)^p R^{-1} \right) \end{aligned}$$

The element (i, i) of

$$\sum_{p=1}^{\infty} \frac{b_p(k)(-1)^{p-1}}{p} (D - I_n)^p$$

is

$$\sum_{p=1}^{\infty} \frac{b_p(k)(-1)^{p-1}}{p}(\lambda_i - 1)^p = ln_k(\lambda_i) \tag{4.12}$$

Then

$$ln_k(A) = R \begin{pmatrix} ln_k(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & ln_k(\lambda_n) \end{pmatrix} R^{-1}.$$

It is converged when $0 < \lambda_i \leq 2$, through (2.7). ■

Proposition 4.7.

$$\frac{d}{dt} \ln_k(tD) = \frac{1}{t} \ln_k(lD) \left(\ln_k(tD)\right)^{-1}.$$

Proof.

$$\begin{aligned} \frac{d}{dt} ln_k(tD) &= \frac{d}{dt}\left[\sum_{p=1}^{\infty} \frac{b_p(k)(-1)^{p-1}}{p}(tD - I_n)^p\right] \\ &= \begin{pmatrix} \frac{d}{dt} ln_k(t\lambda_1) & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dt} ln_k(t\lambda_n) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{t} ln_{2k}(t\lambda_1)(ln_k(t\lambda_1))^{-1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{t} ln_{2k}(t\lambda_n)(ln_k(t\lambda_n))^{-1} \end{pmatrix} \\ &= \frac{1}{t} ln_{2k}(tD)(ln_k(tD))^{-1}. \end{aligned}$$

■

Remark 4.8. For a matrix $B = RDR^{-1}$ applies

$$\frac{d}{dt} ln_k(tB) = R \frac{1}{t} ln_{2k}(tD) \left(ln_k(tD)\right)^{-1} R^{-1}.$$

It is important to note that when $k \to 0$, we obtain

$$\frac{d}{dt} ln(tB) = \frac{1}{t} R\ ln(tD)(ln(tD))^{-1} R^{-1} = \frac{1}{t} I_n.$$

Proposition 4.9. To obtain a diagonalizable matrix $B = RDR^{-1} = R\ diag\{\lambda_1, \dots, \lambda_n\}R^{-1}$, we have to

$$\int ln_k(tB)dt = R\left(\frac{t}{1-k^2}ln_k(tD) - \frac{1}{1-k^2}ln_{2k}(tD)(ln_k(tD))^{-1} + C\right)R^{-1}$$

Proof.

$$\begin{aligned}\int ln_k(tB)dt &= R\left(\int ln_k(tD)dt\right)R^{-1}\\ &= Rdiag\left(\frac{1}{1-k^2}(ln_k(t\lambda_i) - ln_{2k}(t\lambda_i)(ln_k(t\lambda_i))^{-1}) + C_i\right)R^{-1}\\ &= R\left(\frac{t}{1-k^2}ln_k(tD) - \frac{1}{1-k^2}ln_{2k}(tD)(ln_k(tD))^{-1} + C\right)R^{-1}\end{aligned}$$

Then $k \to 0$

$$\begin{aligned}\int ln(tB)dt &= R\left(tln(tD) - tI_n + C\right)R^{-1}\\ &= tRln(tD)R^{-1} - tI_n + C_R\end{aligned}$$

when $C_R = RCR^{-1}$. ∎

5. K-Differential equations

We consider two algebraic structures $(X, \overset{k}{\oplus}, \cdot)$ and $(Y, +, \cdot)$ and the complex of the F functions: $\mathbf{F} : \{f : X \to Y\}$ with $f \subseteq C^{\infty}(\mathbb{R})$. Kanadianis [8] defines the differential $d_k x$ as: $d_k x = \lim_{z\to x} x \overset{k}{\ominus} z$ where $d_k x = dx_k$. Moreover, he defines the k-deviation of the f-function as:

$$\begin{aligned}f'_k = \frac{df(x)}{d_k x} &= \lim_{z\to x}\frac{f(x) - f(z)}{x \overset{k}{\ominus} z}\\ &= \frac{df(x)}{dx_k} = \frac{df(x)}{dx_k}\\ &= \left(\frac{1}{dx_k/dx}\right)\left(\frac{df(x)}{dx}\right)\\ &= \sqrt{1+k^2x^2}\frac{d}{dx}f(x)\end{aligned}$$

with $x, z \in \mathbb{R}$ y $f(x), f(z) \in \mathbb{R}$. When $k \to 0$, the k-deviation is reduced to the usual derivation.

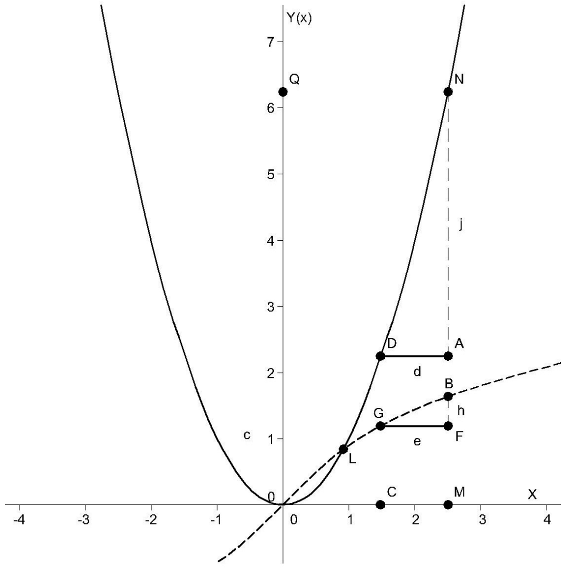

Figure 1: K-derivation concept

According to figure 1, one can observe that the $k-$derivation as defined by G Kaniadakis is interpreted as the reason for change between the image variations of a function. Recognizing the image variations of the x_k deformator we see that when x tends to a x_0

$$\frac{\overline{AN}}{\overline{BF}} = \left(\frac{\overline{AN}}{\overline{AD}}\right)\left(\frac{\overline{AD}}{\overline{BF}}\right) = \left(\frac{\Delta f/\Delta x}{\Delta x_k/\Delta x}\right) = \frac{df(x_0)/dx}{d(x_0)_k/dx}$$
$$= \left(\frac{1}{d(x_0)_k/dx}\right)\left(\frac{df(x_0)}{dx}\right) = \frac{df(x_0)}{d_k x}$$

Proposition 5.1. The k-derivation in point x_0 is the inclination of a hyperbola tangent in point x_0 given by the equation

$$y = f'_k(x_0)(x \overset{\text{k}}{\ominus} x_0) + f(x_0).$$

Proof.

$$y = f'_k(x_0)(x \overset{\text{k}}{\ominus} x_0) + f(x_0) = f'_k(x_0)(x\sqrt{1+k^2x_0^2} - x_0\sqrt{1+k^2x^2}) + f(x_0) \tag{5.13}$$

If we express (5.13) through the formula $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. We have to

$$A = (f'_k(x_0))^2;\ B = -2f'_k(x_0)\sqrt{1+k^2x_0^2};$$
$$C = 1;\ D = 2f'_k(x_0)f(x_0)\sqrt{1+k^2x_0^2};$$
$$E = -2f(x_0);\ F = (f(x_0))^2 - x_0^2(f'_k(x_0))^2.$$

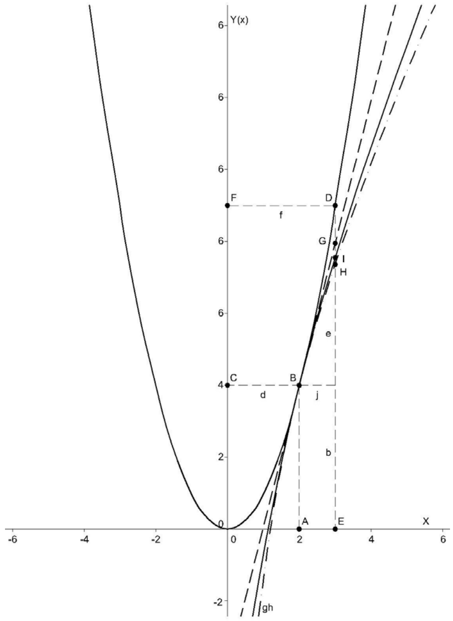

Figure 2: Hyperbola tangent of the function $f(x) = x^2$ in point $x_0 = 2$.

With

$$B^2 - 4AC = 4k^2x_0^2\left(f_k'(x_0)\right)^2 \geq 0.$$

Consequently, it is verified that $y = g(x)$ is the straight tangent to the curve $y = f(x)$ when $k = 0$ and it is the hyperbola for para $k \neq 0$. Moreover,
i.)

$$g(x_0) = f_k'(x_0)\left(x_0\sqrt{1+k^2x_0^2} - x_0\sqrt{1+k^2x^2}\right) + f(x_0) = f'(x_0)(0) + f(x_0) = f(x_0)$$

ii)

$$\begin{aligned} g'(x) &= f_k'(x_0)\left(\sqrt{1+k^2x_0^2} - \frac{x_0xk^2}{\sqrt{1+k^2x^2}}\right) \\ &= f_k'(x_0)\left(\frac{\sqrt{1+k^2x_0^2}\sqrt{1+k^2x_0^2} - x_0x_0k^2}{\sqrt{1+k^2x_0^2}}\right) \\ &= f_k'(x_0)\frac{1}{\sqrt{1+k^2x_0^2}} = \frac{1}{\sqrt{1+k^2x_0^2}} \cdot \sqrt{1+k^2x_0^2}f'(x_0) = f'(x_0). \end{aligned}$$

In figure 2 the function $f(x) = x^2$ is visualized, which is the tangent to the hyperbola $g(x)$ and the point $x_0 = 2$ for the k-values between $(-1, 1)$. ■

Proposition 5.2. When $f, g \in \mathbf{F}$, $B \in \mathbb{R}$ then apply

i) $\frac{d}{d_k x}(f+g) = \frac{d}{d_k x} f + \frac{d}{d_k x} g$

ii) $\frac{d}{d_k x}(f \cdot g) = g\left(\frac{d}{d_k x} f\right) + f\left(\frac{d}{d_k x} g\right)$

iii) $\frac{d}{d_k x}\left[f(g(x))\right] = \frac{1}{d_k x/dx}\left(\frac{d}{dx} f(g(x)) \cdot \frac{d}{dx}(g(x))\right)$

Definition 5.3. When $f \in \mathbf{F}$ where $\mathbf{F} : \{f : X \to Y\}$ with $f \subseteq C^{\infty}(\mathbb{R})$, then

$$\int_k f(x) d_k x = \int \frac{dx_k}{dx} f(x) dx = \int \frac{1}{\sqrt{1+k^2x^2}} f(x) dx,$$

[13] with $\int_0 f(x) d_k x = \int f(x) dx$.

Definition 5.4. A k-differential equation is an expression of the form $E\left(k, x, y, y_k', y_k'', \ldots, y_k^{(n)}\right) = 0$ a represents deformations of a usual differential equation, in such a way that when $k \to 0$, both, the usual differential equation and its solution, reduce.

For example in a k-differential equation $y_k'' - 2y_k' + (1-k^2)y = 0$ it can be verified that one solution is $y = xe_k^x$ and when $k \to 0$ $y_k'' - 2y_k' + (1-k^2)y = 0$ which is reduced to $y_k'' - 2y_k' + y = 0$ and $y = xe_k^x$ which is reduced to $y = xe^x$. Where $y = xe^x$ is solution of $y'' - 2y' + y = 0$.

Proposition 5.5. A separable k-differential equation has the following form $\frac{dy}{d_k x} = f(x) \cdot g(y)$ and a solution is given by

$$\int \left(\frac{1}{g(y)}\right) dy = \int \frac{1}{\sqrt{1+k^2x^2}} f(x) dx.$$

Remark 5.6. The k-exponential is invariant to the k-derivation, which is effective for

$$\begin{aligned}
\frac{dy}{d_k x} &= y \\
\int \frac{1}{y} dy &= \int \frac{1}{\sqrt{1+k^2x^2}} dx \\
&= \int \frac{1}{\sqrt{1+k^2x^2}} dx \\
&= \frac{1}{k} \int \frac{d(kx)}{\sqrt{1+(kx)^2}} \\
&= \frac{1}{k} ln\left|sec\theta + tan\theta\right| + C
\end{aligned}$$

when $kx = tan\theta$, then

$$ln(y) = ln\left|(\sqrt{1+k^2x^2}+kx)^{1/k}\right| + C_1$$

and accordingly

$$y = e^{c_1}(\sqrt{1+k^2x^2}+kx)^{1/k} = C_2 exp_k(x)$$

when $k \to 0$ $\dfrac{dy}{dx} = y$ it results in $y = Ce^x$.

Proposition 5.7. A linear K-diferential equation with constant coefficients has the form $\dfrac{dy}{d_kx} = ay + f(x)$ and its general solution is

$$y = Cexp_{k/a}(ax) + exp_{k/a}(ax)\int_k f(s)\left(exp_{-k/a}(-as)\right)d_ks.$$

Proof. For $f(x) = 0$ we obtain $\dfrac{dy}{d_kx} = ay$ with a general solution $y = C_2exp_{k/a}(ax)$. When $C_2 = v(x)$ then $y = v(x)\ exp_{k/a}(ax)$ then

$$\frac{dy}{d_kx} = \left(\frac{dv(x)}{d_kx}\right)\left(exp_{k/a}(ax)\right) + v(x)\frac{d}{d_kx}\exp_{k/a}(ax).$$

but

$$\frac{d}{d_kx}(exp_{k/a}(ax)) = \sqrt{1+k^2x^2}\frac{d}{dx}(exp_k(x))^a = a(exp_{k/a}(ax)).$$

Therefore,

$$\frac{dy}{d_k(x)} = \sqrt{1+k^2x^2}v'(x)\left(exp_{k/a}(ax)\right) + av(x)\left(exp_{k/a}(ax)\right) \qquad (5.14)$$

Replacing (5.14) in the linear k-diferential equation with constant coeficients we obtain,

$$\sqrt{1+k^2x^2}v'(x)exp_{k/a}(ax) + av(x)(exp_{k/a}(ax)) = av(x)exp_{k/a}(ax) + f(x),$$

wherefrom

$$v(x) = \int_k f(x)\left(exp_{-k/a}(-ax)\right)d_kx.$$

Consequently, the general solution of $\dfrac{dy}{d_kx} = ay + f(x)$ is

$$y = Cexp_{k/a}(ax) + exp_{k/a}(ax)\int_k f(s)\left(exp_{-k/a}(-as)\right)d_ks \qquad (5.15)$$

It is observed that when $k \to 0$, the k-differential equation is reduced to $\dfrac{dy}{dx} = ay + f(x)$ and its respective solution is $y = Cexp(ax) + exp(ax)\displaystyle\int f(y)(exp(-ay))dy.$ ■

6. K-Differential equation systems

Definition 6.1. Given $A \in \mathcal{C}^{n\times n}$ as a constant matrix e Y as a matrix function with in order to $n \times n$ the k-differential matrix equation $\dfrac{dY}{d_k t} = AY$ denominates a homogeneous linear with constant coefficients.

Proposition 6.2. When given $D = diag\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, the matrix function is

$$Y = \begin{pmatrix} C_1 exp_{k/\lambda_1}(\lambda_1 t) & & 0 \\ & \ddots & \\ 0 & & C_n exp_{k/\lambda_n}(\lambda_n t) \end{pmatrix}$$

which is the solution of the k-differential system $\dfrac{dY}{d_k t} = DY$.

Proof.

$$\begin{aligned} \frac{dY}{d_k t} &= \frac{d}{d_k t}\begin{pmatrix} C_1 exp_{k/\lambda_1}(\lambda_1 t) & & 0 \\ & \ddots & \\ 0 & & C_n exp_{k/\lambda_n}(\lambda_n t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{d}{d_k t} C_1 exp_{k/\lambda_1}(\lambda_1 t) & & 0 \\ & \ddots & \\ 0 & & \frac{d}{d_k t} C_n exp_{k/\lambda_n}(\lambda_n t) \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} C_1 exp_{k/\lambda_1}(\lambda_1 t) & & 0 \\ & \ddots & \\ 0 & & C_n exp_{k/\lambda_n}(\lambda_n t) \end{pmatrix} = DY \end{aligned}$$

∎

Proposition 6.3.

$$\frac{dY}{d_k t} = \frac{d}{d_k t}\begin{pmatrix} Y_{11} & \cdots & Y_{1n} \\ \vdots & \ddots & \vdots \\ Y_{n1} & \cdots & Y_{nn} \end{pmatrix} = \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix} \begin{pmatrix} Y_{11} & \cdots & Y_{1n} \\ \vdots & \ddots & \vdots \\ Y_{n1} & \cdots & Y_{nn} \end{pmatrix} = DY$$

obtains as solution

$$Y = \begin{pmatrix} C_{11}exp_{k/d_1}(d_1t) \ddots & \cdots & C_{1n}exp_{k/d_1}(d_1t) \\ \vdots & C_{ij}exp_{k/d_i}(d_it) & \vdots \\ C_{n1}exp_{k/d_n}(d_nt) & \cdots & \ddots C_{nn}exp_{k/d_n}(d_nt) \end{pmatrix}$$

$$= \sum_{i=1}^{n} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ C_{ii} & \cdots & C_{in} \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix} exp_{k/d_i}(d_it) \tag{6.16}$$

Remark 6.4.

i) If $D = I_n$, then $Y(t) = Cexp_k(tI_n)$.

ii) When $k \to 0$, then $\frac{dY}{dt} = DY$ obtains as solution $Y = Cexp(tD)$.

Proposition 6.5. Given $A = PDP^{-1}$ as a diagonalizable matrix, $Y_A = PY(t)P^{-1}$ is the solution of the k-differential equation $\frac{dY(t)}{d_kt} = AY(t)$.

Proof.

$$\frac{d}{d_kt}Y_A(t) = P\frac{d}{d_kt}(Y(t))P^{-1} = PDY(t)p^{-1},$$

but, $A = PDP^{-1}$ implies $AP = PD$, then $\frac{d}{d_kt}Y_A(t) = APY(t)P^{-1} = AY_A(t)$. ■

Proposition 6.6. When A is a diagonalizable matrix where $A = PDP^{-1}$ with $D = diag(\lambda_1, \cdots, \lambda_n)$ then $Y_A = PY(t)P^{-1}$ with $Y_k'(t) = DY$ is solution of $Y_k'(t) = AY$, with $Y(t)$ as in (6.16).

Proof.

$$\begin{aligned} \frac{d}{d_kt}Y_A &= \frac{d}{d_kt}(PY(t)P^{-1}) = PY_k'(t)P^{-1} = PDY(t)P^{-1} \\ &= APY(t)P^{-1} = AY_A(t) \end{aligned}$$

■

Remark 6.7.

i) When $D = I_n,\ Y_A = Pexp_k(tI_n)P^{-1} = exp_k(tA)$

ii) When $k \to 0,\ Y_A = Pexp(tD)P^{-1} = exp(tA)$ is solution of $\frac{dY(t)}{d_kt} = AY$

Proposition 6.8. The k-differential linear system $\dfrac{dY}{d_k t} = AY + F(t)$, where $F(t) = diag\,(f_i(t))$ obtains as a general solution

$$Y(t) = Y_A(t)C + Y_A(t)\int_k Y_A^{-1}(s)F(s)d_k s.$$

Remark 6.9.

i) when $D = I_n$ then $Y = e_k^{tA}C + e_k^{tA}\displaystyle\int_k e_k^{-SA}F(S)dS$

ii) when $k \to 0$; $Y = e^{tA}C + e^{tA}\displaystyle\int e^{-SA}F(S)dS$ is solution of $\dfrac{dY}{dt} = AY + F(t)$.

Proposition 6.10. The system

$$Y_k'(t) = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} Y_{11} & \cdots & Y_{1n} \\ \vdots & & \\ Y_{n1} & \cdots & Y_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}Y_{11} & a_{11}Y_{12} & \cdots & a_{11}Y_{1n} \\ a_{22}Y_{11} & a_{21}Y_{12} & \cdots & a_{21}Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}Y_{11} & a_{n1}Y_{12} & \cdots & a_{n1}Y_{nn} \end{pmatrix}$$

obtains as solution

$$Y(t) = \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \dfrac{a_{21}}{a_{11}}C_{11} & \cdots & \dfrac{a_{21}}{a_{11}}C_{1n} \\ \vdots & \ddots & \vdots \\ \dfrac{a_{n1}}{a_{11}}C_{11} & \cdots & \dfrac{a_{n1}}{a_{11}}C_{1n} \end{pmatrix} exp_{k/a_{(11)}}(a_{11}t).$$

Physical applications

A relativistic linear movement of a mass particle m_0 at rest is moving with a speed of $\overrightarrow{v}$ towards a reference system S

$$\overrightarrow{P} = \frac{m_0\overrightarrow{v}}{\sqrt{1-\frac{v^2}{c^2}}} = m\overrightarrow{v}$$

where $m = \dfrac{m_0}{\sqrt{1-v^2/c^2}}$.
a particular mass of, $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$ defines the Lorentz factor [9], [14].

The relativistic moment is

$$\overrightarrow{P} = m\overrightarrow{v}\gamma.$$

It is known that the relativistic force F exerted on a particle with the moment P is defined as:

$$\begin{aligned}\overrightarrow{F} &= \frac{d\overrightarrow{P}}{dt} = \frac{d(\gamma m_0 \overrightarrow{v})}{dt} \\ &= \frac{d}{dt}\left[\frac{m_0 v}{\sqrt{1-v^2/c^2}}\right] \\ &= \frac{d}{dt}\left[m_0 v(1-v^2/c^2)^{-1/2}\right]\end{aligned}$$

It is to be proven that the relativistic force according to Kaniadakis is:

$$\overrightarrow{F}_k = \sqrt{1+k^2t^2}\left[\frac{m_0}{(1-k^2v^2)^{3/2}}\frac{dv}{dt}\right]$$

Effectively,

$$\begin{aligned}\frac{d\overrightarrow{P}}{dt} &= \frac{d}{dt}\left[m_0 v(1-v^2/c^2)^{-1/2}\right] \\ &= m_0\frac{dv}{dt}(1-v^2/c^2)^{-1/2} + m_0 v\left(\frac{-1}{2}\right)(1-v^2/c^2)^{-3/2}\left(\frac{-2v}{c^2}\right)\frac{dv}{dt} \\ &= \frac{m_0 dv/dt}{\sqrt{1-v^2/c^2}} + \frac{m_0 dv/dt}{(1-v^2/c^2)^{3/2}}\frac{v}{c^2} \\ &= \frac{m_0\frac{dv}{dt}}{\sqrt{1-v^2/c^2}}\Bigg[1+ \\ &\quad \frac{v^2/c^2}{1-v^2/c^2}\Bigg] \quad if \;\; k=\frac{1}{c} \\ &= \frac{m_0\frac{dv}{dt}}{\sqrt{1-k^2v^2}}\left[1+\frac{k^2v^2}{1-k^2v^2}\right] \\ &= \frac{m_0\frac{dv}{dt}}{\sqrt{1-k^2v^2}}\left[\frac{1}{1-k^2v^2}\right] \\ &= \frac{m_0\frac{dv}{dt}}{(1-k^2v^2)^{3/2}}\end{aligned}$$

then

$$\begin{aligned}\overrightarrow{F}_k &= \sqrt{1+k^2t^2}\frac{d\overrightarrow{P}}{dt} \\ &= \sqrt{1+k^2t^2}\left[\frac{m_0\frac{dv}{dt}}{(1-k^2v^2)^{3/2}}\right] \\ \overrightarrow{F}_k &= \sqrt{1+k^2t^2}\left(\frac{m_0}{(1-k^2v^2)^{3/2}}\frac{dv}{dt}\right)\end{aligned}$$

like

$$\frac{P(v_1)}{m_1} \overset{k}{\ominus} \frac{P(v_2)}{m_2} = \frac{P(v_1 \overset{k}{\ominus} v_2)}{m_1}$$

moreover, the kaniadrik hyperbolization is supposed to be expressed through the following form

$$h(x) = f'_k(x_0)(x \overset{k}{\ominus} x_0) + f(x_0).$$

Resolving the k-differential through this expression, the following result is obtained:

$$x \overset{k}{\ominus} x_0 = \frac{h(x) - f(x_0)}{f'_k(x_0)}.$$

For the case of the relativistic linear moment P_1 and P_2 we obtain

$$\frac{P(v_1)}{m_1} \overset{k}{\ominus} \frac{P(v_2)}{m_2} = \frac{h(P(v_1)/m_1) - f(P_2(v_2)/m_2)}{f'_k(P_2(v_2)/m_2)} \\ = \frac{P(v_1 \overset{c}{\ominus} v_2)}{m_1}$$

for the identic function $f(x) = x$ we have to:

$$f_k(x_0) = \sqrt{1 + k^2 x_0^2} f'(x_0) = \sqrt{1 + k^2 x_0^2}.$$

Then

$$\frac{h(P(v_1)/m_1) - P_2(v_2)/m_2}{\sqrt{1 + k^2(P_2(v_2)/m_2)^2}} = \frac{h(P(v_1)/m_1) - P_2(v_2)/m_2}{\sqrt{1 + k^2(v_2/1 - k^2 v_2^2}} \\ = \frac{h(P(v_1)/m_1) - P_2(v_2)/m_2}{\sqrt{1/1 - k^2 v_2^2}} \\ = \sqrt{1 - k^2 v_2^2}\left(h\left(\frac{p(v_1)}{m_1}\right) - \frac{p(v_2)}{m_2}\right) \\ = \sqrt{1 - k^2 v_2^2}\left(h\left(\frac{p(v_1)}{m_1}\right) - \sqrt{1 - k^2 v_2^2}\frac{p(v_2)}{m_2}\right) \\ \times \sqrt{1 - k^2 v_2^2}\left(h\left(\frac{p(v_1)}{m_1}\right) - v_2 = \frac{P(v_1 \overset{c}{\ominus} v_1)}{m_1}\right)$$

The author(s) declare(s) that there is no conflict of interest regarding the publication of this article.

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