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More Exact Solutions of Hirota–Ramani Partial Differential Equations by Applying F-Expansion Method and Symbolic Computation

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Abstract

In this paper, F-expansion method and the extended version of Jacobi elliptic functions that is a straightforward, short, and powerful method is proposed to solve various polynomial nonlinear evolution equations. This method can give more nontrivial solutions of some particular higher order nonlinear partial differential equations. With the aid of symbolic computation, we select the Hirota–Ramani equation with a source to investigate the validity and advantage of the proposed method and construct a frame of soliton, trigonometric and hyperbolic solutions to the equation. Generally, we successfully seek the new exact solutions. Finally, more new trigonometric, hyperbolic, and soliton solutions are construct explicitly.

AMS subject classification:
Keywords: Soliton solutions, Periodic solutions, Hirota equations, F-expansion method, Jacobi elliptic function.

1. Introduction

In the recent decade, the nonlinear partial differential equations have been extensively used to describe physics, biology, chemistry, engineering, etc. Many methods for obtaining exact solutions of nonlinear partial differential equations have been shown and used to solved nonlinear differential equations such as inverse scattering method [2], sine-cosine method [3],[16],[18], tanh method [4], variational iteration method [6],[19], exp-function method [9], homotopy perturbation method [19], algebraic method [12], BÃ¤cklund transformation method [13], Hirota's bilinear method [13],[17], Darboux transformation [11], similarity transformation [4], [12], [13] Jacobian elliptic function expansion method [14],[15], Anstz function method [15], and F-expansion method [16], [17].

The F-expansion method was proposed to obtain more exact travelling wave solutions of nonlinear partial differential equations. Several papers use this method to solve many problems. Abdel-Razek et al. [21] have used the F-expansion method to solve for KdV and some Boussinesq equations. Ren et al. [11] introduced the F-expansion method to obtain solution for the (2+1)-dimensional breaking soliton equation. Zhang and Xia [8], [14] applied the F-expansion method to solve the solution of the (2+1)-dimensional KdV equation. Wang and Li [7] determined the F-expansion method to solve for Hamiltonian amplitude equations.

Hirota equation is a well known physic and quantum mechanics. Some methods have been used to solve exact solutions of this equation. For example, Wang et al. [5] applied projective Riccati equation method to solve this equation. Ji [12] introduced an algebraic method to obtain the soliton solutions, and periodic solutions of Hirota equation. The Hirota Ramani equation is a nonlinear third-order partial differential equation and is extensively used to describe physics, fluid physics, quantum field theory and plasma physics.

In this paper, we propose the F - expansion method with the extended version of Jacobi elliptic functions in Table 1 and apply this method to solve exact travelling wave solutions of the Hirota - Ramani equation [1], [12]. We show all possible results of this study and investigate exact solutions with distinct physical structures. We also verify All results by directly symbolic computation and graph the exact travelling wave solutions of Hirota - Ramani equation [1], [12]

$$u_t(x,t) - u_{xxt}(x,t) + Au_x(x,t)(1 - u_t(x,t)) = 0$$

where A is a real constant.

2. Introduction F - expansion method

In this section, we will introduce the detailed description of the F-expansion method and its extension to all possible cases of Jacobian elliptic functions which applies to a general non-linear partial differential equation (NPDE)

$$P(u, u_x, u_t, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \tag{1}$$

where P is a polynomial in its argument. We define the wave transformation $\zeta = k(x - \omega t)$, where $k > 0$ is the wave number and ω is the travelling wave velocity. We look for the travelling wave solutions of (1) which is written by $u(x,t) = U(\zeta)$ and transform (1) into a non-linear ordinary differential equation (ODE) as

$$O(U, kU', \omega U', kU'', k^2U'', \omega U'', \omega^2 U'', \ldots) = 0 \tag{2}$$

where $U' = dU/d\zeta$ and we assume that the solution of (2) is the form $U(\zeta) = \phi(\zeta) + d$, where d is a constant term. Substituting at (2), it transforms to the non-linear ODE as

$$O(\phi + d, k\phi', \omega\phi', k^2\phi'', \omega^2\phi'', \ldots) = 0. \tag{3}$$

The value of the constant d can be determined by setting the constant term in (3) equal to zero. By F-expansion method we assume that the solution $\phi(\zeta)$ of (3) is in the finite series as

$$\phi(\zeta) = \sum_{i=1}^{r} a_i F(\zeta)^i, \tag{4}$$

where r and $a_i (i = 1, 2, \ldots, n)$ are constants. The value of the number r is given by the equation

$$r = \frac{(s-1)q}{1-p-s}, \tag{5}$$

where the parameter q is the highest order of derivative terms in (3)

$$O\left(\frac{d^q\phi}{d\zeta^q}\right) \quad q = 1, 2, \ldots,$$

and two parameters p, s are in the highest nonlinear term in (3) as

$$O\left(\phi^p \left(\frac{d^q\phi}{d\zeta^q}\right)^s\right), \quad p = 1, 2, \ldots, \quad q = 1, 2, \ldots, \quad s = 0, 1, 2, \ldots$$

and the function $F(\zeta)$ is a general solution of the elliptic equation

$$\left(\frac{dF}{d\zeta}\right)^2 = R + QF^2 + PF^4, \tag{6}$$

where P, Q and R are constants. If the values of P, Q and R are known, the Jacobian elliptic function solution $F(\zeta)$ that depends on the values P, Q and R can be obtained from (6) which can also found in Table1 of Appendix. From Table 1, If we select $P = m^2$, $Q = -(1 + m^2)$, and $R = 1$ and then substitute into (6), so we have

$$\frac{dF}{d\zeta} = \sqrt{(1 - F^2)(1 - m^2F^2)}, \tag{7}$$

where $0 < m < 1$ is the modulus, so we have

$$\zeta = \int_0^{\phi} \frac{1}{\sqrt{1 - m^2 sin^2\theta}} d\theta. \tag{8}$$

Now the Jacobian elliptic function is defined by $sn(\zeta, m) = sin(\phi)$, $cn(\zeta, m) = cos(\phi)$, and $dn(\zeta, m) = \sqrt{1 - m^2 sin^2(\phi)}$, respectively. Table 1 gives the 46 cases of Jacobian elliptic functions. As modulus m approaches to 1, the Jacobian elliptic functions can be rewrite as the hyperbolic functions, $sn(\zeta, 1) = tanh(\zeta)$, $cn(\zeta, 1) = sech(\zeta)$, and $dn(\zeta, 1) = csch(\zeta)$, respectively. But modulus m approaches to 0, they will give the trigonometric functions as $sn(\zeta, 0) = sin(\zeta)$, $cn(\zeta, 0) = cos(\zeta)$, and $dn(\zeta, 0) = 1$, respectively.

The next step of the F-expansion method, we substitute (4) and (6) into the non-linear ODE (3) and collect the coefficients of $F(\zeta)^i$, $i = 1, 2, \ldots, r$. We can derive the set of algebraic equations of a_i, $i = 1, 2, \ldots, r$ by setting each coefficient to zero. By solving these algebraic equations by symbolic computation, we can determine those parameters explicitly. When the constants a_i, $i = 1, 2, \ldots, r$ are obtained. We then substitute these constants and the known general solutions into (4), we can obtain the explicit solution of (1).

3. Exact solutions for the Hirota - Ramani equation

We apply the F - expansion method to find the exact solutions of the Hirota–Ramani equation [1] given by

$$u_t(x, t) - u_{xxt}(x, t) + Au_x(x, t)(1 - u_t(x, t)) = 0, \tag{9}$$

where A is a real constant. Using the wave transformation $\zeta = k(x - \omega t)$, where $k > 0$, the equation (9) can be reduced the following non-linear ordinary differential equations (ODE)

$$(A - \omega)U_\zeta + k^2\omega U_{\zeta\zeta\zeta} + Ak\omega U_\zeta^2 = 0. \tag{10}$$

By setting $U_\zeta = V$, it obtains

$$(A - \omega)V + k^2\omega V'' + Ak\omega V^2 = 0. \tag{11}$$

We let $V = \phi(\zeta) + d$. Under this substitution, (11) becomes

$$(A - \omega + 2Ak\omega d)\phi + Ak\omega\phi^2 + K^2\omega\phi'' + (A - \omega + Ak\omega d)d = 0. \tag{12}$$

We first choose the constant term in the left hand side of (12) equals to zero, i.e.,

$$(A - \omega + 2Ak\omega d)d = 0. \tag{13}$$

Then, the values of d are $d = 0$ or $d = (\omega - A) / (Ak\omega)$. If $d = 0$, In this case, the ODE (12) becomes

$$(A - \omega)\phi + Ak\omega\phi^2 + K^2\omega\phi'' = 0. \tag{14}$$

We assume that the F-expansion solution $\phi(\zeta)$ of (14) is in the finite series (4) as

$$\phi(\zeta) = \sum_{i=0}^{r} a_i F(\zeta)^i.$$

Under balancing between the highest order of derivative term and the non-linear term, the highest order of derivative term is in (14) as

$$\phi'' = \frac{d^2\phi}{d\zeta^2}, \text{ and } \phi^2 = \phi^2 \left(\frac{d^2\phi}{d\zeta^2}\right)^0.$$

So the values of the parameters are $p = 2$, $q = 2$, and $s = 0$, respectively. From (14), so the value of r is $r = 2$, then we take the solution $\phi(\xi)$ of ODE as follows:

$$\phi(\zeta) = a_0 + a_1 F + a_2 F^2, \quad a_2 \neq 0 \tag{15}$$

where a_0, a_1 and a_2 are constants to be determined and the $F(\xi)$ can be solved from the Jacobian elliptic equation in (6) that have been shown in Table 1. Taking derivative with respect to ξ, we have

$$\begin{aligned} \phi'(\zeta) &= a_1 F' + 2a_2 F F' \\ \phi''(\zeta) &= 2a_2 R + a_1 Q F + 4a_2 Q F_2 + 2a_1 P F^3 + 6a_2 P F^4 \end{aligned} \tag{16}$$

Substituting (15) and (16) into (14), it obtains

$$\begin{aligned} &\left(Aa_0 - \omega a_0 + 2Rk^2\omega a_2\right) + \left(Aa_1 - \omega a_1 + 2Ak\omega a_0 a_1 + QK^2\omega a_1\right) F \\ &\quad + \left(Aa_2 - \omega a_2 + Ak\omega a_1^2 + 2Ak\omega a_0 a_2 + 4QK^2\omega a_2\right) F^2 \\ &\quad + \left(2Ak\omega a_1 a_2 + 2Pk^2\omega a_1\right) F^3 + \left(Ak\omega a_2^2 + 6Pk^2\omega a_2\right) F^4 = 0. \end{aligned}$$

Since the function $F(\zeta) \neq 0$, for all ζ, then all coefficient terms of the function F equal to zero, so we can construct the algebraic equations for finding parameters a_0,a_1,a_2 and ω as

$$\begin{aligned} Aa_0 - \omega a_0 + Ak\omega a_0^2 + 2Rk^2\omega a_2 &= 0, \\ Aa_1 - \omega a_1 + 2Ak\omega a_0 a_1 + Qk^2\omega a_1 &= 0, \\ Aa_2 - \omega a_2 + 2Ak\omega a_0 a_2 + 4Qk^2\omega a_2 &= 0, \\ 2Ak\omega a_1 a_2 + 2Pk^2\omega a_1 &= 0, \\ Ak\omega a_2^2 + 6Pk^2\omega a_2 &= 0, \end{aligned}$$

which gives the two possible results

$$\begin{aligned} a_0 &= \frac{2k}{A}\left(-Q+\sqrt{Q^2-3PR}\right), \quad a_1=0, \quad a_2=\frac{-6Pk}{A}, \quad \text{and} \\ \omega &= \frac{A}{1-4k^2\sqrt{Q^2-3PR}}. \\ a_0 &= \frac{2k}{A}\left(-Q-\sqrt{Q^2-3PR}\right), \quad a_1=0, \quad a_2=\frac{-6Pk}{A}, \quad \text{and} \\ \omega &= \frac{A}{1+4k^2\sqrt{Q^2-3PR}}. \end{aligned} \tag{17}$$

Substitute a_0, a_1 and a_2 into (15), we have

$$\begin{aligned} V_1(\zeta) &= \frac{2k}{A}\left(-Q+\sqrt{Q^2-3PR}-3PF^2\right), \\ \text{where } \zeta &= k\left(x-\frac{At}{1-4k^2\sqrt{Q^2-3PR}}\right). \\ V_2(\zeta) &= \frac{2k}{A}\left(-Q-\sqrt{Q^2-3PR}-3PF^2\right), \\ \text{where } \zeta &= k\left(x-\frac{At}{1+4k^2\sqrt{Q^2-3PR}}\right). \end{aligned}$$

Applying Jacobian elliptic function value in Table 1, in case 1, for example we select $P = m^2$, $Q = -\left(1+m^2\right)$ and $R = 1$ then the function $F(\zeta)$ obtains the Jacobi *sn*-function as

$$\begin{aligned} V_1(\zeta) &= \frac{2k}{A}\left(1+m^2+\sqrt{m^4-m^2+1}-3m^2sn^2\zeta\right) \\ \text{where } \zeta &= k\left(x-\frac{At}{1-4k^2\sqrt{m^4-m^2+1}}\right). \\ V_2(\zeta) &= \frac{2k}{A}\left(1+m^2-\sqrt{m^4-m^2+1}-3m^2sn^2\zeta\right) \\ \text{where } \zeta &= k\left(x-\frac{At}{1+4k^2\sqrt{m^4-m^2+1}}\right). \end{aligned}$$

More trigonometric and hyperbolic solution of Hirota-Ramani equation (1) can be obtained when the modulus $m \to 0$ and $m \to 1$ respectively.

4. Trigonometric exact solutions

By limiting form of Jacobi elliptic functions as $m \to 0$, then Jacobi elliptic functions degenerate into trigonometric functions as the following

$$\begin{array}{llll} sn\zeta \longrightarrow sin\zeta, & cn\zeta \longrightarrow cos\zeta, & dn\zeta \longrightarrow 1, & sc\zeta \longrightarrow cot\zeta, \\ sd\zeta \longrightarrow sin\zeta, & cd\zeta \longrightarrow cos\zeta & & \\ ns\zeta \longrightarrow csc\zeta, & nc\zeta \longrightarrow sec\zeta, & nd\zeta \longrightarrow 1, & cs\zeta \longrightarrow tan\zeta, \\ ds\zeta \longrightarrow csc\zeta, & dc\zeta \longrightarrow sec\zeta. & & \end{array}$$

Using The Jacobi elliptic functions in Table 1, so the new trigonometric solutions are given by

Case 1. Choosing $P = 1, Q = -1, R = 0,$ and $F(\zeta) = \sec(\zeta)$, we obtain the trigonometric solutions

$$\begin{aligned} V_1(\zeta) &= -\frac{6k}{A}\sec^2\zeta, \text{ where } \zeta = k\left(x - \frac{At}{1+4k^2}\right), \\ V_2(\zeta) &= \frac{2k}{A}\left(2 - 3\sec^2\zeta\right), \text{ where } \zeta = k\left(x - \frac{At}{1-4k^2}\right). \end{aligned}$$

We knew that $U_\xi = V$, Integrating $V_i(\zeta)$, $i = 1, 2$ with respect to ζ which gives the exact solutions of the Hirota-Ramani equation as the following

$$\begin{aligned} u_1(x,t) &= -\frac{6k}{A}\tan\left(kx - \frac{kAt}{1+4k^2}\right) + c, \\ u_2(x,t) &= \frac{4k}{A}\left(kx - \frac{kAt}{1-4k^2}\right) - \frac{6k}{A}\tan\left(kx - \frac{kAt}{1-4k^2}\right) + c. \end{aligned}$$

Case 2. Choosing $P = 1, Q = -1, R = 0,$ and $F(\zeta) = \csc(\zeta)$ in Table 1, so we have

$$\begin{aligned} u_3(x,t) &= \frac{6k}{A}\cot\left(kx - \frac{kAt}{1+4k^2}\right) + c, \\ u_4(x,t) &= \frac{4k}{A}\left(kx - \frac{kAt}{1-4k^2}\right) + \frac{6k}{A}\cot\left(kx - \frac{kAt}{1-4k^2}\right) + c. \end{aligned}$$

Case 3. Choosing $P = 1/4, Q = 1/2, R = 1/4,$ and $F(\zeta) = \csc(\zeta) \pm \cot(\zeta)$, the new solutions are given by

$$\begin{aligned} u_5(x,t) &= \frac{3k}{2A}\left[\csc\left(kx - \frac{kAt}{1+k^2}\right) \pm \cot\left(kx - \frac{kAt}{1+k^2}\right)\right] + c, \\ u_6(x,t) &= \frac{k}{A}\left(kx - \frac{kAt}{1-k^2}\right) + \frac{3k}{2A}\left[\csc\left(kx - \frac{kAt}{1-k^2}\right) \pm \cot\left(kx - \frac{kAt}{1-k^2}\right)\right] + c, \end{aligned}$$

Case 4. Choosing $P = 1/4$, $Q = 1/2$, $R = 1/4$, and $F(\zeta) = \sec(\zeta) \pm \tan(\zeta)$, the solutions are given by

$$\begin{aligned} u_7(x,t) &= -\frac{3k}{2A}\left[\sec\left(kx - \frac{kAt}{1+k^2}\right) \pm \tan\left(kx - \frac{kAt}{1+k^2}\right)\right] + c, \\ u_8(x,t) &= \frac{k}{A}\left(kx - \frac{kAt}{1-k^2}\right) - \frac{3k}{2A}\left[\sec\left(kx - \frac{kAt}{1-k^2}\right) \pm \tan\left(kx - \frac{kAt}{1-k^2}\right)\right] + c, \end{aligned}$$

Case 5. Choosing $P = 1/4$, $Q = 1/2$, $R = 1/4$, and $F(\zeta) = \sin(\zeta)/(1 \pm \cos(\zeta))$, the solutions are given by

$$\begin{aligned} u_9(x,t) &= -\frac{3k}{A}\tan\frac{1}{2}\left(kx - \frac{kAt}{1+k^2}\right) + c, \\ u_{10}(x,t) &= \frac{k}{A}\left(kx - \frac{kAt}{1-k^2}\right) - \frac{3k}{A}\tan\frac{1}{2}\left(kx - \frac{kAt}{1-k^2}\right) + c, \\ u_{11}(x,t) &= \frac{3k}{A}\cot\frac{1}{2}\left(kx - \frac{kAt}{1+k^2}\right) + c, \\ u_{12}(x,t) &= \frac{k}{A}\left(kx - \frac{kAt}{1-k^2}\right) + \frac{3k}{A}\cot\frac{1}{2}\left(kx - \frac{kAt}{1-k^2}\right) + c, \end{aligned}$$

Case 6. Choosing $P = a^2/4$, $Q = 1/2$, $R = 1/(4a^2)$, and $F(\zeta) = \cos(\zeta)/(a + a\sin(\zeta))$, the solutions are given by

$$\begin{aligned} u_{13}(x,t) &= \frac{6k}{A\left(\tan\frac{1}{2}\left(kx - \frac{kAt}{1+k^2}\right) + 1\right)} + c, \\ u_{14}(x,t) &= \frac{k\left[\left(kx - \frac{kAt}{1-k^2}\right)\tan\frac{1}{2}\left(kx - \frac{kAt}{1-k^2}\right) + \left(kx - \frac{kAt}{1-k^2}\right) + 6\right]}{A\left(tan\frac{1}{2}\left(kx - \frac{kAt}{1-k^2}\right) + 1\right)} + c, \end{aligned}$$

Case 7. Choosing $P = b^2/4$, $Q = 1/2$, $R = 1/(4b^2)$, and
$F(\zeta) = \left(\sqrt{1 - c^2/b^2} + \sin(\zeta)\right)/(b\cos(\zeta) + c)$,

$$\begin{aligned} u_{15}(x,t) &= \frac{k}{A}\frac{(b-c)^2\frac{1}{2}\left(kx - \frac{kAt}{1-k^2}\right)\tan^2\frac{1}{2}\left(kx - \frac{kAt}{1-k^2}\right) + 6bc\tan\frac{1}{2}\left(kx - \frac{kAt}{1-k^2}\right)}{(b^2-c^2)tan\frac{1}{2}\left(kx - \frac{kAt}{1-k^2}\right) + (c^2-b^2)} \\ &+ \frac{k}{A}\frac{(c^2-b^2)\left(kx - \frac{kAt}{1+k^2}\right) + 6b^2\sqrt{1-c^2/b^2}}{(b^2-c^2)\tan\frac{1}{2}\left(kx - \frac{kAt}{1+k^2}\right) + (c^2-b^2)} + c \\ u_{16}(x,t) &= \frac{6k}{A}\left(\frac{b(b-c)\tan\frac{1}{2}\left(kx - \frac{kAt}{1+k^2}\right) + b^2\sqrt{1-c^2/b^2}}{(b^2-c^2)\tan\frac{1}{2}\left(kx - \frac{kAt}{1+k^2}\right) + (c^2-b^2)}\right) + c \end{aligned}$$